Wigner function of Airy wave packet

Airy wave packet

$$\psi(x,t) = \mathrm{Ai}\left[\frac{B}{\hbar^{2/3}}\left(x - \frac{B^3t^2}{4m^2}\right)\right]\exp\left[i\,\frac{B^3t}{2m\hbar}\left(x - \frac{B^3t^2}{6m^2}\right)\right]$$

$B = \hbar = m = 1$:

$$\psi(x,t) = \mathrm{Ai}\left[x - \frac{t^2}{4}\right]\exp\left[i\,\frac{t}{2}\left(x - \frac{t^2}{6}\right)\right]$$

Wigner function

$$W(x,p) = \frac{1}{\pi}\int_{-\infty}^{\infty}\langle x-y|\rho|x+y\rangle\exp[2ipy]\,\mathrm{d}y = \frac{1}{\pi}\int_{-\infty}^{\infty}\psi^*(x+y)\psi(x-y)\exp[2ipy]\,\mathrm{d}y$$

Wigner function of Airy wave packet

$$\begin{aligned}
W(x,p) = {} & \frac{1}{\pi}\int_{-\infty}^{\infty}\psi^*(x+y)\psi(x-y)\exp[2ipy]\,\mathrm{d}y = \\
= {} & \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{Ai}\left[(x+y) - \frac{t^2}{4}\right]\exp\left[-i\,\frac{t}{2}\left((x+y) - \frac{t^2}{6}\right)\right]\mathrm{Ai}\left[(x-y) - \frac{t^2}{4}\right]\exp\left[i\,\frac{t}{2}\left((x-y) - \frac{t^2}{6}\right)\right]\exp[2ipy]\,\mathrm{d}y = \\
= {} & \left\{\exp\left[-i\,\frac{t}{2}\left((x+y) - \frac{t^2}{6}\right)\right]\exp\left[i\,\frac{t}{2}\left((x-y) - \frac{t^2}{6}\right)\right]\exp[2ipy] = \exp\left[i\left(-\frac{t}{2}(x+y) + \frac{t^3}{12} + \frac{t}{2}(x-y) - \frac{t^3}{12} + 2py\right.\right.\right. \\
= {} & \exp\left[i(2p-t)y\right]\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{Ai}\left[\left(x - \frac{t^2}{4}\right) + y\right]\mathrm{Ai}\left[\left(x - \frac{t^2}{4}\right) - y\right]\exp\left[i(2p-t)y\right]\,\mathrm{d}y = \\
= {} & \left\{\begin{array}{ccc} y & = & s - \left(x - \frac{t^2}{4}\right) \\ \mathrm{d}y & = & \mathrm{d}s \end{array}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{Ai}\left[s\right]\mathrm{Ai}\left[2\left(x - \frac{t^2}{4}\right) - s\right]\exp\left[i(2p-t)\left(s - \left(x - \frac{t^2}{4}\right)\right)\right]\,\mathrm{d}s
\end{aligned}$$

$2p - t \equiv \beta$:

$$W(x,p) = \frac{\exp\left[-i\left(x - \frac{t^2}{4}\right)\beta\right]}{\pi}\int_{-\infty}^{\infty}\mathrm{Ai}[s]\mathrm{Ai}\left[2\left(x - \frac{t^2}{4}\right) - s\right]\exp[i\beta s]\,\mathrm{d}s$$

$$\begin{aligned}
\mathcal{F}[W(x,p)](z,p) = {} & \frac{1}{\pi}\int_{-\infty}^{\infty}\exp\left[-i\left(x - \frac{t^2}{4}\right)\beta\right]\left(\int_{-\infty}^{\infty}\mathrm{Ai}[s]\mathrm{Ai}\left[2\left(x - \frac{t^2}{4}\right) - s\right]\exp[i\beta s]\,\mathrm{d}s\right)\exp[-ixz]\,\mathrm{d}x = \\
= {} & \left\{\begin{array}{ccc} x' & = & x - \frac{t^2}{4} \\ \mathrm{d}x' & = & \mathrm{d}x \end{array}\right\} = \frac{1}{\pi}\int_{-\infty}^{\infty}\exp\left[-ix'\beta\right]\left(\int_{-\infty}^{\infty}\mathrm{Ai}[s]\mathrm{Ai}\left[2x' - s\right]\exp[i\beta s]\,\mathrm{d}s\right)\exp\left[-i\left(x' + \frac{t^2}{4}\right)z\right]\,\mathrm{d}x' = \\
= {} & \frac{\exp\left[-i\frac{t^2}{4}z\right]}{\pi}\int_{-\infty}^{\infty}\exp[i\beta s]\mathrm{Ai}[s]\left(\int_{-\infty}^{\infty}\exp\left[-i(\beta+z)x'\right]\mathrm{Ai}\left[2x' - s\right]\,\mathrm{d}x'\right)\,\mathrm{d}s = \\
= {} & \left\{\begin{array}{ccc} u & = & 2x' - s \\ x' & = & \frac{u+s}{2} \\ \mathrm{d}x' & = & \frac{\mathrm{d}u}{2} \end{array}\right\} = \\
= {} & \frac{\exp\left[-i\frac{t^2}{4}z\right]}{2\pi}\int_{-\infty}^{\infty}\exp[i\beta s]\mathrm{Ai}[s]\left(\int_{-\infty}^{\infty}\exp\left[-i(\beta+z)\frac{u+s}{2}\right]\mathrm{Ai}\left[u\right]\,\mathrm{d}u\right)\,\mathrm{d}s = \\
= {} & \frac{\exp\left[-i\frac{t^2}{4}z\right]}{2\pi}\left(\int_{-\infty}^{\infty}\exp\left[-i\frac{(z-\beta)}{2}s\right]\mathrm{Ai}[s]\,\mathrm{d}s\right)\left(\int_{-\infty}^{\infty}\exp\left[-i\frac{(z+\beta)}{2}u\right]\mathrm{Ai}\left[u\right]\,\mathrm{d}u\right) = \\
= {} & \frac{\exp\left[-i\frac{t^2}{4}z\right]}{2\pi}\mathcal{F}\left[\mathrm{Ai}\left(\frac{z-\beta}{2}\right)\right]\mathcal{F}\left[\mathrm{Ai}\left(\frac{z+\beta}{2}\right)\right] = \\
= {} & \frac{\exp\left[-i\frac{t^2}{4}z\right]}{2\pi}\exp\left[\frac{i}{3}\left(\frac{z-\beta}{2}\right)^3\right]\exp\left[\frac{i}{3}\left(\frac{z+\beta}{2}\right)^3\right] = \\
= {} & \frac{1}{2\pi}\exp\left[i\left(\frac{(z-\beta)^3 + (z+\beta)^3}{24} - \frac{t^2}{4}z\right)\right] =
\end{aligned}$$

$$= \frac{1}{2\pi} \exp\left[i\left(\frac{z^3 - 3z^2\beta + 3z\beta^2 - \beta^3 + z^3 + 3z^2\beta + 3z\beta^2 + \beta^3}{24} - \frac{t^2}{4}z\right)\right] =$$
$$= \frac{1}{2\pi} \exp\left[i\left(\frac{z^3 + 3z\beta^2}{12} - \frac{t^2}{4}z\right)\right] = \frac{1}{2\pi} \exp\left[i\left(\frac{\left(\frac{z}{4^{1/3}}\right)^3}{3} + 4^{1/3}\left(\frac{\beta^2}{4} - \frac{t^2}{4}\right)\frac{z}{4^{1/3}}\right)\right]$$

$$\begin{aligned}
W(x,p) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \exp\left[i\left(\frac{\left(\frac{z}{4^{1/3}}\right)^3}{3} + 4^{1/3}\left(\frac{\beta^2}{4} - \frac{t^2}{4}\right)\frac{z}{4^{1/3}}\right)\right] \exp\left[i4^{1/3}x\frac{z}{4^{1/3}}\right] \mathrm{d}z = \\
&= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} \exp\left[i\left(\frac{\left(\frac{z}{2^{2/3}}\right)^3}{3} + 2^{2/3}\left(\frac{\beta^2}{4} + x - \frac{t^2}{4}\right)\frac{z}{2^{2/3}}\right)\right] \mathrm{d}z = \left\{\begin{array}{ccc} z' & = & \frac{z}{2^{2/3}} \\ \mathrm{d}z' & = & \frac{\mathrm{d}z}{2^{2/3}} \end{array}\right\} = \\
&= \frac{2^{2/3}}{(2\pi)^2}\int_{-\infty}^{\infty} \exp\left[i\left(\frac{z'^3}{3} + 2^{2/3}\left(\frac{\beta^2}{4} + x - \frac{t^2}{4}\right)z'\right)\right] \mathrm{d}z' = \\
&= \frac{2^{2/3}}{2\pi}\,\mathrm{Ai}\left[2^{2/3}\left(\frac{\beta^2}{4} + x - \frac{t^2}{4}\right)\right] = \\
&= \frac{2^{2/3}}{2\pi}\,\mathrm{Ai}\left[2^{2/3}\left(\frac{(2p-t)^2}{4} + x - \frac{t^2}{4}\right)\right] = \\
&= \frac{2^{2/3}}{2\pi}\,\mathrm{Ai}\left[2^{2/3}\left(\frac{4p^2 - 4pt + t^2}{4} + x - \frac{t^2}{4}\right)\right] =
\end{aligned}$$

$$W(x,p) = \frac{1}{2^{1/3}\pi}\,\mathrm{Ai}\left[2^{2/3}(p^2 - pt + x)\right]$$