# 1 Linear theory in a weak field approximation

(More detailed technical and explanatory remarks are marked in red.)

Creating a mathematical formalism describing gravitational waves is extremely difficult, among other things, because of its nonlinearity. In this case, we consider spacetime ripples very weak because we are far from the source. Following this assumption, we can linearize the gravitational field as a slightly deformed Minkowski flat spacetime $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ with a small perturbation $h_{\mu\nu}$ (we use the convention $+---$ here, other authors may use the opposite, while the physics stays unaffected),

$$g_{\mu\nu} \simeq \eta_{\mu\nu} + h_{\mu\nu} + \mathcal{O}(h_{\mu\nu})^2, \quad |h_{\mu\nu}| \ll 1. \tag{1}$$

We raise and lower the indices of this terms by $\eta_{\mu\nu}$: following the principle,

$$h^{\mu\nu} = \eta^{\mu\alpha}\eta^{\nu\beta}h_{\alpha\beta}, \quad h = \eta^{\mu\nu}h_{\mu\nu}, \tag{2}$$

we obtain the “upper index” linearization

$$g^{\mu\nu} \simeq \eta^{\mu\nu} - h^{\mu\nu}. \tag{3}$$

It’s because $g^{\mu\nu}$ is obtained from $g_{\mu\nu}$ as the inverse matrix, i.e., via

$$g^{\mu\nu}g_{\nu\sigma} = \delta^{\mu}_{\ \sigma}. \tag{4}$$

Now we formally distinguish “lower” and “upper” indexed perturbations as $h$ and $\tilde{h}$,

$$g_{\nu\sigma} = \eta_{\nu\sigma} + h_{\nu\sigma}, \quad g^{\mu\nu} = \eta^{\mu\nu} + \tilde{h}^{\mu\nu}, \tag{5}$$

noting again that $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$. Following this, we get

$$g^{\mu\nu}g_{\nu\sigma}(= \delta^{\mu}_{\ \sigma}) = (\eta^{\mu\nu} + \tilde{h}^{\mu\nu})(\eta_{\nu\sigma} + h_{\nu\sigma}) = \delta^{\mu}_{\ \sigma} + h^{\mu}_{\ \sigma} + \tilde{h}^{\mu}_{\ \sigma} + \mathcal{O}(\tilde{h}^{\mu\nu}h_{\nu\sigma}), \tag{6}$$

which means (or, after raising the lower index again with $\eta^{\nu\sigma}$),

$$\tilde{h}^{\mu}_{\ \sigma} = -h^{\mu}_{\ \sigma}, \quad \tilde{h}^{\mu\nu} = -h^{\mu\nu}. \tag{7}$$

So, this results from the consistently applied formalism of lowering and raising indices with $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$, respectively.

Now we want to find the equation of motion of the perturbations $h_{\mu\nu}$ by examining Einstein’s equations in the first order. We adopt the Christoffel symbols to the linearized gravity (since the derivatives of $\eta_{\mu\nu}$ are zero) as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}\right) = \frac{1}{2}\left(\partial_\mu h^{\rho}_{\ \nu} + \partial_\nu h^{\rho}_{\ \mu} - \partial^\rho h_{\mu\nu}\right). \tag{8}$$

$$\Gamma^{\rho}_{\mu\rho} = \frac{1}{2}\eta^{\rho\sigma}\left(\partial_\mu h_{\rho\sigma} + \partial_\rho h_{\mu\sigma} - \partial_\sigma h_{\mu\rho}\right) = \frac{1}{2}\left(\partial_\mu h^{\rho}_{\ \rho} + \partial_\rho h^{\rho}_{\ \mu} - \partial_\sigma h^{\sigma}_{\ \mu}\right). \tag{9}$$

The Ricci tensor (since the Christoffel symbols are the first order quantities; we employ only the derivatives of $\Gamma$ and neglect the $\Gamma^2$ terms) will then take the form

$$R_{\mu\nu} \simeq \partial_\rho \Gamma^{\rho}_{\mu\nu} - \partial_\nu \Gamma^{\rho}_{\mu\rho} \simeq \frac{1}{2}\left(\partial_\rho\partial_\mu h^{\rho}_{\ \nu} + \partial_\rho\partial_\nu h^{\rho}_{\ \mu} - \partial_\mu\partial_\nu h^{\rho}_{\ \rho} - \partial_\rho\partial^\rho h_{\mu\nu}\right). \tag{10}$$

The term $h^\rho{}_\rho = \eta^{\rho\sigma} h_{\sigma\rho} = h$, and the operator $\partial_\rho \partial^\rho$ represents the scalar product of the four-derivatives in Minkowski space with the raised index $\partial^\rho = \eta^{\rho\sigma}\partial_\sigma$, that is,

$$\partial_\rho \partial^\rho = c^{-2}\partial_t^2 - \partial_x^2 - \partial_y^2 - \partial_z^2 = c^{-2}\partial_t^2 - \nabla^2 = \Box, \tag{11}$$

where the "box" shortly symbolizes the d'Alembert operator (also called the d'Alembertian). Applying the $\eta^{\mu\nu}R_{\mu\nu}$ contraction of Ricci tensor, we obtain the Ricci scalar $R$ as

$$R = \frac{1}{2}\left(\partial_\rho\partial_\mu h^{\mu\rho} + \partial_\rho\partial_\nu h^{\nu\rho} - \partial_\mu\partial^\mu h^\rho{}_\rho - \partial_\rho\partial^\rho h^\mu{}_\mu\right) = \partial_\rho\partial_\mu h^{\mu\rho} - \partial_\rho\partial^\rho h^\mu{}_\mu. \tag{12}$$

Using the contraction of $h^\rho{}_\rho = h$ and the d'Alembertian symbol, we can write the Ricci tensor and Ricci scalar of the linearized perturbed spacetime in the more convenient form as

$$R_{\mu\nu} = \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu}\right), \quad R = \partial_\rho\partial_\mu h^{\mu\rho} - \Box h. \tag{13}$$

We directly obtain the Einstein tensor $G_{\mu\nu}$ of this linearized flat spacetime perturbations by summing the Ricci tensor and Ricci scalar,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R = \frac{1}{2}\left(\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\mu h^{\mu\rho} + \eta_{\mu\nu}\Box h\right). \tag{14}$$

We can thus simply write the equation of the *linearized gravity*,

$$\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho\partial_\mu h^{\mu\rho} + \eta_{\mu\nu}\Box h = \frac{16\pi G}{c^4}T_{\mu\nu}. \tag{15}$$

As a next step, it is convenient to introduce a "trace-reversed" (in 4-dimensional spacetime) tensor $\phi_{\mu\nu}$ defined as

$$\phi_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{16}$$

which manifestly gives $\phi = -h$ and so $h_{\mu\nu} = \phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi$ (for the same reason, we may regard the Einstein tensor as simply the trace-reversed Ricci tensor). The metric perturbation $h_{\mu\nu}$ and the trace-reversed perturbation $\phi_{\mu\nu}$ thus contain the same information. In $n$-dimensional spacetime it would be

$$h = h^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu} = h_{00} - h_{11} - h_{22} - h_{33} - \ \dots\ - h_{nn}, \tag{17}$$

$$\phi = \phi^\mu{}_\mu = \eta^{\mu\nu}h_{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}h = h - \frac{1}{2}nh, \tag{18}$$

noting that $\eta^{\mu\nu}\eta_{\mu\nu} = n$ (summation convention). If $n = 4$, then $\phi = h - 2h = -h$. We can now insert the tensor $\phi_{\mu\nu}$ into Eq. (15), getting

$$\begin{aligned}\partial_\rho\partial_\mu\left[\eta^{\rho\mu}\left(\phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi\right)\right] + \partial_\rho\partial_\nu\left[\eta^{\rho\nu}\left(\phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi\right)\right] + \partial_\mu\partial_\nu\phi - \Box\left(\phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi\right) - \\ -\eta_{\mu\nu}\partial_\rho\partial_\mu\left[\eta^{\mu\nu}\eta^{\mu\rho}\left(\phi_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\phi\right)\right] - \eta_{\mu\nu}\Box\phi = \frac{16\pi G}{c^4}T_{\mu\nu} \Rightarrow\end{aligned} \tag{19}$$

$$\begin{aligned}\partial_\rho\partial_\mu\phi^\rho{}_\nu - \frac{1}{2}\partial_\mu\partial_\nu\phi + \partial_\rho\partial_\nu\phi^\rho{}_\mu - \frac{1}{2}\partial_\mu\partial_\nu\phi + \partial_\mu\partial_\nu\phi - \Box\phi_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\Box\phi - \\ -\eta_{\mu\nu}\partial_\rho\partial_\mu\phi^{\mu\rho} + \frac{1}{2}\eta_{\mu\nu}\Box\phi - \eta_{\mu\nu}\Box\phi = \frac{16\pi G}{c^4}T_{\mu\nu} \Rightarrow\end{aligned} \tag{20}$$

$$\partial_\rho \partial_\mu \phi^\rho{}_\nu + \partial_\rho \partial_\nu \phi^\rho{}_\mu - \Box \phi_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\mu \phi^{\mu\rho} = \frac{16\pi G}{c^4} T_{\mu\nu}, \tag{21}$$

we thus reduced the left-hand side of the equation of the linearized gravity to four terms; by lowering the upper index $\rho$ in the term $\phi^{\mu\nu}$, we can even reduce the left-hand side to two terms,

$$\partial_\rho \partial_\nu \phi^\rho{}_\mu - \Box \phi_{\mu\nu} = \frac{16\pi G}{c^4} T_{\mu\nu}, \tag{22}$$

Another step in the solution of this problem is to "choose a gauge." This is similar to the "electromagnetic" Lorentz gauge condition on the vector potential $A_\mu$ (by setting $\phi' = \phi - \psi_t$, $\vec{A}' = \vec{A} + \vec{\nabla}\psi$, where $\psi$ is an arbitrary function), in which $\nabla_\mu A^\mu = 0$. To illustrate the gauge from basic principles, let's introduce two coordinate systems $x_\mu$ and $x'_\mu$, deviating from each other by a very small displacement $\xi^\mu$ (four functions of the order $h_{\mu\nu}$ in a 4-dimensional spacetime),

$$x^{\mu\prime} = x^\mu + \xi^\mu. \tag{23}$$

Then, obviously,

$$\frac{\partial x^\mu}{\partial x^{\nu\prime}} = \delta^\mu_\nu - \partial_\nu \xi^\mu, \quad \frac{\partial x^{\mu\prime}}{\partial x^\nu} = \delta^\mu_\nu + \partial_\nu \xi^\mu, \tag{24}$$

and the first-order displacement is

$$\begin{aligned} g'_{\mu\nu}(x') = \frac{\partial x^\rho}{\partial x^{\mu\prime}} \frac{\partial x^\sigma}{\partial x^{\nu\prime}} g_{\rho\sigma}(x) &= \left(\delta^\rho{}_\mu - \partial_\mu \xi^\rho\right)\left(\delta^\sigma{}_\nu - \partial_\nu \xi^\sigma\right)\left(\eta_{\rho\sigma} + h_{\rho\sigma}\right) \\ &\simeq \eta_{\mu\nu} + h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu = \eta_{\mu\nu} + h'_{\mu\nu}. \end{aligned} \tag{25}$$

Thus, the metric transforms as

$$h'_{\mu\nu} = h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu \tag{26}$$

and, following the contraction $h' = \eta^{\mu\nu} h'_{\mu\nu} = \eta^{\mu\nu}(h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu) = h^\mu_\mu - \partial_\mu \xi^\mu - \partial_\nu \xi^\nu = h - 2\partial_\alpha \xi^\alpha$, we can immediately generalize the gauge condition in the trace-reversed perturbation as

$$\phi'_{\mu\nu} = \phi_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu + \eta_{\mu\nu} \partial_\lambda \xi^\lambda. \tag{27}$$

In general relativity, solving Einstein's equations, it is common practice to choose a *harmonic gauge* condition (also known as Lorentz, Hilbert, de Donder, or Fock gauge) on the coordinate system, where (absolutely analogously to the Lorentz gauge in EM)

$$\Box x^\mu = 0. \tag{28}$$

Since $\Box = \nabla^\alpha \nabla_\alpha$ (recalling that the covariant derivative $\nabla_\alpha x^\mu = \partial_\alpha x^\mu + \Gamma^\mu_{\alpha\nu} x^\nu$, $\nabla_\alpha \omega_\mu = \partial_\alpha \omega_\mu - \Gamma^\nu_{\alpha\mu} \omega_\nu$ where $x_\mu$ is the four-vector and $\omega_\mu$ is the one-form why? because the covariant derivative of a scalar is simply the partial derivative, so let's assume an ad-hoc symbolized expression $\nabla_\alpha \omega_\mu = \partial_\alpha \omega_\mu + \widetilde{\Gamma}^\nu_{\alpha\mu} \omega_\nu$ and let's impose the covariant derivative on a scalar $\nabla_\alpha \left(\omega_\mu x^\mu\right) = x^\mu \left(\nabla_\alpha \omega_\mu\right) + \omega_\mu \left(\nabla_\alpha x^\mu\right)$, giving $x^\mu(\partial_\alpha \omega_\mu + \widetilde{\Gamma}^\sigma_{\alpha\mu} \omega_\sigma) + \omega_\mu \left(\partial_\alpha x^\mu + \Gamma^\mu_{\alpha\nu} x^\nu\right)$, while, simultaneously, $\nabla_\alpha \left(\omega_\mu x^\mu\right) = \partial_\alpha \left(\omega_\mu x^\mu\right) = x^\mu \left(\partial_\alpha \omega_\mu\right) + \omega_\mu \left(\partial_\alpha x^\mu\right)$; this cancels the connection coefficient (Christoffel symbols) terms, $\widetilde{\Gamma}^\sigma_{\alpha\mu} \omega_\sigma x^\mu = -\Gamma^\mu_{\alpha\nu} x^\nu \omega_\mu$, so, since $\widetilde{\Gamma}^\sigma_{\alpha\mu} \omega_\sigma x^\mu \equiv \widetilde{\Gamma}^\mu_{\alpha\nu} \omega_\mu x^\nu$, this finally proves that $\widetilde{\Gamma}^\mu_{\alpha\nu} =$

$-\Gamma^{\mu}_{\alpha\nu}$), we may this condition (noting that it is imposed on a scalar coordinate function $x^\mu$ within the orthogonal four-coordinate system where obviously $\partial_\alpha x^\mu = \delta^\mu_\alpha$) explicitly expand as

$$0 = \Box x^\mu = \eta^{\rho\sigma}(\partial_\rho \partial_\sigma x^\mu - \Gamma^\lambda_{\rho\sigma}\partial_\lambda x^\mu) = -\eta^{\rho\sigma}\Gamma^\lambda_{\rho\sigma}. \tag{29}$$

At this point, it is necessary to strictly keep in mind that the four functions $x^\mu$ are just functions, not components of a vector; since the covariant derivative of a scalar function is just the partial derivative, $\nabla_\alpha x^\mu = \partial_\alpha x^\mu$, we simply arrive at the expression (29). Expanding the Eq. (29) explicitly in the weak field limit,

$$0 = \frac{1}{2}\eta^{\rho\sigma}\eta^{\lambda\alpha}(\partial_\rho h_{\sigma\alpha} + \partial_\sigma h_{\rho\alpha} - \partial_\alpha h_{\rho\sigma}) = \partial_\rho h^\rho_\alpha - \frac{1}{2}\partial_\alpha h. \tag{30}$$

The first term on the left-hand side of Eq. (15) then transforms to $\frac{1}{2}\partial_\nu\partial_\mu h = \frac{1}{2}\eta_{\mu\nu}\Box h$, while the fifth term transforms to $-\partial_\rho\partial_\mu h^\rho_\nu = -\frac{1}{2}\eta_{\mu\nu}\Box h$; that is, Eq. (15) by imposing the Lorentz gauge simplifies to

$$\Box h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\Box h = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{31}$$

or

$$\Box \phi_{\mu\nu} = -\frac{16\pi G}{c^4}T_{\mu\nu}, \tag{32}$$

while the vacuum form of Einstein equation $R_{\mu\nu} = 0$ (together with the gauge invariance expressed by Eq. (23) and the Lorentz gauge condition, Eq. (28)) super-reduces Eq. (31) to

$$\Box h_{\mu\nu} = 0, \qquad \Box\phi_{\mu\nu} = 0, \qquad \Box\xi^\mu = 0. \tag{33}$$

Moreover, by raising indices and substituting the trace-reversed tensor $\phi_{\mu\nu}$ into Eq. (30), we obtain the alternative important form of the Lorentz gauge condition,

$$0 = (\partial_\rho h^\rho_\alpha - \frac{1}{2}\partial_\alpha h)\eta^{\alpha\nu} = \partial_\rho h^{\rho\nu} - \frac{1}{2}\partial^\nu h = \partial_\rho(\phi^{\rho\nu} - \frac{1}{2}\eta^{\rho\nu}\phi) + \frac{1}{2}\partial^\nu\phi \Rightarrow$$
$$\partial_\rho\phi^{\rho\nu} = 0. \tag{34}$$

Now, we apply the linearized gravity to a gravitational field of an isolated mass in the Newtonian limit. In this case, we assume that the energy-momentum tensor (see explanations in Example 2.5 in the textbook "Prakticke_pocetni_metody_pro_fyziky") is dominated by the energy density $\rho c^2$ ($T_{\mu\nu} = \rho c^2 \delta_{\mu 0}\delta_{\nu 0}$), the matter is practically static, or it moves slowly enough so we may neglect the time derivatives (then $\Box = -\nabla^2$ in the convention $+---$ of the flat spacetime), and the spacetime is "asymptotically flat," that is, it behaves as the Minkowski spacetime at large distances. Then Eq. (32) says

$$\nabla^2\phi_{00} = \frac{16\pi G}{c^2}\rho \tag{35}$$

and, after implementing the gravitational Poisson equation $\nabla^2\Phi = 4\pi G\rho$ (we hereafter consistently distinguish $\Phi$ as a gravitational potential from a trace-reversed perturbation $\phi$),

$$\phi_{00} = \frac{4\Phi}{c^2}. \tag{36}$$

Since the other components of $\phi_{\mu\nu}$ are negligible, then $\phi = \phi_{00}$, and $h_{i0} = \phi_{i0} - \frac{1}{2}\eta_{i0}\phi = 0$ (where $i = 1, 2, 3$ are the spatial components of the metric). The "inverse" Eq. (16) thus immediately gives

$$h_{\mu\nu} = \frac{2\Phi}{c^2}\delta_{\mu\nu}. \tag{37}$$

Finally, following Eq. (1), the metric form of a perturbed spacetime in this weak-field limit is

$$\mathrm{d}s^2 = \left(c^2 + 2\Phi\right)\mathrm{d}t^2 - \left(1 + \frac{2\Phi}{c^2}\right)\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right). \tag{38}$$

We now describe the formalism of the weak-field limit application to gravitational radiation. Let's suppose the perturbation within the vacuum solution $\Box\phi_{\mu\nu} = 0$ radiates plane waves in the form

$$\phi_{\mu\nu} = A_{\mu\nu}\,\mathrm{e}^{\mathrm{i}k_\alpha x^\alpha} = A_{\mu\nu}\,\mathrm{e}^{-\mathrm{i}(k_i x^i - \omega t)}, \tag{39}$$

where $A_{\mu\nu}$ is a constant and symmetric spacetime tensor of second order (consisting thus of ten independent components, called polarization tensor including information of the amplitude and the polarization properties of the gravitational waves), and $k_\alpha$ is the wavevector; $k_0 = k^0 = \omega/c$, $k_i = -k^i$. Then, the flat-space d'Alembertian imposed on a scalar complex 4-exponential yields

$$0 = \Box\phi_{\mu\nu} = \eta^{\alpha\beta}\partial_\alpha\partial_\beta\phi_{\mu\nu} = \eta^{\alpha\beta}\partial_\alpha(\mathrm{i}k_\beta\phi_{\mu\nu}) = -\eta^{\alpha\beta}k_\alpha k_\beta\phi_{\mu\nu} = -k_\alpha k^\alpha\phi_{\mu\nu}. \tag{40}$$

Because at least some of the $\phi_{\mu\nu}$ components must be nonzero (otherwise, we do not have any subject to deal with), we have a solution of a null wavevector,

$$k_\alpha k^\alpha = 0, \tag{41}$$

which immediately shows that the gravitational waves propagate with the speed of light. Moreover, since $\omega^2 = c^2\delta_{ij}k^i k^j = c^2k^2$ ($k_\alpha k^\alpha = \omega^2 - c^2|\vec{k}|^2 = 0$) (where $k^2 = |\vec{k}|^2$), this relation explicitly suggests

$$v_{\text{phase}} = \frac{\omega}{k} = c, \qquad v_{\text{group}} = \frac{\partial\omega}{\partial k} = c, \tag{42}$$

that is, the group as well as the phase velocity of the gravitational waves are equal to the speed of light $c$.

This simple plane wave, of course does not describe the complete or general solution; (possibly) an infinite number of distinct plane waves can be superposed and solve the linear wave equations (33). Imposing the Lorentz gauge condition (34) on Eq. (39), we see that

$$0 = \partial_\mu(A^{\mu\nu}\,\mathrm{e}^{\mathrm{i}k_\alpha x^\alpha}) = \mathrm{i}k_\mu A^{\mu\nu}\,\mathrm{e}^{\mathrm{i}k_\alpha x^\alpha}, \tag{43}$$

which is fulfilled if and only if

$$k_\mu A^{\mu\nu} = 0. \tag{44}$$

Thus, we may regard the wavevector $k_\mu$ as orthogonal to $A^{\mu\nu}$; this condition reduces the number of independent components of $A^{\mu\nu}$ from ten to six. Explicitly, recalling that due to the symmetry of $A_{\mu\nu}$ ($A_{\mu\nu} = A_{\nu\mu}$),

$$
\begin{aligned}
c\left(k_\mu A^{\mu 0}\right) &= \omega A^{00} - c\left(k_1 A^{10} + k_2 A^{20} + k_3 A^{30}\right) = 0, && (45)\\
c\left(k_\mu A^{\mu 1}\right) &= \omega\left(A^{01} \equiv A^{10}\right) - c\left(k_1 A^{11} + k_2 A^{21} + k_3 A^{31}\right) = 0, && (46)\\
c\left(k_\mu A^{\mu 2}\right) &= \omega\left(A^{02} \equiv A^{20}\right) - c\left[k_1\left(A^{12} \equiv A^{21}\right) + k_2 A^{22} + k_3 A^{32}\right] = 0, && (47)\\
c\left(k_\mu A^{\mu 3}\right) &= \omega\left(A^{03} \equiv A^{30}\right) - c\left[k_1\left(A^{13} \equiv A^{31}\right) + k_2\left(A^{23} \equiv A^{32}\right) + k_3 A^{33}\right] = 0, && (48)
\end{aligned}
$$

showing that in this case, the ten originally independent components $A^{00}$, $A^{10}$, $A^{20}$, $A^{30}$, $A^{11}$, $A^{21}$, $A^{31}$, $A^{22}$, $A^{32}$, $A^{33}$, reduce to only six, $A^{11}$, $A^{21}$, $A^{31}$, $A^{22}$, $A^{32}$, and $A^{33}$ (the same applies for symmetric counterparts of the nondiagonal ones). From the last equation (33) and analogously to Eq. (39), we claim the solution for the displacement

$$\xi_\mu = B_\mu\, \mathrm{e}^{\mathrm{i}k_\sigma x^\sigma}, \tag{49}$$

where $B_\mu$ are constant coefficients and $k_\alpha$ is the wave four-vector. Then, following (27) with Eq. (39) plugged in and with Eq. (49) for the transformation of displacement, we obtain the gravitational wave amplitude changes as

$$A'_{\mu\nu} = A_{\mu\nu} - \mathrm{i}k_\mu B_\nu - \mathrm{i}k_\nu B_\mu + \mathrm{i}\eta_{\mu\nu} k_\lambda B^\lambda, \tag{50}$$

This, after raising indices by $\eta^{\mu\nu}$ changes to

$${A'}^{\mu}_{\ \mu} = A^\mu_{\ \mu} - \mathrm{i}k^\nu B_\nu - \mathrm{i}k^\mu B_\mu + 4\mathrm{i}k_\lambda B^\lambda = A^\mu_{\ \mu} + 2\mathrm{i}k_\lambda B^\lambda. \tag{51}$$

which can also be simply modified to

$${A'}^{\mu}_{\ \mu} = A^\mu_{\ \mu} + 2\mathrm{i}k_\lambda B^\lambda. \tag{52}$$

This can be rearranged as

$${A'}^{\mu}_{\ \mu} = A^\mu_{\ \mu} + 2\mathrm{i}k_\lambda B_\sigma \eta^{\sigma\lambda}, \tag{53}$$

which we may explicitly write as (employing only the relevant terms)

$$
\begin{aligned}
{A'}^{\mu}_{\ \mu} &= A^\mu_{\ \mu} + 2\mathrm{i}\left(k_0 B_0 \eta^{00} + k_1 B_1 \eta^{11} + k_2 B_2 \eta^{22} + k_3 B_3 \eta^{33}\right) = && (54)\\
&= A^\mu_{\ \mu} + 2\mathrm{i}\left(k_0 B_0 - k_1 B_1 - k_2 B_2 - k_3 B_3\right), && (55)
\end{aligned}
$$

Following Eqs. (50) and (52), we can construct the matrix transformation for selected amplitudes,

$$\begin{pmatrix} \frac{1}{2}{A'}^{\mu}_{\ \mu} \\ A'_{01} \\ A'_{02} \\ A'_{03} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}A^\mu_{\ \mu} \\ A_{01} \\ A_{02} \\ A_{03} \end{pmatrix} + \mathrm{i} \begin{pmatrix} \frac{\omega}{c} & -k_1 & -k_2 & -k_3 \\ -k_1 & -\frac{\omega}{c} & 0 & 0 \\ -k_2 & 0 & -\frac{\omega}{c} & 0 \\ -k_3 & 0 & 0 & -\frac{\omega}{c} \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}, \tag{56}$$

where, because we may choose the coordinate shift amplitudes $B_\mu$ however we need, we can also adjust the coordinate system so that

$${A'}^{\mu}_{\ \mu} = 0, \qquad A'_{0\nu} = 0, \tag{57}$$

to which we convert the "old" $A_{\mu\nu}$ coefficients. We can also extend the solution (44) via

$$k^\mu A'_{\mu\nu} = k^\mu A_{\mu\nu} - \mathrm{i}k^\mu k_\mu B_\nu - \mathrm{i}k^\mu k_\nu B_\mu + \mathrm{i}\eta_{\mu\nu}k^\mu k_\lambda B^\lambda = \\ = 0 - 0 - \mathrm{i}k_\nu \left(k^\mu B_\mu - k_\lambda B^\lambda\right) = 0, \quad \text{that is} \quad k^\mu A'_{\mu\nu} = 0. \tag{58}$$

For clarity, we readjust the relation (56) as

$$\begin{pmatrix} \frac{1}{2}A^\mu{}_\mu \\ A_{01} \\ A_{02} \\ A_{03} \end{pmatrix} = \mathrm{i} \begin{pmatrix} -\frac{\omega}{c} & k_1 & k_2 & k_3 \\ k_1 & \frac{\omega}{c} & 0 & 0 \\ k_2 & 0 & \frac{\omega}{c} & 0 \\ k_3 & 0 & 0 & \frac{\omega}{c} \end{pmatrix} \begin{pmatrix} B_0 \\ B_1 \\ B_2 \\ B_3 \end{pmatrix}. \tag{59}$$

In any case, the first equation (57) brings another constraint to the already announced six independent components $A^{\mu\nu}$, reducing them to five. The second equation (57) should bring four more constraints; however, since one of these is already dependent via Eq. (44), we have only three additional constraints at this point. The total number of independent coefficients $A^{\mu\nu}$ is thus reduced to two. The following suggestions will yet more clearly explain this.

## 2 The transverse - traceless (TT) gauge

We can now choose the coordinate system oriented so that the plane gravitational wave is propagating in the direction of the third spatial axis $x^\mu = x^3$. In this case, according to the null vector solution (41), we have

$$k_\mu k^\mu = k_0 k^0 + k_3 k^3 = \frac{\omega}{c}\frac{\omega}{c} + x_3(-x_3) = 0 \Rightarrow x_3 = \frac{\omega}{c} \tag{60}$$

$$k^\mu = \left(\frac{\omega}{c}, 0, 0, x^3\right) = \left(\frac{\omega}{c}, 0, 0, \frac{\omega}{c}\right). \tag{61}$$

In this case, the condition (58) now gives $k^0 A'_{0\nu} + k^3 A'_{3\nu} = 0$, which, together with the second equation (57), means

$$A'_{3\nu} = 0, \tag{62}$$

reducing the number of independent terms in the $A'_{\mu\nu}$ to four, $A'_{11}$, $A'_{12}$, $A'_{21}$, and $A'_{22}$. Moreover, $A'_{21} = A'_{12}$ due to the symmetry of $A'_{\mu\nu}$ and the first equation (57) implies that $A'_{\mu\nu}$ is traceless, $A'^{\mu}{}_{\mu} = \eta^{\mu\nu}A'_{\mu\nu} = A'_{00} - A'_{11} - A'_{22} - A'_{33} = 0$, $A'_{22} = -A'_{11}$, and

$$A'_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A'_{11} & A'_{12} & 0 \\ 0 & A'_{12} & -A'_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{63}$$

The plane is within this configuration completely described by frequency $\omega$ and the two components $A'_{11}$ and $A'_{12}$. We call this particular gauge the *Transverse Traceless* (TT) *gauge* (or radiation gauge). Now, since the trace-reversed perturbation $\phi_{\mu\nu}$ is in this gauge traceless (because $A_{\mu\nu}$ is), and it is equal to the trace-reverse of $h_{\mu\nu}$, we have

$$\phi^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}. \tag{64}$$

We can, therefore, use both quantities equally in this gauge.

Let's rename the two numbers to forms that will better illustrate the principles (will be explained soon), $A_+ = A'_{11}$ and $A_\times = A'_{12}$, then

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_+ & A_\times & 0 \\ 0 & A_\times & -A_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad h_{\mu\nu} = \boldsymbol{A}\,\mathrm{e}^{ik_\sigma x^\sigma} \tag{65}$$

The quantities $A_+$ and $A_\times$ are invariant under gauge transformations according to Eq. (50), so we may use them to compute the amplitudes of an arbitrary gravitational wave. The metric associated with a gravitational wave within this gauge is

$$\begin{aligned} \mathrm{d}s^2 &= g_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \\ &= c^2\,\mathrm{d}t^2 - (1 + A_+\mathrm{e}^{ik_\sigma x^\sigma})(\mathrm{d}x^1)^2 - 2A_\times \mathrm{e}^{ik_\sigma x^\sigma}\mathrm{d}x^1\mathrm{d}x^2 - (1 - A_+\mathrm{e}^{ik_\sigma x^\sigma})(\mathrm{d}x^2)^2 - (\mathrm{d}x^3)^2. \end{aligned} \tag{66}$$

Let's consider a linearly polarized wave with polarization $+$, and an ensemble of test particles at coordinates $x^1$, $x^2$, $x^3$. As the wave passes, the particles separated in the 1st direction oscillate in this direction. The particles separated in the 2nd direction do the same in the counter-phase. The distance element in the 3rd direction (the direction of propagation) remains unaffected by the passage of the gravitational wave. The distance elements in the 1st and 2nd direction are thus given by

$$\mathrm{d}x^1 \approx \left(1 + \frac{1}{2}A_+\mathrm{e}^{ik_\sigma x^\sigma}\right)\mathrm{d}x^1_0 \qquad \mathrm{d}x^2 \approx \left(1 - \frac{1}{2}A_+\mathrm{e}^{ik_\sigma x^\sigma}\right)\mathrm{d}x^2_0. \tag{67}$$

A $\times$ polarized wave oscillates along the axes rotated by 45 degrees. The 3rd direction remains again unaffected. We designate $+$ and $\times$ polarizations rather than "horizontal" or "vertical," as we do for light. Here should be added pictures of polarization-dependent oscillations.

These polarization modes are thus invariant under rotation of 180 degrees; since the spin $S$ is generally given by the rotational angle $\theta$ invariance as $S = 360°/\theta$, we may suppose that the "gravitons" predicted in the quantum gravitation theory will have the spin 2. Of course, such particles have not yet been detected (and maybe they will never be in the future). However, relevant quantum field theories suggest their existence as massless particles (since they propagate with the speed of light) with the corresponding spin.

Now, let's revoke Eq. (32) as the starting point for describing the coupling of the gravitational radiation with its source (supposed to be of a material nature). The solution to such an equation can be obtained using Green's function in precisely the same way as the analogous problem in electromagnetism. However, before we come to this, we join the short explanatory notes on "retarded" solutions as they are defined in, e.g., the Liénard-Wiechert potentials in electrodynamics (in this sense, there can be replaced a mass for a charge, or vice versa). The term "retarded" is used in this context in the sense of "propagation delays." Consider a particle of charge $q$ moving along a trajectory $\boldsymbol{r} - \boldsymbol{r}'(t')$ whose velocity is $\boldsymbol{u}(t') = \dot{\boldsymbol{r}}'(t')$. Its charge and current is

$$q = \int q\,\delta(\boldsymbol{r} - \boldsymbol{r}'(t'))\,\mathrm{d}^3\boldsymbol{r}, \quad q\boldsymbol{u} = \int q\boldsymbol{u}\,\delta(\boldsymbol{r} - \boldsymbol{r}'(t'))\,\mathrm{d}^3\boldsymbol{r}, \tag{68}$$

where the charge and current densities are $\rho(\boldsymbol{r}, t) = q\delta(\boldsymbol{r} - \boldsymbol{r}'(t'))$ and $\boldsymbol{j}(\boldsymbol{r}, t) = q\boldsymbol{u}\delta(\boldsymbol{r} - \boldsymbol{r}'(t'))$, respectively, and where the general property of the Dirac $\delta$-function is a localization of an integral given by

$$\int f(x)\,\delta(x-x_0)\,\mathrm{d}x = f(x_0). \tag{69}$$

We now define the retarded potentials (in the system of SI units) as

$$\phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\int\frac{[\rho]\,\mathrm{d}^3\boldsymbol{r'}}{|\boldsymbol{r}-\boldsymbol{r'}|}, \quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\int\frac{[\boldsymbol{j}]\,\mathrm{d}^3\boldsymbol{r'}}{|\boldsymbol{r}-\boldsymbol{r'}|}. \tag{70}$$

where the quantities in the square brackets mean that they are evaluated at the retarded time $t-|\boldsymbol{r}-\boldsymbol{r'}|/c$, which refers to conditions at the point $\boldsymbol{r'}$ that existed at a time earlier by $|\boldsymbol{r}-\boldsymbol{r'}|/c$ than $t$ where $|\boldsymbol{r}-\boldsymbol{r'}|/c$ is the time required for light to travel between $\boldsymbol{r'}$ and $\boldsymbol{r}$.

We now evaluate these retarded potentials from Eq. (70) via the charge and current densities given by integrands of Eq. (68). The scalar potential is then

$$\begin{aligned}\phi(\boldsymbol{r},t) &= \frac{1}{4\pi\epsilon_0}\int\mathrm{d}^3\boldsymbol{r'}\int\frac{\rho(\boldsymbol{r'},t')}{|\boldsymbol{r}-\boldsymbol{r'}|}\delta\left[t'-\left(t-\frac{|\boldsymbol{r}-\boldsymbol{r'}|}{c}\right)\right]\mathrm{d}t' = \\ &= \frac{q}{4\pi\epsilon_0}\int R^{-1}(t')\,\delta\left[t'-t+\frac{R(t')}{c}\right]\mathrm{d}t', \qquad (71)\\ \boldsymbol{A}(\boldsymbol{r},t) &= \frac{\mu_0 q}{4\pi}\int\boldsymbol{u}(t')R^{-1}(t')\,\delta\left[t'-t+\frac{R(t')}{c}\right]\mathrm{d}t', \qquad (72)\end{aligned}$$

where $\boldsymbol{R}(t')=\boldsymbol{r}-\boldsymbol{r'}(t')$ and $R(t')=|\boldsymbol{R}(t')|$.

The argument of the $\delta$-function vanishes for a value of $t'=t_{\mathrm{ret}}$ (retarded time) given by

$$t_{\mathrm{ret}} = t - \frac{R(t_{\mathrm{ret}})}{c}. \tag{73}$$

We substitute a new variable $t''=t'-t+R(t')/c$ whose differential

$$\mathrm{d}t'' = \left[1+\dot{R}(t')\right]\mathrm{d}t' = \left[1-\frac{1}{c}\boldsymbol{n}(t')\cdot\boldsymbol{u}(t')\right]\mathrm{d}t', \tag{74}$$

where we obtain the latter by differentiating the identity $R^2(t') = 2R(t')\dot{R}(t') = -2\boldsymbol{R}(t')\cdot\boldsymbol{u}(t')$, where $\dot{\boldsymbol{R}}(t')=-\boldsymbol{u}(t')$ and the unit vector $\boldsymbol{n}=\boldsymbol{R}/R$. Equations (71) and (72) take the form

$$\phi(\boldsymbol{r},t) = \frac{q}{4\pi\epsilon_0}\int R^{-1}(t')\left[1-\frac{1}{c}\boldsymbol{n}(t')\cdot\boldsymbol{u}(t')\right]^{-1}\delta t''\,\mathrm{d}t'', \tag{75}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0 q}{4\pi}\int\boldsymbol{u}(t')R^{-1}(t')\left[1-\frac{1}{c}\boldsymbol{n}(t')\cdot\boldsymbol{u}(t')\right]^{-1}\delta t''\,\mathrm{d}t''. \tag{76}$$

Setting $t''=0$ or equivalently $t'=t_{\mathrm{ret}}$ simplifies the notation to

$$\phi(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\kappa R}\right], \quad \boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi}\left[\frac{q\boldsymbol{u}}{\kappa R}\right], \tag{77}$$

where we keep the "bracket" notation for retarded potentials and where

$$\kappa(t_{\mathrm{ret}}) = 1-\frac{1}{c}\boldsymbol{n}(t_{\mathrm{ret}})\cdot\boldsymbol{u}(t_{\mathrm{ret}}) = 1-\boldsymbol{\beta}(t_{\mathrm{ret}})\cdot\boldsymbol{n}(t_{\mathrm{ret}}). \tag{78}$$

Equations (77) are the Liénard-Wiechert Potentials. They differ from static electromagnetic potentials by the factor $\kappa(t_{\rm ret})$ that becomes very important at velocities close to the speed of light $c$, where it concentrates the potentials into a narrow cone about the particle's velocity (relativistic beaming effect). Another difference is that the quantities are evaluated at the retarded time $t_{\rm ret}$, which enables a particle to radiate. The potentials fall off as $\propto 1/r$ so that the fields would decrease $\propto 1/r^2$ if the differentiation of potentials acted exclusively on the factor $\propto 1/r$. The retardation involves an implicit dependence on position via the definition of retarded time, and the differentiation for this dependence transforms the $1/r$ behavior of the potentials into the fields themselves. This allows radiation energy to flow to an infinite distance.

The alternative (and maybe better) explanation of this stuff is the following, using the principles of special relativity: In the primed frame, the charge is at rest, so (see electro- and magnetostatics),

$$\phi' = \frac{1}{4\pi\epsilon_0}\frac{q}{R'}, \quad \text{that is, since } \mu_0\epsilon_0 = c^{-2}, \quad \frac{\phi'}{c} = \frac{\mu_0}{4\pi}\frac{qc}{R'}, \quad \boldsymbol{A'} = 0, \tag{79}$$

where the latter two terms are the components of the four-potential $A^\alpha = (\phi/c, \boldsymbol{A})$, while the other relevant four-quantities are $u^\alpha = \gamma(c, \boldsymbol{u})$, and $R^\alpha$ is the position four-vector of the inter-event distance, formed analogously to the four-position definition $r^\alpha = (ct, \boldsymbol{r})$ as

$$R^\alpha = \left[c\left(t-t'\right), \boldsymbol{r}-\boldsymbol{r'}\right] \equiv \left[c\left(t-t'\right), \boldsymbol{R'}\right]. \tag{80}$$

We can, however, generalize the four-potential (using the four-notation, in the non-primed quantities) consistently as

$$A^\alpha = \frac{\mu_0}{4\pi} q \frac{cu^\alpha}{u_\beta R^\beta}, \tag{81}$$

where, reminding that $c(t-t') = |\boldsymbol{r}-\boldsymbol{r'}| \equiv R$,

$$u_\beta R^\beta = \gamma(c, -\boldsymbol{u}) \cdot \left[c\left(t-t'\right), \boldsymbol{R}\right] = \gamma c\left(R - \boldsymbol{\beta}\cdot\boldsymbol{R}\right) = \gamma c R\left(1-\boldsymbol{\beta}\cdot\boldsymbol{n}\right), \tag{82}$$

where $\boldsymbol{n} \equiv \boldsymbol{R}/R$ is a unit vector in the observer's direction, $\boldsymbol{\beta} \equiv \boldsymbol{u}/c$ is the normalized velocity of the particle, and $\gamma \equiv 1/(1-u^2/c^2)^{1/2}$ is the Lorentz factor. In the reference frame of the charge (in which $\beta = 0$ and $\gamma = 1$), the expression (82) is reduced to $cR$ so we see that Eq. (81) is correctly reduced to Eq. (79). We now express the components of Eq. (81) in terms of the lab frame (in which $t' = t_{\rm ret}$ and $R = R_{\rm ret}$),

$$\phi(\boldsymbol{r}, t) = \frac{1}{4\pi\epsilon_0} q \left[\frac{1}{R\left(1-\boldsymbol{\beta}\cdot\boldsymbol{n}\right)}\right]_{\rm ret}, \quad \boldsymbol{A}(\boldsymbol{r}, t) = \frac{\mu_0}{4\pi} qc \left[\frac{\boldsymbol{\beta}}{R\left(1-\boldsymbol{\beta}\cdot\boldsymbol{n}\right)}\right]_{\rm ret}. \tag{83}$$

These formulas for the Liénard-Wiechert potentials are identical to those in Eq. (77).

Let's now do one more formal calculation whose result we will need anyway. We differentiate the geometric relation

$$c(t - t_{\rm ret}) = R_{\rm ret} \equiv |\boldsymbol{r}(t) - \boldsymbol{r'}(t_{\rm ret})| \tag{84}$$

over $t_{\rm ret}$ and then, independently, over $t$, assuming that $\boldsymbol{r}$ is fixed. Differentiating both sides of the identity $R_{\rm ret}^2 = \boldsymbol{R}_{\rm ret} \cdot \boldsymbol{R}_{\rm ret}$ over $t_{\rm ret}$, we have

$$2R_{\rm ret}\frac{\partial R_{\rm ret}}{\partial t_{\rm ret}} = 2\boldsymbol{R}_{\rm ret} \cdot \frac{\partial \boldsymbol{R}_{\rm ret}}{\partial t_{\rm ret}}. \tag{85}$$

Regarding the fixed $\boldsymbol{r}$, then $\partial \boldsymbol{R}_{\text{ret}}/\partial t_{\text{ret}} \equiv \partial(\boldsymbol{r}-\boldsymbol{r}')/\partial t_{\text{ret}} = -\partial \boldsymbol{r}'/\partial t_{\text{ret}} \equiv -\boldsymbol{u}_{\text{ret}}$, Eq. (85) yields

$$\frac{\partial R_{\text{ret}}}{\partial t_{\text{ret}}} = \frac{\boldsymbol{R}_{\text{ret}}}{R_{\text{ret}}} \cdot \frac{\partial \boldsymbol{R}_{\text{ret}}}{\partial t_{\text{ret}}} = -\left(\boldsymbol{n}\cdot\boldsymbol{u}\right)_{\text{ret}} . \tag{86}$$

Differentiating in Eq. (84) the same $R_{\text{ret}}$ over $t$ gives

$$\frac{\partial R_{\text{ret}}}{\partial t} = c - c\frac{\partial t_{\text{ret}}}{\partial t}, \tag{87}$$

while using the chain rule, we have

$$\frac{\partial R_{\text{ret}}}{\partial t} = \frac{\partial R_{\text{ret}}}{\partial t_{\text{ret}}}\frac{\partial t_{\text{ret}}}{\partial t} = -\left(\boldsymbol{n}\cdot\boldsymbol{u}\right)_{\text{ret}}\frac{\partial t_{\text{ret}}}{\partial t}. \tag{88}$$

Combining Eqs. (87) and (88) gives

$$\frac{\partial t_{\text{ret}}}{\partial t} = \frac{c}{c-\left(\boldsymbol{n}\cdot\boldsymbol{u}\right)_{\text{ret}}} = \left[\frac{1}{1-\boldsymbol{\beta}\cdot\boldsymbol{n}}\right]_{\text{ret}} . \tag{89}$$

The Green's function $\mathcal{G}(x^\alpha - y^\alpha)$ for the D'Alembertian operator $\Box$ is the solution of the wave equation in the presence of a delta-function source: $\Box_x \mathcal{G}(x^\alpha - y^\alpha) = \delta^{(4)}(x^\alpha - y^\alpha)$, where $\Box_x$ denotes the D'Alembertian for the coordinates $x^\alpha$. The general solution of Eq. (32) can be then written as the convolution see the general principles of Fourier transformation

$$\phi_{\mu\nu}(x^\alpha) = -\frac{16\pi G}{c^4}\int \mathcal{G}(x^\alpha - y^\alpha)\, T_{\mu\nu}(y^\alpha)\,\mathrm{d}^4 y. \tag{90}$$

We express the retarded Green's function, which represents the accumulated effects of signals to the past of the point under consideration as

$$\mathcal{G}(x^\alpha - y^\alpha) = -\frac{1}{4\pi\left|\boldsymbol{x}-\boldsymbol{y}\right|}\delta\left[\left(x^0-y^0\right) - \frac{\left|\boldsymbol{x}-\boldsymbol{y}\right|}{c}\right]\theta(x^0-y^0). \tag{91}$$

where $t = x^0$ and the boldface denote the spatial vectors $\boldsymbol{x} = (x^1, x^2, x^3)$ and $\boldsymbol{y} = (y^1, y^2, y^3)$, with the norm $\left|\boldsymbol{x}-\boldsymbol{y}\right| = \left[\delta_{ij}(x^i-y^i)(x^j-y^j)\right]^{1/2}$ while the function $\theta(x^0-y^0)$ equals 1 when $x^0 > y^0$, and zero otherwise, to protect the time causality.

Inserting (91) into (90), we can perform the integral over $y'$ using the delta function,

$$\phi_{\mu\nu}(t,\boldsymbol{x}) = \frac{4G}{c^4}\int \frac{1}{\left|\boldsymbol{x}-\boldsymbol{y}\right|}\, T_{\mu\nu}\left(t - \frac{\left|\boldsymbol{x}-\boldsymbol{y}\right|}{c}, \boldsymbol{y}\right)\,\mathrm{d}^3 y, \tag{92}$$

the retarded time now refers to $t_{\text{ret}} = \left|\boldsymbol{x}-\boldsymbol{y}\right|/c$. We may consider Eq. (92) in the following way: the perturbation of the gravitational field at the "spacetime point" $(t, \boldsymbol{x})$ is a sum of the energy and momentum impulses generated at points on the past light cone. Illustrative picture?

Let's now consider the case of the gravitational radiation emitted by a sufficiently distant isolated source comprised of nonrelativistic matter; these approximations will be consistently explained further on. We employ the Fourier transform solutions, which always make life easier in case of waves or oscillations. Reminding the general principles: given a whatever function of spacetime $f(t, \boldsymbol{x})$, we can construct its Fourier transform (and inverse) for time variable (using the symmetric scaling convention) as,

$$\widehat{f}(\omega,\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t,\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t, \qquad f(t,\boldsymbol{x}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}(\omega,\boldsymbol{x})\,\mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega, \tag{93}$$

When applying the same to the function $\phi_{\mu\nu}$ including the Green's function solution, we have

$$\begin{aligned}\widehat{\phi}_{\mu\nu}(\omega,\boldsymbol{x}) &= \int \phi_{\mu\nu}(t,\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}t = \frac{4G}{c^4\sqrt{2\pi}}\int \mathrm{d}t\int \frac{T_{\mu\nu}\left(t-\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c},\boldsymbol{y}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{e}^{-\mathrm{i}\omega t}\,\mathrm{d}^3y = \\ &= \frac{4G}{c^4\sqrt{2\pi}}\int \mathrm{d}\widetilde{t}\int \frac{T_{\mu\nu}\left(\widetilde{t},\boldsymbol{y}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{e}^{-\mathrm{i}\omega\left(\widetilde{t}+\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}\right)}\,\mathrm{d}^3y = \\ &= \frac{4G}{c^4\sqrt{2\pi}}\int T_{\mu\nu}\left(\widetilde{t},\boldsymbol{y}\right)\,\mathrm{e}^{-\mathrm{i}\omega\widetilde{t}}\,\mathrm{d}\widetilde{t}\int \frac{\mathrm{e}^{-\mathrm{i}\omega\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}}}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{d}^3y = \frac{4G}{c^4}\int \frac{\widehat{T}_{\mu\nu}\left(\omega,\boldsymbol{y}\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\,\mathrm{e}^{-\mathrm{i}\omega\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}}\,\mathrm{d}^3y, \end{aligned} \tag{94}$$

where $\widetilde{t} = t - |\boldsymbol{x}-\boldsymbol{y}|/c$. The first row of equation (94) defines the Fourier transform, the second reflects the equation (92), the third transforms the variable $t$ to $\widetilde{t}$, and defines again the Fourier transform. We now approximate the source as isolated, far away, and slowly moving. We consider the center of the source at a spatial distance $r$, with the opposite parts at distances $r+\delta r$ where $\delta r \ll r$. Since it is slowly moving, most of the radiation emitted will be at frequencies $\omega$ sufficiently low that $\delta r \ll \omega^{-1}$. (Essentially, light traverses the source much faster than the components of the source itself do.)

Under these approximations, the term $\mathrm{e}^{-\mathrm{i}\omega\frac{|\boldsymbol{x}-\boldsymbol{y}|}{c}}/|\boldsymbol{x}-\boldsymbol{y}|$ can be replaced by $\mathrm{e}^{-\mathrm{i}\omega\frac{r}{c}}/r$ and brought outside the integral. This leaves us with

$$\widehat{\phi}_{\mu\nu}(\omega,\boldsymbol{x}) = \frac{4G}{c^4}\frac{\mathrm{e}^{-\mathrm{i}\omega\frac{r}{c}}}{r}\int \widehat{T}_{\mu\nu}(\omega,\boldsymbol{y})\,\mathrm{d}^3y. \tag{95}$$

We do not need to calculate all the components of $\widehat{\phi}_{\mu\nu}(\omega,\boldsymbol{x})$, since the harmonic gauge condition $\partial_\mu\phi^{\mu\nu}(t,\boldsymbol{x}) = 0$ in Fourier space implies

$$\widehat{\phi}^{0\nu} = \frac{\mathrm{i}}{\omega}\partial_i\widehat{\phi}^{i\nu}. \tag{96}$$

We, therefore, only need to concern ourselves with the spacelike components of $\widehat{\phi}_{\mu\nu}(\omega,\boldsymbol{x})$. From (95), we want to take the integral of the spacelike components of $\widehat{T}_{\mu\nu}(\omega,\boldsymbol{y})$. We integrate by parts,

$$\int \widehat{T}^{ij}(\omega,\boldsymbol{y})\,\mathrm{d}^3y = \int \partial_k\left(y^i\widehat{T}^{kj}\right)\mathrm{d}^3y - \int y^i\left(\partial_k\widehat{T}^{kj}\right)\mathrm{d}^3y, \tag{97}$$

where the first term is a surface integral which will vanish since the source is isolated, while the second can be related to $\widehat{T}^{0j}$ by the Fourier-space version of $\partial_\mu T^{\mu\nu} = 0$,

$$\partial_k\widehat{T}^{k\mu} = -\mathrm{i}\omega\widehat{T}^{0\mu}. \tag{98}$$

Thus,

$$\begin{aligned}\int \widehat{T}^{ij}(\omega,\boldsymbol{y})\,\mathrm{d}^3y &= \mathrm{i}\omega\int y^i\widehat{T}^{0j}\,\mathrm{d}^3y = \frac{\mathrm{i}\omega}{2}\int\left(y^i\widehat{T}^{0j}+y^j\widehat{T}^{0i}\right)\mathrm{d}^3y = \\ &= \frac{\mathrm{i}\omega}{2}\int\left[\partial_k\left(y^iy^j\widehat{T}^{0k}\right) - y^iy^j\left(\partial_k\widehat{T}^{0k}\right)\right]\mathrm{d}^3y = -\frac{\omega^2}{2}\int y^iy^j\widehat{T}^{00}\,\mathrm{d}^3y. \end{aligned} \tag{99}$$

The third integral of Eq. (99) is justified since we know that the left-hand side is symmetric in $i$ and $j$, while the fourth and fifth integrals are simply repetitions of reverse integration by parts and conservation of $T^{\mu\nu}$. It is conventional to define the *quadrupole moment tensor* of the energy density of the source,

$$q_{ij}(t) = \int y^i y^j T^{00}(t, \boldsymbol{y})\,\mathrm{d}^3 y, \tag{100}$$

a constant tensor on each surface of constant time. In terms of the Fourier transform of the quadrupole moment, our solution takes on the compact form

$$\widehat{\phi}_{ij}(\omega, \boldsymbol{x}) = -\frac{2G\omega^2}{c^4}\frac{\mathrm{e}^{-\mathrm{i}\omega\frac{r}{c}}}{r}\widehat{q}_{ij}(\omega), \tag{101}$$

or, transforming back to $t$ by the inverse Fourier transform, we can absorb the factor $-\omega^2$ into a second time derivative, as well as $\mathrm{e}^{-\mathrm{i}\omega r}$ by transforming to the retarded time, following the previously used steps in reverse. Thus, we finally arrive at the quadrupole formula

$$\begin{aligned}\phi_{ij}(t, \boldsymbol{x}) &= -\frac{1}{\sqrt{2\pi}}\frac{2G}{c^4 r}\int \omega^2 \mathrm{e}^{\mathrm{i}\omega(t-\frac{r}{c})}\,\widehat{q}_{ij}(\omega)\,\mathrm{d}\omega = \frac{1}{\sqrt{2\pi}}\frac{2G}{c^4 r}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\int \mathrm{e}^{\mathrm{i}\omega t_{\mathrm{ret}}}\,\widehat{q}_{ij}(\omega)\,\mathrm{d}\omega = \\ &= \frac{2G}{c^4 R}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\left[q_{ij}(t_{\mathrm{ret}})\right].\end{aligned} \tag{102}$$

The gravitational wave produced by an isolated nonrelativistic object is, therefore, proportional to the second derivative of the quadrupole moment of the energy density at the point where the observer's past light cone intersects the source.

In contrast, the leading contribution to electromagnetic radiation comes from the changing dipole moment of the charge density. The difference can be traced back to the universal nature of gravitation. A changing dipole moment corresponds to the motion of the center of density - charge density in the case of electromagnetism and energy density in the case of gravitation. While there is nothing to stop the center of charge of an object from oscillating, the oscillation of the center of mass of an isolated system violates the conservation of momentum. (You can shake a body up and down, but you and the Earth shake slightly in the opposite direction to compensate.) The quadrupole moment, which measures the shape of the system, is generally smaller than the dipole moment, and for this reason (as well as the weak coupling of matter to gravity) gravitational radiation is typically much weaker than electromagnetic radiation.

It is always illustrative to take a general solution and apply it to a specific case of interest. One case of interest is the gravitational radiation emitted by a binary star (two stars in orbit around each other). For simplicity, let us consider two stars of mass $M$ in a circular orbit in the $x^1$-$x^2$ plane, at a distance $R$ from their common center of mass. Circular orbits in the Newtonian approximation (where the orbital period of both stars is $T = 2\pi R/V$) are characterized by equating the gravitational and centrifugal forces,

$$\frac{GM^2}{(2R)^2} = \frac{MV^2}{R}, \qquad \text{giving} \qquad V = \sqrt{\frac{GM}{4R}} \quad \text{and} \quad \Omega = \sqrt{\frac{GM}{4R^3}}, \tag{103}$$

where $\Omega$ is the angular frequency (hereafter, we distinguish $\Omega$ as the angular frequency of the orbiting system and $\omega$ as the angular frequency of the gravitational wave, where $\omega = 2\Omega$ due to two "tidal maxima" of the wave during one orbital period). Then, we can express the trajectories of stars A and B as

$$x^1_A = R\cos\Omega t, \quad x^2_A = R\sin\Omega t, \quad x^1_B = -R\cos\Omega t, \quad x^2_B = -R\sin\Omega t, \tag{104}$$

while the corresponding energy density is

$$T^{00}(t,\boldsymbol{x}) = M\delta(x^3)\left[\delta(x^1 - R\cos\Omega t)\,\delta(x^2 - R\sin\Omega t) + \delta(x^1 + R\cos\Omega t)\,\delta(x^2 + R\sin\Omega t)\right]. \tag{105}$$

Following this, we can integrate Eq. (100) (using the principle given in Eq. (69)),

$$q_{11} = 2MR^2\cos^2\Omega t, \quad q_{22} = 2MR^2\sin^2\Omega t, \quad q_{12} = q_{21} = 2MR^2\cos\Omega t\sin\Omega t, \quad q_{i3} = 0. \tag{106}$$

We get the components of the metric perturbation from (101) as

$$\phi_{ij}(t,\boldsymbol{x}) = -\frac{8GM}{c^4 r}\Omega^2 R^2\begin{pmatrix}\cos 2\Omega t_{\rm ret} & \sin 2\Omega t_{\rm ret} & 0\\ \sin 2\Omega t_{\rm ret} & -\cos 2\Omega t_{\rm ret} & 0\\ 0 & 0 & 0\end{pmatrix}. \tag{107}$$

The remaining components of $\phi_{\mu\nu}$ could be derived from the harmonic gauge condition.

Another approach to construct the quadrupole moment tensor $q_{ij}(t)$ is based on the evaluation of the "second moment of mass" (moment of inertia) of the nonrelativistic isolated two-body system (of two masses $m_1$ and $m_2$ where $M = m_1 + m_2$) in the center-of-mass frame where $\boldsymbol{x}_0 = \boldsymbol{x}_1 - \boldsymbol{x}_2$ is the relative coordinate, the center-of-mass coordinate $\boldsymbol{x}_{\rm CM}$ and the reduced mass $\mu$ are (usually) given as

$$\boldsymbol{x}_{\rm CM} = \frac{m_1\boldsymbol{x}_1 + m_2\boldsymbol{x}_2}{M}, \qquad \mu = \frac{m_1 m_2}{M}. \tag{108}$$

Then, the second moment of mass can be written as

$$q_{ij} = m_1 x_1^i x_1^j + m_2 x_2^i x_2^j = M x_{\rm CM}^i x_{\rm CM}^j + \mu x_0^i x_0^j. \tag{109}$$

Following the last expression, we can expand this as

$$\begin{aligned} M\frac{m_1x_1^i + m_2x_2^i}{M}\frac{m_1x_1^j + m_2x_2^j}{M} + \frac{m_1m_2}{M}(x_1^i - x_2^i)(x_1^j - x_2^j) = \\ = \frac{m_1^2x_1^ix_1^j + m_1m_2x_1^ix_2^j + m_1m_2x_1^jx_2^i + m_2^2x_2^ix_2^j}{M} + \frac{m_1m_2}{M}(x_1^ix_1^j - x_1^jx_2^i - x_1^ix_2^j + x_2^ix_2^j) \\ = \frac{1}{M}\left[m_1\left(m_1+m_2\right)x_1^ix_1^j + m_2\left(m_1+m_2\right)x_2^ix_2^j\right] = m_1x_1^ix_1^j + m_2x_2^ix_2^j. \end{aligned} \tag{110}$$

Therefore, if we choose the origin of the coordinate system at $\boldsymbol{x}_{\rm CM} = 0$, the quadrupole moment becomes the same as that of a particle of mass $\mu$ described by the coordinate $\boldsymbol{x}_0(t)$. In this CM frame, the mass density is then

$$\rho(t,\boldsymbol{x}) = \mu\delta^{(3)}\left(\boldsymbol{x} - \boldsymbol{x}_0(t)\right), \tag{111}$$

and the quadrupole moment is

$$q_{ij}(t) = \mu x_0^i(t)x_0^j(t). \tag{112}$$

We can thus relate the gravitational radiation emitted by such a two-body system to the mass quadrupole moment of this system whose relative coordinate performs a given periodic motion, say simple harmonic oscillations.

# 3 Energy and momentum of gravitational waves emitted from a two-body system

To derive the energy emitted via gravitational radiation, let us consider vacuum Einstein's equations to second order and see how the result can be interpreted as an energy-momentum tensor for the gravitational field. Once again, if we consider the metric as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, then at first order we have $G^{(1)}_{\mu\nu}[\eta + h] = 0$, where $G^{(1)}_{\mu\nu}$ is the Einstein's tensor expanded to first order in $h_{\mu\nu}$. These equations determine $h_{\mu\nu}$ up to (unavoidable) gauge transformations, so to satisfy the equations to second order, we have to add a higher-order perturbation, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + h^{(2)}_{\mu\nu}$. The second-order version of Einstein's equations consists of all terms either quadratic in $h_{\mu\nu}$ or linear in $h^{(2)}_{\mu\nu}$. Since any cross terms would be of at least third order, we have

$$G^{(1)}_{\mu\nu}[\eta + h^{(2)}] + G^{(2)}_{\mu\nu}[\eta + h] = 0. \tag{113}$$

Here, $G^{(2)}_{\mu\nu}$ is part of the Einstein tensor, which is of second order in the metric perturbation. It can be computed from the second-order Ricci tensor (involving the nonlinear terms of the Ricci tensor (not only the linear ones as we did before in the case of the vacuum solution used to find the solution of perturbations), which is given by

$$R^{(2)}_{\mu\nu} = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho} + \Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} - \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}. \tag{114}$$

The solution of the first nonlinear term within the TT gauge is

$$\Gamma^\rho_{\rho\lambda}\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu}\left[\frac{1}{2}\eta^{\rho\sigma}\left(\partial_\rho h_{\lambda\sigma} + \partial_\lambda h_{\rho\sigma} - \partial_\sigma h_{\rho\lambda}\right)\right] = \Gamma^\lambda_{\mu\nu}\frac{1}{2}\left(\partial_\rho h^\rho_\lambda + \partial_\lambda h^\rho_\rho - \partial_\sigma h^\sigma_\lambda\right) = 0, \tag{115}$$

because the first and third terms cancel (they are the same in the transverse gauge, using the symmetry in the metric $\eta^{\alpha\beta} = \eta^{\beta\alpha}$ and following, therefore, the identities $\eta^{\rho\sigma}\partial_\rho h_{\lambda\sigma} = \eta^{\sigma\rho}\partial_\sigma h_{\lambda\rho}$), and the middle term is zero due to the traceless choice of the gauge. The second nonlinear term can be expanded as

$$\begin{aligned}
-\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} &= -\frac{1}{4}\eta^{\rho\sigma}\left(\partial_\mu h_{\lambda\sigma} + \partial_\lambda h_{\mu\sigma} - \partial_\sigma h_{\mu\lambda}\right)\eta^{\lambda\sigma}\left(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}\right) = \\
&= -\frac{1}{4}\left(\partial_\mu h^\rho_\lambda + \partial_\lambda h^\rho_\mu - \partial^\rho h_{\mu\lambda}\right)\eta^{\lambda\sigma}\left(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}\right) = \\
&= -\frac{1}{4}\left(\partial_\mu h^{\rho\sigma} + \partial^\sigma h^\rho_\mu - \partial^\rho h^\sigma_\mu\right)\left(\partial_\nu h_{\rho\sigma} + \partial_\rho h_{\nu\sigma} - \partial_\sigma h_{\nu\rho}\right) = \\
&= -\frac{1}{4}\left[(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) + (\partial_\mu h^{\rho\sigma})(\partial_\rho h_{\nu\sigma}) - (\partial_\mu h^{\rho\sigma})(\partial_\sigma h_{\nu\rho}) + (\partial^\sigma h^\rho_\mu)(\partial_\nu h_{\rho\sigma}) + \right. \\
&\quad + (\partial^\sigma h^\rho_\mu)(\partial_\rho h_{\nu\sigma}) - (\partial^\sigma h^\rho_\mu)(\partial_\sigma h_{\nu\rho}) - (\partial^\rho h^\sigma_\mu)(\partial_\nu h_{\rho\sigma}) - (\partial^\rho h^\sigma_\mu)(\partial_\rho h_{\nu\sigma}) + \\
&\quad \left. + (\partial^\rho h^\sigma_\mu)(\partial_\sigma h_{\nu\rho})\right],
\end{aligned} \tag{116}$$

the second term cancels with the third term, and the fourth term cancels with the seventh term due to the obvious symmetry of the metric, as was already described above. Equation (116) thus simplifies to

$$\begin{aligned}
-\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} = &-\frac{1}{4}\left[(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) + (\partial^\sigma h^\rho_\mu)(\partial_\rho h_{\nu\sigma}) - (\partial^\sigma h^\rho_\mu)(\partial_\sigma h_{\nu\rho}) - \right. \\
&\left. -(\partial^\rho h^\sigma_\mu)(\partial_\rho h_{\nu\sigma}) + (\partial^\rho h^\sigma_\mu)(\partial_\sigma h_{\nu\rho})\right],
\end{aligned} \tag{117}$$

where, however, the second and the fifth terms are for the same reason identical due to the symmetry of indices, $\partial^\sigma h^\rho_\mu = \partial^\rho h^\sigma_\mu$ as well as $\partial_\rho h_{\nu\sigma} = \partial_\sigma h_{\nu\rho}$, while for the third and the fourth terms we can also apply this argument. We may thus yet simplify this identity to

$$-\Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho} = -\frac{1}{4}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) + \frac{1}{2}(\partial^\sigma h^\rho_\mu)(\partial_\rho h_{\nu\sigma}) - \frac{1}{2}(\partial^\sigma h^\rho_\mu)(\partial_\sigma h_{\nu\rho}). \tag{118}$$

We can rewrite the second and the third term on the right-hand side of Eq. (118), using the rule for the derivative of the product of two functions as

$$\frac{1}{2}(\partial^\sigma h^\rho_\mu)(\partial_\rho h_{\nu\sigma}) = \frac{1}{2}\partial^\sigma(h^\rho_\mu \partial_\rho h_{\nu\sigma}) - \frac{1}{2}h^\rho_\mu \partial_\rho \partial^\sigma h_{\nu\sigma}, \tag{119}$$

$$-\frac{1}{2}(\partial^\sigma h^\rho_\mu)(\partial_\sigma h_{\nu\rho}) = -\frac{1}{2}\partial^\sigma(h^\rho_\mu \partial_\sigma h_{\nu\rho}) + \frac{1}{2}h^\rho_\mu \partial^\sigma \partial_\sigma h_{\nu\rho}, \tag{120}$$

where, due to the gauge condition (34), the last term in (119) vanishes. Now, let's label the two linear terms of the Ricci tensor from Eq. (10) as being the "connections" of the second order (due to the derivatives of the Christoffel's symbols); we can then rewrite the complete second-order Ricci tensor (see Eq. (13) for its linear part only) in this sense as

$$\begin{aligned} R^{(2)}_{\mu\nu} =& \frac{1}{2}\Big[\partial_\rho\partial_\mu h^\rho{}_\nu + \partial_\rho\partial_\nu h^\rho{}_\mu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \\ & -\frac{1}{2}(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) + \partial^\sigma(h^\rho_\mu \partial_\rho h_{\nu\sigma}) - \partial^\sigma(h^\rho_\mu \partial_\sigma h_{\nu\rho}) + h^\rho_\mu \Box h_{\nu\rho}\Big], \end{aligned} \tag{121}$$

where the d'Alembertian in the last term comes from $\partial^\sigma\partial_\sigma = \partial_\sigma\partial^\sigma = \Box$. We now evaluate the second-order Ricci scalar by contracting the second row of Eq. (121) by $\eta^{\mu\nu}$ (since the first-order expression, that is, the first row in Eq. (121), was already calculated in Eq. (13)),

$$R^{(2)} = \partial_\rho\partial_\mu h^{\mu\rho} - \Box h - \frac{1}{2}\left[\frac{1}{2}(\partial^\nu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma}) + \partial^\sigma(h^{\nu\rho}\partial_\rho h_{\nu\sigma}) - \partial^\sigma(h^{\nu\rho}\partial_\sigma h_{\nu\rho}) + h^{\nu\rho}\Box h_{\nu\rho}\right]. \tag{122}$$

Now, since $G^{(2)}_{\mu\nu} = R^{(2)}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{(2)}$ and, following Eq. (113) where we can put $G^{(1)}_{\mu\nu}(\eta + h^{(2)}) = \frac{8\pi G}{c^4}t_{\mu\nu}$, we simply define

$$G^{(2)}_{\mu\nu}(\eta + h) = -\frac{8\pi G}{c^4}t_{\mu\nu}, \tag{123}$$

we formally distinguish $t_{\mu\nu}$ as the energy-momentum tensor specifically for the gravitational field in the weak field regime.

We will not describe here the Bianchi identity and other mathematical tools which may lead us too far aside (see Caroll and others for the detailed stuff), stating only that it implies $\nabla^\mu R_{\rho\mu} = \frac{1}{2}\nabla_\rho R$ which is equivalent to energy conservation $\nabla^\mu G_{\mu\nu} = 0$ and, therefore, $\nabla^\mu T_{\mu\nu} = 0$. For this reason, we have in the flat space (weak field regime) the modified energy-momentum conservation condition

$$\partial_\mu t^{\mu\nu} = 0. \tag{124}$$

By averaging over a macroscopic region, $[\int_a^b f(x)\,\mathrm{d}x]/(b-a)$, we can yet simplify the Einstein tensor. At this point, it is important to note that we are working with functions constructed from small perturbations. The difference in the values $a$ and $b$ of the function at any point will also be small. If the wavelength of the gravitational wave is much larger than the region $b - a$ we are considering, $f(a)$ and $f(b)$ will also be practically the same. So let's go through the right-hand sides of equations (121) and (122) term by term and see which ones can be neglected (noting again that we "are" in the TT gauge): The first four terms and the last term in (121) vanish due to the conditions $\partial_\mu h^\mu_\alpha = 0$, $h = h^\alpha_\alpha = 0$, and $\Box h_{\mu\nu} = 0$ (see the explanations above). In contrast, the second and the third terms from the end vanish due to neglecting the average of the total derivative, as described closely above. The same principles apply in Eq. (122) where also the third term (first term in square bracket) vanishes due to the same macroscopic condition within the averaged value (remembering that $u'v = (uv)' - uv'$ where the middle averaged term vanishes, so $u'v \simeq -uv'$), that is,

$$-\frac{1}{4}\left\langle(\partial^\nu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma})\right\rangle = \frac{1}{4}\left\langle h^{\rho\sigma}\Box h_{\rho\sigma}\right\rangle = 0. \tag{125}$$

Finally, we are only left with a single term (the fifth term on the right-hand side of Eq. (121)) for the effective stress-energy of gravitational waves,

$$R^{(2)}_{\mu\nu}(\eta + h) = -\frac{1}{4}\left\langle(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma})\right\rangle = -\frac{8\pi G}{c^4}t_{\mu\nu}, \tag{126}$$

that is,

$$t_{\mu\nu} = \frac{c^4}{32\pi G}\left\langle(\partial_\mu h^{\rho\sigma})(\partial_\nu h_{\rho\sigma})\right\rangle, \tag{127}$$

where we only readjusted Eq. (126) and transferred the "Einstein gravity coefficient" to the opposite side of the equation.

Regarding the derived energy-momentum tensor, we proceed with the $T_{00}$ component that describes the energy density. Therefore, we can find the total energy of the gravitational radiation contained within the surface $\Sigma$ of constant time. Using the symmetries of the system, we put the source of the radiation into the origin, which is especially important when we work with the retarded time,

$$E = \int_\Sigma t_{00}\,\mathrm{d}^3\boldsymbol{x}. \tag{128}$$

We can also calculate the energy loss $\Delta E$ due to radiation through a sphere $S$ with a radius $R$ per second because the $T_{0\mu}$ components describe the energy flux in the direction $\mu$ as

$$\Delta E = \int_S t_{0\mu}n^\mu\,\mathrm{d}^2\boldsymbol{x}\,\mathrm{d}t, \tag{129}$$

where the integral is taken over a spacelike two-sphere in infinity and some interval in time, and $n^\mu$ is a unit spacelike vector normal to $S$. After a little rearrangement and a transcription of the isotropic situation to spherical coordinates where the only component $t_{0\mu}n^\mu$ is $t_{0r}$, we can write the same equation (not in the infinity but at least in a considerable distance $r$) as

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_S t_{0r}r^2\,\mathrm{d}\Omega. \tag{130}$$

Then, according to Eq. (127),

$$t_{0r} = \frac{c^4}{32\pi G}\left\langle(\partial_0 h^{\alpha\beta}_{TT})(\partial_r h^{TT}_{\alpha\beta})\right\rangle, \tag{131}$$

where we further employ the derivation of the trace-reversed tensor (101) based on the second-order time derivative of the quadrupole momentum, noting that $\phi_{\mu\nu} = h_{\mu\nu}$ in the TT gauge. We also advancingly use the fact that the amount of radiated energy (129) can be written in terms of the radiated power $P$ as

$$\Delta E = \int P \, \mathrm{d}t, \tag{132}$$

where the power is then given by the integrand of Eq. (130) in spherical coordinates.

Now, we need to impose the TT gauge to the trace-reversed perturbation so it changes to the weak field perturbation, and we can use it in equation (127). We have to find a TT tensor $q^{ij}_{TT}$ constructed from $q^{ij}$. We can use this to find $T_{0\mu}$, after which we can change $q^{ij}_{TT}$ back to $q^{ij}$ without information loss. First, we begin by projecting $q^{ij}$ on its traceless component $Q^{ij}$ (whose $\mathrm{Tr}\, Q_{ij} = Q^i_i = 0$),

$$Q^{ij} = q^{ij} - \frac{1}{3}\delta^{ij}\delta_{kl}q^{kl}, \tag{133}$$

second, to make $Q^{ij}$ transversal, we want to project its components on a transversal image. Therefore, we will use the projection operator

$$P^b_a(\boldsymbol{x}) = \delta^b_a - \frac{x^b x_a}{r^2}, \quad \text{which is equivalent to} \quad P^b_a(\boldsymbol{n}) = \delta^b_a - n^b n_a, \tag{134}$$

where $\boldsymbol{x} = \left(x^1, x^2, x^3\right)$ so $x^i x_i = r^2$, and $\boldsymbol{n} = \left(\frac{x^1}{r}, \frac{x^2}{r}, \frac{x^3}{r}\right)$ so $n^i n_i = 1$, that acts as

$$Q^{ij}_{TT} = P^i_a Q_{ab} P^j_b - \frac{1}{2} P_{ab} Q_{ab} P^{ij}. \tag{135}$$

After projecting $Q^{ij}$ on its transversal image, you can use the projection again, but it stays the same transversal image. Since a general property of a projection operator is

$$P^2 = P \wedge P^b_a P^a_c = P^b_c, \tag{136}$$

$$P^2 = \left(\delta^b_a - \frac{x^b x_a}{r^2}\right)\left(\delta^a_c - \frac{x^a x_c}{r^2}\right) = \delta^b_a\delta^a_c - \delta^b_a\frac{x^a x_c}{r^2} - \delta^a_c\frac{x^b x_a}{r^2} + \frac{x^b x_a}{r^2}\frac{x^a x_c}{r^2} =$$
$$= \delta^b_c - \frac{x^b x_c}{r^2} - \frac{x^b x_c}{r^2} + \frac{x^b r^2 x_c}{r^4} = \delta^b_c - \frac{x^b x_c}{r^2} = P, \quad \text{in other words} \tag{137}$$

$$P^2 = \left(\delta^b_a - n^b n_a\right)\left(\delta^a_c - n^a n_c\right) = \delta^b_a\delta^a_c - \delta^b_a n^a n_c - \delta^a_c n^b n_a + n^b n_a n^a n_c =$$
$$= \delta^b_c - n^b n_c - n^b n_c + n^b n_c = \delta^b_c - n^b n_c = P \tag{138}$$

using this, we can check that $q^{ij}_{TT} = Q^{ij}_{TT}$ is traceless and transversal: Let's first check that $Q^{ij}_{TT}$ is transversal, that is, $x \cdot Q^{ij}_{TT} = 0$ in the following way,

$$x_i Q^{ij}_{TT} = (x_i P^i_a) Q_{ab} P^j_b - \frac{1}{2} P_{ab} Q_{ab} (x_i P^{ij}), \quad \text{checking in particular that}$$
$$x_i P^i_a = x_i \left(\delta^i_a - \frac{x^i x_a}{r^2}\right) = x_a - \frac{x^i x_i}{r^2} x_a = x_a - x_a = 0, \quad \text{in other words} \tag{139}$$
$$n_i P^i_a = n_i \left(\delta^i_a - n^i n_a\right) = x_a - n^i n_i n_a = n_a - n_a = 0, \tag{140}$$
$$x_i P^{ij} = x_i \left(\delta^{ij} - \frac{x^i x^j}{r^2}\right) = x^j - \frac{x_i x^i}{r^2} x^j = x^j - x^j = 0. \quad \text{This is also} \tag{141}$$
$$n_i P^{ij} = n_i \left(\delta^{ij} - n^i n^j\right) = n^j - n_i n^i n^j = n^j - n^j = 0. \tag{142}$$

Second, we check that $Q_{TT}^{ij}$ is traceless, that is, $Q^i_i = 0$, by

$$
\begin{aligned}
Q^i_i &= g_{ij} Q^{ij}_{TT} = P^i_a Q_{ab} P_{ib} - \frac{1}{2} P_{ab} Q_{ab} P^i_i, \quad \text{where} \\
P^i_a P_{ib} &= \left(\delta^i_a - \frac{x^i x_a}{r^2}\right)\left(\delta_{ib} - \frac{x_i x_b}{r^2}\right) = \delta_{ab} - \frac{x_a x_b}{r^2} = P_{ab}, \\
P^i_a P_{ib} &= \left(\delta^i_a - n^i n_a\right)\left(\delta_{ib} - n_i n_b\right) = \delta_{ab} - n_a n_b = P_{ab}, \\
\frac{1}{2} P^i_i &= \frac{1}{2}\left(\delta^i_i - \frac{x^i x_i}{r^2}\right) = \frac{1}{2}(3-1) = 1, \quad \text{that is,} \\
\frac{1}{2} P^i_i &= \frac{1}{2}\left(\delta^i_i - n^i n_i\right) = \frac{1}{2}(3-1) = 1, \\
Q^i_i &= P_{ab} Q_{ab} - P_{ab} Q_{ab} = 0.
\end{aligned} \tag{143}
$$

Now, inserting Eq. (101) into Eq. (131), we continue as

$$
\partial_0 h^{\alpha\beta}_{TT} = \frac{2G}{c^4 r} \frac{\partial}{\partial t}\left[\ddot{Q}^{\alpha\beta}_{TT}(t_{\text{ret}})\right] = \frac{2G}{c^4 r} \frac{\partial t_{\text{ret}}}{\partial t} \frac{\partial}{\partial t_{\text{ret}}}\left[\ddot{Q}^{\alpha\beta}_{TT}(t_{\text{ret}})\right], \tag{144}
$$

$$
\partial_r h^{TT}_{\alpha\beta} = \frac{2G}{c^4} \frac{\partial}{\partial r}\left[\frac{1}{r}\ddot{Q}^{\alpha\beta}_{TT}(t_{\text{ret}})\right] = \frac{2G}{c^4 r} \frac{\partial t}{\partial r} \frac{\partial t_{\text{ret}}}{\partial t} \frac{\partial}{\partial t_{\text{ret}}}\left[\ddot{Q}^{\alpha\beta}_{TT}(t_{\text{ret}})\right] - \frac{2G}{c^4 r^2}\left[\ddot{Q}^{TT}_{\alpha\beta}(t_{\text{ret}})\right], \tag{145}
$$

where we use Eq. (89) for the derivative $\partial t_{\text{ret}}/\partial t$ which however, due to negligible $\beta$ term, we regard as 1. Due to the spherical symmetry (one radial spatial coordinate $r$), we set the radial derivative $\partial_r$ of the quadrupole equal to $c$ times $\partial_t$ (the time derivative), except the direct spatial derivative of $r$ in the denominator. We thus obtain

$$
\partial_0 h^{\alpha\beta}_{TT} = \frac{2G}{c^4 r}\left[\dddot{Q}^{\alpha\beta}_{TT}(t_{\text{ret}})\right] \tag{146}
$$

$$
\partial_r h^{TT}_{\alpha\beta} = \frac{2G}{c^5 r}\left[\dddot{Q}^{TT}_{\alpha\beta}(t_{\text{ret}})\right] - \frac{2G}{c^4 r^2}\left[\ddot{Q}^{TT}_{\alpha\beta}(t_{\text{ret}})\right]. \tag{147}
$$

Because we are far from the source, we may neglect the last term in Eq. (147). We can thus write an expression for the power $P = \mathrm{d}E/\mathrm{d}t$ radiated by a gravitational source,

$$
\frac{\mathrm{d}E}{\mathrm{d}t} = \left\langle \int_S \frac{c^4}{32\pi G} \frac{4G^2}{c^9 r^2} r^2 \dddot{Q}^{\alpha\beta}_{TT} \dddot{Q}^{TT}_{\alpha\beta}\, \mathrm{d}\Omega \right\rangle = \left\langle \int_S \frac{G}{8\pi c^5} \dddot{Q}^{\alpha\beta}_{TT} \dddot{Q}^{TT}_{\alpha\beta}\, \mathrm{d}\Omega \right\rangle. \tag{148}
$$

Since we want to achieve a general solution of the integral, we transform this back to a non-transversal form. We use again the property of the projection operator $P^2 = P$,

$$
\dddot{Q}^{TT}_{ij} \dddot{Q}^{ij}_{TT} = \left(P^a_i P^b_j - \frac{1}{2} P^{ab} P_{ij}\right)\left(P^i_c P^j_d - \frac{1}{2} P_{cd} P^{ij}\right) \dddot{Q}_{ab} \dddot{Q}^{cd}. \tag{149}
$$

We can split the two brackets with the projection operator into four terms

$$
\begin{aligned}
P^a_i P^b_j P^i_c P^j_d = P^a_c P^b_d, \quad -\frac{1}{2} P^a_i P^b_j P_{cd} P^{ij} = -\frac{1}{2} P_{cd} P^{ab}, \quad -\frac{1}{2} P^{ab} P_{ij} P^i_c P^j_d = -\frac{1}{2} P_{cd} P^{ab}, \\
\frac{1}{4} P^{ab} P_{ij} P_{cd} P^{ij} = \frac{1}{4} P_{cd} P^{ab} P^i_i = \frac{1}{2} P_{cd} P^{ab},
\end{aligned} \tag{150}
$$

(see Eq. (143) for justification of the last equation's solution) we need to solve the two operators,

$$P^a_c P^b_d = \left(\delta^a_c - \frac{x^a x_c}{r^2}\right)\left(\delta^b_d - \frac{x^b x_d}{r^2}\right) = \delta^a_c\delta^b_d - \delta^a_c\frac{x^b x_d}{r^2} - \delta^b_d\frac{x^a x_c}{r^2} + \frac{x^a x_c x^b x_d}{r^4}, \tag{151}$$

$$P_{cd}P^{ab} = \left(\delta_{cd} - \frac{x_c x_d}{r^2}\right)\left(\delta^{ab} - \frac{x^a x^b}{r^2}\right) = \delta^{ab}\delta_{cd} - \delta^{ab}\frac{x_c x_d}{r^2} - \delta_{cd}\frac{x^a x^b}{r^2} + \frac{x^a x^b x_c x_d}{r^4}. \tag{152}$$

Substituting this into Eq. (149) gives

$$P^a_c P^b_d \dddot{Q}_{ab}\dddot{Q}^{cd} = \dddot{Q}_{ab}\dddot{Q}^{ab} - \frac{x^b x^d}{R^2}\dddot{Q}_{ab}\dddot{Q}^a_d - \frac{x^a x^c}{R^2}\dddot{Q}_{ab}\dddot{Q}^b_c + \frac{x^a x^c x^b x^d}{R^4}\dddot{Q}_{ab}\dddot{Q}_{cd}, \tag{153}$$

$$-\frac{1}{2}P_{cd}P^{ab}\dddot{Q}_{ab}\dddot{Q}^{cd} = -\frac{1}{2}\dddot{Q}\dddot{Q} + \frac{x_c x_d}{2R^2}\dddot{Q}\dddot{Q}^{cd} + \frac{x^a x^b}{2R^2}\dddot{Q}_{ab}\dddot{Q} - \frac{x^a x^b x^c x^d}{2R^4}\dddot{Q}_{ab}\dddot{Q}_{cd}. \tag{154}$$

Remembering that the quadrupole moment of mass distribution is still traceless ($Q = 0$), we can rewrite equation (149) as

$$\dddot{Q}^{ij}_{TT}\dddot{Q}^{TT}_{ij} = \dddot{Q}_{ab}\dddot{Q}^{ab} - \frac{x^b x^d}{R^2}\dddot{Q}_{ab}\dddot{Q}^a_d - \frac{x^a x^c}{R^2}\dddot{Q}_{ab}\dddot{Q}^b_c + \frac{x^a x^c x^b x^d}{2R^4}\dddot{Q}_{ab}\dddot{Q}_{cd}. \tag{155}$$

We must integrate this over $\mathrm{d}\Omega$ to obtain the total power. To perform the integration, we yet introduce the solutions for a surface $S$ with radius $R$ (cf., e.g., Michele Maggiore, Gravitational waves, 2008, pp. 104-105)

$$\int \frac{\mathrm{d}\Omega}{4\pi} x^a x^b = \frac{1}{3}R^2\eta^{ab}, \quad \int \frac{\mathrm{d}\Omega}{4\pi} x^a x^b x^c x^d = \frac{1}{15}R^4\left(\eta^{ab}\eta^{cd} + \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}\right). \tag{156}$$

These identities and their generalization to an arbitrary number of $n$'s can be found as follows. For an odd number of $n_i$ the integral vanishes because the integrand is odd under parity. For an even number of $n_i$, we use the fact that the tensor $n_{i1}\, n_{i2} \ldots n_{i2k}$ is symmetric, and therefore its integral can only depend on the symmetrized product of Kronecker deltas. Once the tensor structure is fixed, the overall constant is obtained by contracting all indices. This all gives

$$\int \frac{\mathrm{d}\Omega}{4\pi}\, n_{i_1}\, n_{i_2} \ldots n_{i_{2k}} = \frac{1}{(2k+1)!!}\left(\delta_{i_1 i_2}\delta_{i_3 i_4} \ldots \delta_{i_{2k-1} i_{2k}} + \ldots\right), \tag{157}$$

where !! denote the "double factorial" $n(n-2)(n-4)\ldots$ and the final dots denote all possible pairing of indices. In particular, we thus have

$$\int \frac{\mathrm{d}\Omega}{4\pi} n^a n^b = \frac{1}{3}\eta^{ab}, \quad \int \frac{\mathrm{d}\Omega}{4\pi} n^a n^b n^c n^d = \frac{1}{15}\left(\eta^{ab}\eta^{cd} + \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}\right). \tag{158}$$

which is analogous to Eq. (156).

We can now use these particular solutions to calculate the integral of the total energy release (power of the gravitational radiation)

$$\begin{aligned}\int \frac{\mathrm{d}\Omega}{4\pi}\dddot{Q}^{ij}_{TT}\dddot{Q}^{TT}_{ij} &= \\ &= \dddot{Q}_{ab}\dddot{Q}^{ab} - \frac{\eta^{bd}}{3}\dddot{Q}_{ab}\dddot{Q}^a_d - \frac{\eta^{ac}}{3}\dddot{Q}_{ab}\dddot{Q}^b_c + \frac{\eta^{ab}\eta^{cd} + \eta^{ac}\eta^{bd} + \eta^{ad}\eta^{bc}}{30}\dddot{Q}_{ab}\dddot{Q}_{cd} = \\ &= \dddot{Q}_{ab}\dddot{Q}^{ab} - \frac{2}{3}\dddot{Q}_{ab}\dddot{Q}^{ab} + \frac{1}{30}\left(\dddot{Q}\dddot{Q} + \dddot{Q}_{ab}\dddot{Q}^{ab} + \dddot{Q}_{ab}\dddot{Q}^{ba}\right) = \frac{2}{5}\dddot{Q}_{ab}\dddot{Q}^{ab},\end{aligned} \tag{159}$$

where we use again $Q = 0$ and $Q^{ab} = Q^{ba}$. Inserting this into Eq. (148), we finally find the famous quadrupole radiation formula, first derived by A. Einstein,

$$\frac{\mathrm{d}E_{\mathrm{gw}}}{\mathrm{d}t} = \frac{G}{5c^5}\left\langle \dddot{Q}_{ab}\dddot{Q}^{ab}\right\rangle, \tag{160}$$

where, again, $\dddot{Q}_{ab}$ must be evaluated at the retarded time $t_{\mathrm{ret}}$. Sometimes, in explicit computations, it is more practical to use $q_{ij}$ instead of $Q_{ij}$, then we have $\frac{\mathrm{d}E_{\mathrm{gw}}}{\mathrm{d}t} = \frac{G}{5c^5}\left\langle \dddot{q}_{ab}\dddot{q}^{ab} - \frac{1}{3}\dddot{q}_{cc}q^{cc}\right\rangle$ which is not traceless. Some authors, e.g., Landau and Lifshitz, Vol. II (1979), or Carroll (1997), define the quadrupole moment with a different normalization, $Q^{LL}_{ij} = \int \mathrm{d}^3x\, \rho(t, \boldsymbol{x})(3x_i x^j - r^2\delta^{ij})$, where the superscript "LL" stands for Landau & Lifshitz. This is larger by the factor 3 than the definition in Eq. (160); the quadrupole formula, therefore, is $\frac{\mathrm{d}E_{\mathrm{gw}}}{\mathrm{d}t} = \frac{G}{45c^5}\left\langle \dddot{Q}_{ab}\dddot{Q}^{ab}\right\rangle$, and all other equations involving $Q_{ij}$ must be rescaled similarly.

Let's now consider the evolution of a binary star on a circular orbit in the $xy$ plane. The stars have masses $m_1$ and $m_2$ and separation $a$; they thus have the reduced mass $\mu$ and they orbit each other with an angular frequency $\omega$ and the total orbital energy $E_{\mathrm{orb}}$ of the binary system,

$$\mu = \frac{m_1 m_2}{M}, \qquad \Omega = \sqrt{\frac{GM}{a^3}}, \qquad E_{\mathrm{orb}} = \frac{1}{2}\left(m_1 a_1^2 + m_2 a_2^2\right)\Omega^2 - \frac{G m_1 m_2}{a} = -\frac{G\mu M}{2a}, \tag{161}$$

where $M = m_1 + m_2$ is the total mass and where the term in the bracket in the last equation is equal $\mu a^2$ in the center-of-mass coordinate system, as is already described in Eq. (110). We now follow the principles manifested by Eq. (112) of the quadrupole moment ($q_{ij}(t) = \mu x_0^i(t)x_0^j(t)$ for reminder); we consider for the moment an isolated binary system with masses $m_1$ and $m_2$, and we assume that the relative coordinate $x_0$ is performing a circular orbit with constant radius $R$. We choose the system's trajectory in $xy$ plane, so in the center-of-mass frame, the $x_0^i$ components are given as

$$x_0(t_{\mathrm{ret}}) = R\cos\Omega t_{\mathrm{ret}}, \qquad y_0(t_{\mathrm{ret}}) = R\sin\Omega t_{\mathrm{ret}}, \qquad z_0(t_{\mathrm{ret}}) = 0, \tag{162}$$

so, the components of the mass quadrupole moment are

$$q_{11}(t_{\mathrm{ret}}) = \mu R^2\cos^2\Omega t_{\mathrm{ret}}, \quad q_{22}(t_{\mathrm{ret}}) = \mu R^2\sin^2\Omega t_{\mathrm{ret}}, \quad q_{12}(t_{\mathrm{ret}}) = \mu R^2\sin\Omega t_{\mathrm{ret}}\cos\Omega t_{\mathrm{ret}}, \tag{163}$$

(where $q_{21}(t_{\mathrm{ret}}) = q_{12}(t_{\mathrm{ret}})$) while the other components vanish. In this sense, we have the time second derivatives

$$\begin{aligned} \ddot{q}_{11}(t_{\mathrm{ret}}) = -2\mu R^2\Omega^2\cos 2\Omega t_{\mathrm{ret}}, \qquad \ddot{q}_{22}(t_{\mathrm{ret}}) = 2\mu R^2\Omega^2\cos 2\Omega t_{\mathrm{ret}}, \\ \ddot{q}_{12}(t_{\mathrm{ret}}) = -2\mu R^2\Omega^2\sin 2\Omega t_{\mathrm{ret}}, \end{aligned} \tag{164}$$

and the time third derivatives

$$\begin{aligned} \dddot{q}_{11}(t_{\mathrm{ret}}) = 4\mu R^2\Omega^3\sin 2\Omega t_{\mathrm{ret}}, \qquad \dddot{q}_{22}(t_{\mathrm{ret}}) = -4\mu R^2\Omega^3\sin 2\Omega t_{\mathrm{ret}}, \\ \dddot{q}_{12}(t_{\mathrm{ret}}) = -4\mu R^2\Omega^3\cos 2\Omega t_{\mathrm{ret}}. \end{aligned} \tag{165}$$

The consistent matrix notation of the previous thus will be

$$\dddot{q}_{ij}(t_{\mathrm{ret}}) = 4\mu R^2\Omega^3\begin{pmatrix} \sin 2\Omega t_{\mathrm{ret}} & -\cos 2\Omega t_{\mathrm{ret}} & 0 \\ -\cos 2\Omega t_{\mathrm{ret}} & -\sin 2\Omega t_{\mathrm{ret}} & 0 \\ 0 & 0 & 0\end{pmatrix}_{ij}. \tag{166}$$

Since this matrix is traceless, according to Eq. (133), it also equals $\dddot{Q}_{ij}$. Plugging it into the quadrupole radiation formula (160) where we may the $\mathrm{d}E_{\mathrm{gw}}/\mathrm{d}t$ regard as the "luminosity" of the gravitational waves radiation $L_{\mathrm{gw}}$, we have

$$L_{\mathrm{gw}} = \frac{G}{5c^5}\left\langle \dddot{Q}_{ab}\dddot{Q}^{ab}\right\rangle = \frac{32G\mu^2R^4\Omega^6}{5c^5} = \frac{32G^4\mu^2M^3}{5c^5R^5}, \tag{167}$$

where the additional factor 2 comes from summing over all products of the corresponding elements $\langle\sin^2 2\Omega t_{\mathrm{ret}} + \cos^2 2\Omega t_{\mathrm{ret}} + \cos^2 2\Omega t_{\mathrm{ret}} + \sin^2 2\Omega t_{\mathrm{ret}}\rangle$ of the two identical matrices from Eq. (166).

As the gravitating system loses energy by emitting radiation, the distance between the two bodies shrinks at a rate

$$L_{\mathrm{gw}} = -\frac{\mathrm{d}E_{\mathrm{orb}}}{\mathrm{d}t} = -\frac{G\mu M}{2R^2}\frac{\mathrm{d}R}{\mathrm{d}t}, \quad \text{that is,} \quad \frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{64G^3\mu M^2}{5c^5R^3}. \tag{168}$$

The orbital frequency increases according to the decrease in the orbital distance as

$$\frac{\dot{f}}{f} = \frac{\dot{\Omega}}{\Omega} = -\frac{3\dot{R}}{2R}, \tag{169}$$

and, therefore, if the present separation of the two stars is $R_{\mathrm{init}}$, then from Eq. (168), the binary system will merge in the gravitational wave inspiral time

$$t_{\mathrm{GW}} = -\frac{5c^5}{64G^3\mu M^2}\int_{R_{\mathrm{init}}}^{0} R^3\,\mathrm{d}R = \frac{5c^5R_{\mathrm{init}}^4}{256G^3\mu M^2} = \frac{5c^5M^{1/3}}{256G^{5/3}m_1m_2\Omega_{\mathrm{init}}^{8/3}}. \tag{170}$$

As long as $\dot{\Omega} \ll \Omega^2$, we are in the quasi-circular motion regime. From Eq. (169), we see that $\dot{R} = -\frac{2}{3}R\frac{\dot{\Omega}}{\Omega} = -\frac{2}{3}R\Omega\frac{\dot{\Omega}}{\Omega^2} = -\frac{2}{3}V_\phi\frac{\dot{\Omega}}{\Omega^2}$, then, as long as the condition $\dot{\Omega} \ll \Omega^2$ is fulfilled, $|\dot{R}|$ is much smaller than the tangential orbital velocity $V_\phi = R\Omega$. The approximation of a circular orbit with a slowly varying radius is then applicable.

To evaluate the amplitude of the gravitational wave, we first apply the $P$ operator from Eq. (134) to the traceless form of Eq. (102),

$$h_{ij}^{TT} = \frac{2G}{c^4r}\ddot{Q}_{ij}(t_{\mathrm{ret}}), \tag{171}$$

remembering that in the TT gauge $h_{ij} = \phi_{ij}$ and $\ddot{Q}_{ij} = \ddot{q}_{ij}$. First, we realize that if the propagation direction of a gravitational wave is $\boldsymbol{z}$, then the $P_{ij}$ operator becomes (because both $n_x$ and $n_y$ are zero, and $n^in_j = 0$ if $i \neq j$, so $P_x^x = \delta_x^x = 1$, $P_y^y = \delta_y^y = 1$, and $P_z^z = \delta_z^z - n^zn_z = 1 - 1 = 0$)

$$P_{ij} = P_i^j = P^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}_{ij}, \quad \text{so} \quad P_i^a\ddot{q}_{ab}P_b^j - \frac{1}{2}P_{ab}\,\ddot{q}_{ab}P^{ij} = \ddot{q}_{ij} - \frac{1}{2}\ddot{q}_{aa}P^{ij} =$$

$$= \begin{pmatrix} \ddot{q}_{11} & \ddot{q}_{12} & 0 \\ \ddot{q}_{21} & \ddot{q}_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2}\operatorname{Tr}(\ddot{q})\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (\ddot{q}_{11} - \ddot{q}_{22})/2 & \ddot{q}_{12} & 0 \\ \ddot{q}_{21} & -(\ddot{q}_{11} - \ddot{q}_{22})/2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{172}$$

We directly see the two polarization amplitudes of the gravitational waves propagating in the $z$ direction

$$h_+(t,\boldsymbol{z}) = \frac{G}{c^4r}(\ddot{q}_{11} - \ddot{q}_{22}), \quad h_\times(t,\boldsymbol{z}) = \frac{2G}{c^4r}\,\ddot{q}_{12}, \tag{173}$$

evaluated at $t_{\rm ret}$. To calculate the amplitude of a wave that in the coordinate frame with axes $(x, y, z)$ propagates to an arbitrary direction $\boldsymbol{n}$, we introduce another orthogonal frame $(x', y', z')$ that is identical with $(x, y, z)$ at the beginning and performs two rotations, first around the axis $z \equiv z'$ and subsequently around the axis $x'$, after which both frames are inclined in both angular directions by angles $\phi$ and $\theta$. If the wave now propagates in the primed frame along the axis $z'$, then the vector $\boldsymbol{n}$ has coordinates $n'_i = (0, 0, 1)$ while in the unprimed frame $n_i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Equation (173) will in the primed frame differ by the argument $(t, \boldsymbol{n})$ of the amplitudes (instead of $(t, \boldsymbol{z})$) and the $\ddot{q}$'s now will be primed. We can construct the rotational matrix $R$ of two rotations around axes $z'$ and $x'$, respectively, simply as

$$\mathcal{R} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\phi & \cos\theta\sin\phi & \sin\theta\sin\phi \\ -\sin\phi & \cos\theta\cos\phi & \sin\theta\cos\phi \\ 0 & -\sin\theta & \cos\theta \end{pmatrix}, \tag{174}$$

and solve the transformation for $\ddot{q}'$ as $\ddot{q}'_{ij} = (\mathcal{R}^T \ddot{q}\, \mathcal{R})_{ij}$, where $\mathcal{R}^T$ is the transposed matrix $\mathcal{R}$. Applying this to the relevant terms of the primed version of Eq. (173) and merging symmetric terms into single ones (with doubled values), we obtain

$$\ddot{q}'_{11} = \ddot{q}_{11}\cos^2\phi - \ddot{q}_{12}\sin 2\phi + \ddot{q}_{22}\sin^2\phi, \tag{175}$$

$$\ddot{q}'_{12} = \ddot{q}_{11}\frac{\cos\theta\sin 2\phi}{2} + \ddot{q}_{12}\cos\theta\cos 2\phi - \ddot{q}_{13}\sin\theta\cos\phi - \ddot{q}_{22}\frac{\cos\theta\sin 2\phi}{2} + \ddot{q}_{23}\sin\theta\sin\phi \tag{176}$$

$$\begin{aligned} \ddot{q}'_{22} = \ddot{q}_{11}\cos^2\theta\sin^2\phi + \ddot{q}_{12}\cos^2\theta\sin 2\phi - \ddot{q}_{13}\sin 2\theta\sin\phi + \ddot{q}_{22}\cos^2\theta\cos^2\phi - \\ - \ddot{q}_{23}\sin 2\theta\cos\phi + \ddot{q}_{33}\sin^2\theta, \end{aligned} \tag{177}$$

and, therefore,

$$\begin{aligned} h_+(t, \boldsymbol{n}) &= \frac{G}{c^4 r}\left|\ddot{q}_{11}\left(\cos^2\phi - \cos^2\theta\sin^2\phi\right) - \ddot{q}_{12}\sin 2\phi\left(1 + \cos^2\theta\right) + \ddot{q}_{13}\sin 2\theta\sin\phi + \right. \\ &\quad \left. + \ddot{q}_{22}\left(\sin^2\phi - \cos^2\theta\cos^2\phi\right) + \ddot{q}_{23}\sin 2\theta\cos\phi - \ddot{q}_{33}\sin^2\theta\right|, \\ h_\times(t, \boldsymbol{n}) &= \frac{G}{c^4 r}\left|(\ddot{q}_{11} - \ddot{q}_{22})\cos\theta\sin 2\phi + 2\,\ddot{q}_{12}\cos\theta\cos 2\phi - 2\,\ddot{q}_{13}\sin\theta\cos\phi + \right. \\ &\quad \left. + 2\,\ddot{q}_{23}\sin\theta\sin\phi\right|. \end{aligned} \tag{178}$$

This equation allows us to calculate the angular distribution of the quadrupole radiation once $q_{ij}$ is given.

Let's now again remind that we consider the evolution of a binary star on a circular orbit in the $xy$ plane, so if we plug Eq. (3) into (178), after some arithmetic, we have

$$h_+(t, \boldsymbol{n}) = \frac{4G\mu R^2\Omega^2}{c^4 r}\left(\frac{1 + \cos^2\theta}{2}\right)\cos\left(2\Omega t_{\rm ret} + 2\phi\right), \tag{179}$$

$$h_\times(t, \boldsymbol{n}) = \frac{4G\mu R^2\Omega^2}{c^4 r}\cos\theta\sin\left(2\Omega t_{\rm ret} + 2\phi\right), \tag{180}$$

and, substituting the frequency $f$ of gravitational wave (that is twice the orbital frequency) and realizing that in this circular orbit approximation where $\Omega\Delta t_{\rm ret} = \Delta\phi$, we can shift the origin of time so that $\Omega t_{\rm ret} + \phi \to \Omega t_{\rm ret}$, we get a modification of the above,

$$h_+(t, \boldsymbol{n}) = \frac{4G\mu R^2(\pi f)^2}{c^4 r}\left(\frac{1 + \cos^2\theta}{2}\right)\cos\left(2\pi f t_{\rm ret}\right), \tag{181}$$

$$h_\times(t, \boldsymbol{n}) = \frac{4G\mu R^2(\pi f)^2}{c^4 r}\cos\theta\sin\left(2\pi f t_{\rm ret}\right). \tag{182}$$

We see that observing the gravitational waves binary system source pole-on ($\theta = 0$, $\cos\theta = 1$), the amplitudes of $h_+$ and $h_\times$ are identical while the phase is shifted by $\pi/2$; the wave is thus circularly polarized. On the other hand, if we observe the system equator-on ($\theta = \pi/2$, $\cos\theta = 0$), the amplitude $h_\times$ vanishes, and the amplitude $h_+$ is of half magnitude, the wave is linearly polarized.

We make another consideration; following the angular acceleration and substituting Eq. (168), we find that

$$\dot{\Omega} = -\frac{3\Omega}{2R}\dot{R} = \frac{96 G^{7/2}\mu M^{5/2}}{5c^5 R^{11/2}} = \frac{96 G^{5/3}\left(\mu M^{2/3}\right)\Omega^{11/3}}{5c^5}. \tag{183}$$

Now, we rearrange this equation to express the explicit mass-containing term (highlighted by the bracket) in the dimension of mass [g]; this means that we have to raise the expression $\mu M^{2/3}$ by the power 3/5, transfer it to the left-hand side of the equation, and leave on the right-hand side the quantities $\Omega$ and $\dot{\Omega}$ observable due to the propagation of gravitational waves. We get

$$\mu^{3/5} M^{2/5} = \left(\frac{5}{3}\right)^{3/5} \frac{c^3}{8G}\,\dot{\Omega}^{3/5}\Omega^{-11/5}, \tag{184}$$

where the quantity on the left-hand side is the "chirp mass,"

$$\mathcal{M}_\mathrm{c} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}} \tag{185}$$

that can thus be measured from observations.

Implementing the chirp mass as a canonical quantity, we define the chirp as a rapid frequency increase $\dot{f}$. We follow Eq. (183) keeping in mind that the orbiting angular frequency is twice lower than the frequency of the wave (which has two peaks during one orbit), so $\Omega = \pi f$ (not $2\pi f$); after little arithmetic we write its usual form

$$\dot{f} = \frac{96}{5}\frac{c^3}{G}\frac{f}{\mathcal{M}_\mathrm{c}}\left(\frac{G}{c^3}\pi f \mathcal{M}_\mathrm{c}\right)^{8/3}. \tag{186}$$

Integrating Eq. (186) as $\int \frac{\mathrm{d}f}{f^{11/3}} = \frac{96}{5}\frac{c^3}{G}\frac{1}{\mathcal{M}_\mathrm{c}}\left(\frac{G}{c^3}\pi\mathcal{M}_\mathrm{c}\right)^{8/3}\int_t^{t_\mathrm{coal}} \mathrm{d}t$, we see that $f$ formally diverges at a finite value of time (time of coalescence) that we denote $t_\mathrm{coal}$; in terms of the time remaining to $t_\mathrm{coal}$, the frequency of a gravitational wave is

$$f(t) = \frac{1}{\pi}\left(\frac{c^3}{G\mathcal{M}_\mathrm{c}}\right)^{5/8}\left[\frac{5}{256\,(t_\mathrm{coal} - t)}\right]^{3/8}. \tag{187}$$

Inserting Eqs. (186) and (187) into Eq. (169), after a little algebra, we evaluate the expression

$$\frac{\dot{R}}{R} = -\frac{2\dot{f}}{3f} = -\frac{1}{4\,(t_\mathrm{coal} - t)}, \tag{188}$$

which, after integration of the radius according to time, gives

$$\int_{R_0}^{R}\frac{\mathrm{d}R}{R} = -\int_{t_0}^{t}\frac{\mathrm{d}t'}{4\,(t_\mathrm{coal} - t')}, \quad \text{that is} \quad R(t) = R_0\left(\frac{t_\mathrm{coal} - t}{t_\mathrm{coal} - t_0}\right)^{1/4}, \tag{189}$$

where $R_0$ is the value of $R$ at the initial time $t_0$. This means that after a long phase of relatively smoothly decreasing $R$, there is a plunge phase where the approximation of a quasi-circular orbit is no longer valid. By inversion of (186), we can also express the chirp mass as

$$\mathcal{M}_\mathrm{c} = \frac{c^3}{G}\left(\frac{5}{96}\pi^{-8/3}f^{-11/3}\dot{f}\right)^{3/5}. \tag{190}$$

Substituting the chirp mass $\mathcal{M}_c$ into the angle-independent part of Eq. (181) or (182), we obtain (after some algebra and renaming the general distance $r$ to the luminosity distance $D$) the scaling amplitude

$$h_0 = \frac{4}{D}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3}\left(\frac{\pi f}{c}\right)^{2/3}, \tag{191}$$

In this lowest order Newtonian approximation, the amplitudes depend on masses $m_1$ and $m_2$ only through the chirp mass $\mathcal{M}_c$. By simple inversion, we can thus calculate the luminosity distance

$$D = \frac{4}{h_0}\left(\frac{G\mathcal{M}_c}{c^2}\right)^{5/3}\left(\frac{\pi f}{c}\right)^{2/3}, \tag{192}$$

which is a method of measuring the luminosity distance using only gravitational wave observables. This is extremely useful as an independent distance indicator in astronomy.

The amplitude of the emitted gravitational waves depends on the angle between the line of sight and the axis of angular momentum; the formula for the amplitude contains angular factors of order 1. The relative strength of the two polarizations depends on that angle as well. If three (or more) detectors observe the same signal, it is possible to reconstruct the full waveform and deduce many details about the orbit of the binary system. As the oldest canonical example, the well-studied pulsar PSR 1913+16 (the Hulse-Taylor pulsar) served a long time. It is expected to merge after $\sim 3.5 \times 10^8$ years. The binary system is roughly 5 kpc away from the Earth; the masses of the two neutron stars are estimated to be $\sim 1.4\, M_\odot$ each, and the present period of the system is $\sim 7$ h and 45 min. The predicted rate of period change is $\dot{T} = -2.4 \times 10^{-12}$ sec/sec, while the corresponding observed value is in excellent agreement with the predictions, i.e., $\dot{T} = (-2.3 \pm 0.22) \times 10^{-12}$ sec/sec; finally, the present amplitude of gravitational waves is of the order of $h \sim 10^{-23}$ at a frequency of $\sim 7 \times 10^{-5}$ Hz.