Solution of the bonus example from distance exams (spring 2020):

Task:
Calculate the volume of the body $\mathcal{V}(x,y,z)$: $\left\{z \in \langle \frac{x+R}{2}, H - \sqrt{x^2+y^2}\rangle \wedge \sqrt{x^2+y^2}\big|_{z=0} = R\right\}$. Give the result as a single term (i.e., not as a sum of several different terms) using parameters R and H. Sketch the given body. What is the ratio of volumes of the given body and the same body where, however, the bottom level of the z-coordinate will be zero?

Solution:
It follows from the assignment that the given body $\mathcal{V}$ is bounded from above by a conical surface of height H equal to the radius of the base R and from below by a planar surface which is parallel to the y direction, intersects the axis of the cone at half its height, and passes through the base of the cone at point $x = -R$. The specified body is highlighted by a colored area in the schematic Figure 1.

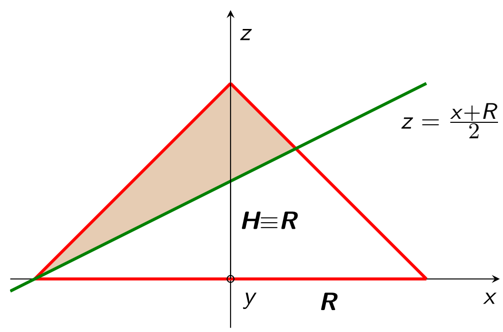

Figure 1: Schematic representation of the specified body in the plane xz. The contour of the cone with the base radius R and the height $H \equiv R$ is shown in red while the lower limiting planar surface in green. The resulting body is highlighted by an ocher (brown) area.

The volume of the body is calculated by integration, preferably in the cylindrical coordinate system, i.e., $\mathcal{V}(x,y,z) \to \mathcal{V}(r,\phi,z)$. The following integration limits result directly from the assignment:

$$\phi \in \langle 0, 2\pi\rangle\,, \quad z \in \left\langle \frac{r\cos\phi + R}{2},\, R - r\right\rangle. \tag{1}$$

It will be somewhat more complicated with the upper limit of integration for the cylindrical radial coordinate r, which in this case is not constant and is given by the intersection of the "green" planar surface with the "red" conical surface; by comparing the equations of these two surfaces (i.e., from the equality of the upper and lower bounds for z) it follows:

$$r \in \left\langle 0,\, \frac{R}{\cos\phi + 2}\right\rangle. \tag{2}$$

The only integral independent of the other coordinates is the integral over ϕ. The explicit form of the triple integral for the volume of a given body (see Equation 7.6 in the Computing Practice script) will therefore be

$$V = \int_0^{2\pi}\left[\int_0^{\frac{R}{\cos\phi+2}} r\left(\int_{\frac{r\cos\phi+R}{2}}^{R-r} \mathrm{d}z\right)\mathrm{d}r\right]\mathrm{d}\phi, \tag{3}$$

where r in the integrand of the "middle" integral (i.e., the integral of the radial coordinate, enclosed in square brackets), is a Jacobian of the cylindrical system.

Both the "inner" integrals (over z and r) are easy to solve, so we get

$$V = \frac{R^3}{12}\int_0^{2\pi}\frac{\mathrm{d}\phi}{(\cos\phi+2)^2}. \tag{4}$$

This integral can be solved by the universal substitution $\mathrm{tg}\,(\phi/2) = t$, so $\cos\phi = (1-t^2)/(1+t^2)$, $\mathrm{d}\phi = 2\,\mathrm{d}t/(1+t^2)$, or $\sin\phi = 2t/(1+t^2)$ (at this point, it was also allowed to use suitable analytical software, for example Wolfram Alpha). Once substituted, the integrand $\mathcal{I}$ of Equation (4), including the sticked out constant expressions, will have the form

$$\mathcal{I} = \frac{R^3\left(1+t^2\right)}{6\left(3+t^2\right)^2}. \tag{5}$$

However, a fundamental step, requiring a certain “skill“ or experience, is associated with the transformation of integration limits. If we substitute the existing limits $\phi = 0$ and $\phi = 2\pi$ into the universal substitution, we get zero upper and lower limits for the new variable t (which is related to the fact that the function $\mathrm{tg}\,(\phi/2)$ in the interval $\langle 0, 2\pi\rangle$ contains the inner singularity exactly at the center of this interval, at the point π). If we take a closer look at the profile of the function $(\cos\phi+2)^{-2}$ (try to plot it), we see that it is periodic with the period 2π (with minima in even and maxima in odd multiples of π) and its functional values are always positive. Therefore, if we shift both limits simultaneously by any interval, the integral (area under the curve) of such a function must be the same. Changing the limits $\phi = 0$, $\phi = 2\pi$ in Equation (4) to $\phi = -\pi$, $\phi = \pi$ (in this interval the function $\mathrm{tg}\,(\phi/2)$ does not contain internal singularity, singularities are only within the both limits), we get the integral for the new variable t in the form

$$V = \frac{R^3}{6}\int_{-\infty}^{\infty} \frac{1+t^2}{\left(3+t^2\right)^2}\,\mathrm{d}t. \tag{6}$$

Such an integral can already be solved relatively easily. In the first step, we will expand it, for example, in the following way,

$$V = \frac{R^3}{6}\int_{-\infty}^{\infty}\left[\frac{3+t^2}{\left(3+t^2\right)^2} - \frac{2}{\left(3+t^2\right)^2}\right]\mathrm{d}t = \frac{R^3}{6}\int_{-\infty}^{\infty}\left[\frac{1}{3+t^2} - \frac{2}{\left(3+t^2\right)^2}\right]\mathrm{d}t. \tag{7}$$

We solve the first term in the integrand of the last equation by a simple substitution $t = \sqrt{3}\,u$. The best substitution for the second term is $t = \sqrt{3}\ \mathrm{tg}\,v$ (and therefore $\mathrm{d}t = \sqrt{3}\ \mathrm{d}v/\cos^2 v$), which will lead to the next sum of two integrals in the form

$$V = \frac{R^3}{6}\left(\frac{1}{\sqrt{3}}\int_{-\infty}^{\infty}\frac{\mathrm{d}u}{1+u^2} - \frac{2}{3\sqrt{3}}\int_{-\pi/2}^{\pi/2}\cos^2 v\ \mathrm{d}v\right), \tag{8}$$

After simple integration and substitution, we get the result,

$$V = \frac{\pi R^3}{9\sqrt{3}}. \tag{9}$$

The volume of the considered body is therefore $(3\sqrt{3})^{-1}$ times the volume of the whole cone with input parameters.