Solve the inhomogeneous wave equation using the Green's function:

Let's now consider the inhomogeneous wave equation in three spatial dimensions,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right) u(t,\boldsymbol{x}) = f(t,\boldsymbol{x}), \tag{1}$$

where we hereafter denote the d'Alembertian operator (the term in bracket) conventionally as $\Box$. We now define the Green's function $G(t,\boldsymbol{x},t',\boldsymbol{x}')$ as a function which satisfies

$$\Box\, G(t,\boldsymbol{x},t',\boldsymbol{x}') = \delta(t-t')\,\delta^3(\boldsymbol{x}-\boldsymbol{x}'), \tag{2}$$

where the time derivative and Laplacian act on the unprimed variables.

Without any loss of generality, we can set $t'=0$ and $\boldsymbol{x}'=0$ so $G(t,\boldsymbol{x})$ is the response of the system to a point-impulse delivered at $t=0$, $\boldsymbol{x}=0$. We begin to solve for $G(t,\boldsymbol{x})$ by taking the Fourier transform in time and space (see Chapter 11 in the textbook Computing Practice, while the sign convention can be opposite in other texts)

$$\widehat{G}(\omega,\boldsymbol{k}) = \int \mathrm{d}t \int \mathrm{d}^3\boldsymbol{x}\; G(t,\boldsymbol{x})\,\mathrm{e}^{-\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}, \quad G(t,\boldsymbol{x}) = \frac{1}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}^3\boldsymbol{k}\; \widehat{G}(\omega,\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}. \tag{3}$$

We also note the consequences of the Fourier transform of the Dirac delta function and its inverse,

$$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{e}^{-\mathrm{i}\xi x}\,\mathrm{d}x = \mathrm{e}^{-\mathrm{i}\xi\cdot 0} = 1 \quad \text{so then} \quad \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\xi x}\,\mathrm{d}\xi = \delta(x). \tag{4}$$

Applying this all to Eq. (2), we get

$$\Box\,\frac{1}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}^3\boldsymbol{k}\; \widehat{G}(\omega,\boldsymbol{k})\,\mathrm{e}^{\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})} = \frac{1}{(2\pi)^4}\int \mathrm{d}\omega \int \mathrm{d}^3\boldsymbol{k}\; \mathrm{e}^{\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})} \tag{5}$$

and, after differentiation of the left-hand side of Eq. (5) where, since it is a function of different variables, the d'Alembertian does not affect the Fourier transformed Green's function $\widehat{G}(\omega,\boldsymbol{k})$, it is

$$\int \mathrm{d}\omega \int \mathrm{d}^3\boldsymbol{k}\; \widehat{G}(\omega,\boldsymbol{k})\left(k^2 - \frac{\omega^2}{c^2}\right)\mathrm{e}^{\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})} = \int \mathrm{d}\omega \int \mathrm{d}^3\boldsymbol{k}\; \mathrm{e}^{\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}, \tag{6}$$

where $k = |\boldsymbol{k}|$. Comparing the left- and right-hand side of Eq. (6), we get

$$\widehat{G}(\omega,\boldsymbol{k}) = \frac{1}{k^2-\omega^2/c^2}. \tag{7}$$

Inserting Eq. (7) into the second equation in Eq. (3), we find the Green's function $G(t,\boldsymbol{x})$ by integration

$$G(t,\boldsymbol{x}) = \frac{1}{(2\pi)^4}\iint \frac{\mathrm{e}^{\mathrm{i}(\omega t-\boldsymbol{k}\cdot\boldsymbol{x})}}{k^2-\omega^2/c^2}\,\mathrm{d}^3\boldsymbol{k}\,\mathrm{d}\omega. \tag{8}$$

Converting to spherical coordinates ($x,y,z \to k,\theta,\phi$), where we identify the x-axis with the third component of the wave vector k_3, i.e. $k=|\boldsymbol{k}|$, $x=|\boldsymbol{x}|$, $k_3||\boldsymbol{x}$ and $\boldsymbol{k}\cdot\boldsymbol{x} = kx\cos\theta$, we transform Eq. (8) as

$$G(t,\boldsymbol{x}) = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega t}\,\mathrm{d}\omega \int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} \frac{\mathrm{e}^{-ikx\cos\theta}}{k^2-\omega^2/c^2}\,k^2\sin\theta\,\mathrm{d}k\,\mathrm{d}\theta\,\mathrm{d}\phi. \tag{9}$$

Noting that the azimuthal integral $\int \mathrm{d}\phi = 2\pi$, we first integrate separately the polar angular part of the integral (9) by substituting $-\mathrm{i}kx\cos\theta = t$, $\mathrm{i}kx\sin\theta\,\mathrm{d}\theta = \mathrm{d}t$ and involving the Euler identities, that is

$$\int_0^{\pi} \mathrm{e}^{-\mathrm{i}kx\cos\theta}\sin\theta\,\mathrm{d}\theta = \frac{1}{\mathrm{i}kx}\int_{-\mathrm{i}kx}^{\mathrm{i}kx} \mathrm{e}^{t}\,\mathrm{d}t = \frac{\mathrm{e}^{\mathrm{i}kx}-\mathrm{e}^{-\mathrm{i}kx}}{\mathrm{i}kx} \quad \left(= \frac{2}{kx}\sin kx\right). \tag{10}$$

By converting this solution into the immediately preceding integral in Eq. (9), we may write this integral as

$$\frac{1}{\mathrm{i}x}\left(\int_0^{\infty}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{k^2-\omega^2/c^2}\,\mathrm{d}k - \int_0^{\infty}\frac{k\,\mathrm{e}^{-\mathrm{i}kx}}{k^2-\omega^2/c^2}\,\mathrm{d}k\right). \tag{11}$$

We can formally simplify the last integral by substitution $k \to -k$, $\mathrm{d}k \to -\mathrm{d}k$ in the second term, then its upper integration limit changes from ∞ to $-\infty$; we can swap the limits and change the sign in front of the integral, obtaining

$$\frac{1}{\mathrm{i}x}\left[\int_0^{\infty}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{k^2-\omega^2/c^2}\,\mathrm{d}k + \int_{-\infty}^{0}\frac{(-k)\,\mathrm{e}^{\mathrm{i}kx}}{k^2-\omega^2/c^2}(-\,\mathrm{d}k)\right] = \frac{1}{\mathrm{i}x}\int_{-\infty}^{\infty}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{k^2-\omega^2/c^2}\,\mathrm{d}k. \tag{12}$$

We solve Eq. (12) using the Residual theorem (see Eq. (11.20) and the accompanying text in the textbook Computing Practice). Unfortunately, the contour integral is undefined since it goes right through isolated poles on the real axis at $k = \pm\omega/c$. We get around this obstacle by moving the poles slightly off the real axis (in the "vertical sense", i.e., along the imaginary axis) by adding a "tiny shift" $\mathrm{i}\epsilon$ in the limit $\epsilon \to 0$. Let's modify the last term of Eq. (12) to

$$\frac{1}{\mathrm{i}x}\int_{-\infty}^{\infty}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{k^2-(\omega/c+\mathrm{i}\epsilon)^2}\,\mathrm{d}k = \frac{1}{\mathrm{i}x}\int_{-\infty}^{\infty}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{(k-\omega/c-\mathrm{i}\epsilon)\,(k+\omega/c+\mathrm{i}\epsilon)}\,\mathrm{d}k. \tag{13}$$

Both the poles are first-order, so according to the Residual theorem, we get the solution of the contour integral in the form

$$2\pi\mathrm{i}\,\mathrm{Res}_{-(\omega/c+\mathrm{i}\epsilon)} = \frac{2\pi}{x}\lim_{k\to-(\omega/c+\mathrm{i}\epsilon)}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{(k-\omega/c-\mathrm{i}\epsilon)} = \frac{2\pi}{x}\frac{(\omega/c+\mathrm{i}\epsilon)\;\mathrm{e}^{-\mathrm{i}(\omega/c+\mathrm{i}\epsilon)x}}{2\,(\omega/c+\mathrm{i}\epsilon)} \tag{14}$$

and, since $\epsilon \to 0$, it is

$$2\pi\mathrm{i}\,\mathrm{Res}_{-(\omega/c+\mathrm{i}\epsilon)} = \frac{\pi}{x}\,\mathrm{e}^{-\mathrm{i}\omega x/c}. \tag{15}$$

Inserting everything into Eq. (9) (with the term 2π from the ϕ integration), we arrive at a form strikingly similar to the inverse Fourier transform of the delta function in Eq. (4),

$$G(t,x) = \frac{\pi}{x(2\pi)^3}\int_{-\infty}^{\infty}\mathrm{e}^{\mathrm{i}\omega t-\mathrm{i}\omega x/c}\,\mathrm{d}\omega = \frac{\delta(t-x/c)}{4\pi x}. \tag{16}$$

Finally, we express the latter in the full 4-dimensional form as

$$G(t,\boldsymbol{x},t',\boldsymbol{x}') = \frac{\delta\left(t-t'-\dfrac{|\boldsymbol{x}-\boldsymbol{x}'|}{c}\right)}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|}. \tag{17}$$