1 Solve the 3D inhomogeneous wave equation using the Green's function:

Let's now consider the inhomogeneous wave equation in three spatial dimensions (defined on a domain with specified initial or boundary conditions),

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta\right)u(t, \mathbf{x}) = f(t, \mathbf{x}),\tag{1}$$

where we hereafter denote the d'Alembertian operator (the term in bracket) conventionally as \Box . We now define the Green's function $G(t, \mathbf{x}, t', \mathbf{x}')$ as a function which satisfies

$$\Box G(t, \mathbf{x}, t', \mathbf{x}') = \delta(t - t') \,\delta^3(\mathbf{x} - \mathbf{x}'), \tag{2}$$

where the time derivative and Laplacian act on the unprimed variables.

Without any loss of generality, we can set t' = 0 and $\mathbf{x}' = 0$ so $G(t, \mathbf{x})$ is the response of the system to a point-impulse delivered at t = 0, $\mathbf{x} = 0$. We begin to solve for $G(t, \mathbf{x})$ by taking the Fourier transform in time and space (see Chapter 11 in the textbook Computing Practice, while the sign convention can be opposite in other texts)

$$\widehat{G}(\omega, \mathbf{k}) = \int \mathrm{d}t \int \mathrm{d}^3 \mathbf{x} \ G(t, \mathbf{x}) \,\mathrm{e}^{-\mathrm{i}(\omega t - \mathbf{k} \cdot \mathbf{x})}, \quad G(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \int \mathrm{d}\omega \int \mathrm{d}^3 \mathbf{k} \ \widehat{G}(\omega, \mathbf{k}) \,\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k} \cdot \mathbf{x})}. \tag{3}$$

We also note the consequences of the Fourier transform of the Dirac delta function and its inverse,

$$\int_{-\infty}^{\infty} \delta(x) e^{-i\xi x} dx = e^{-i\xi \cdot 0} = 1 \quad \text{so then} \quad \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\xi x} d\xi = \delta(x).$$
(4)

Applying this all to Eq. (2), we get

$$\Box \frac{1}{(2\pi)^4} \int d\omega \int d^3 \boldsymbol{k} \ \hat{G}(\omega, \boldsymbol{k}) e^{i(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})} = \frac{1}{(2\pi)^4} \int d\omega \int d^3 \boldsymbol{k} \ e^{i(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})}$$
(5)

and, after differentiation of the left-hand side of Eq. (5) where, since it is a function of different variables, the d'Alembertian does not affect the Fourier transformed Green's function $\hat{G}(\omega, \mathbf{k})$, it is

$$\int d\omega \int d^3 \boldsymbol{k} \ \hat{G}(\omega, \boldsymbol{k}) \left(k^2 - \frac{\omega^2}{c^2} \right) e^{i(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})} = \int d\omega \int d^3 \boldsymbol{k} \ e^{i(\omega t - \boldsymbol{k} \cdot \boldsymbol{x})}, \tag{6}$$

where $k = |\mathbf{k}|$. Comparing the left- and right-hand side of Eq. (6), we get

$$\widehat{G}(\omega, \mathbf{k}) = \frac{1}{k^2 - \omega^2/c^2}.$$
(7)

Inserting Eq. (7) into the second equation in Eq. (3), we find the Green's function $G(t, \mathbf{x})$ by integration

$$G(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \iint \frac{\mathrm{e}^{\mathrm{i}(\omega t - \mathbf{k} \cdot \mathbf{x})}}{k^2 - \omega^2/c^2} \,\mathrm{d}^3 \mathbf{k} \,\mathrm{d}\omega.$$
(8)

Converting to spherical coordinates $(x,y,z \to k,\theta,\phi)$, where we identify the x-axis with the third component of the wave vector k_3 , i.e. $k = |\mathbf{k}|, x = |\mathbf{x}|, k_3||\mathbf{x}|$ and $\mathbf{k} \cdot \mathbf{x} = kx \cos \theta$, we transform Eq. (8) as

$$G(t, \mathbf{x}) = \frac{1}{(2\pi)^4} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \int_{0}^{\infty} \int_{0}^{\pi} \int_{0}^{2\pi} \frac{e^{-ikx\cos\theta}}{k^2 - \omega^2/c^2} k^2 \sin\theta \, \mathrm{d}k \, \mathrm{d}\theta \, \mathrm{d}\phi.$$
(9)

Noting that the azimuthal integral $\int d\phi = 2\pi$, we first integrate separately the polar angular part of the integral (9) by substituting $-ikx \cos \theta = t$, $ikx \sin \theta d\theta = dt$ and involving the Euler identities, that is

$$\int_{0}^{\pi} e^{-ikx\cos\theta} \sin\theta \,\mathrm{d}\theta = \frac{1}{ikx} \int_{-ikx}^{ikx} e^{t} \,\mathrm{d}t = \frac{e^{ikx} - e^{-ikx}}{ikx} \quad \left(=\frac{2}{kx}\sin kx\right). \tag{10}$$

By converting this solution into the immediately preceding integral in Eq. (9), we may write this integral as

$$\frac{1}{ix} \left(\int_{0}^{\infty} \frac{k e^{ikx}}{k^2 - \omega^2/c^2} dk - \int_{0}^{\infty} \frac{k e^{-ikx}}{k^2 - \omega^2/c^2} dk \right).$$
(11)

We can formally simplify the last integral by substitution $k \to -k$, $dk \to -dk$ in the second term, then its upper integration limit changes from ∞ to $-\infty$; we can swap the limits and change the sign in front of the integral, obtaining

$$\frac{1}{\mathrm{i}x} \left[\int_{0}^{\infty} \frac{k \,\mathrm{e}^{\mathrm{i}kx}}{k^2 - \omega^2/c^2} \,\mathrm{d}k + \int_{-\infty}^{0} \frac{(-k) \,\mathrm{e}^{\mathrm{i}kx}}{k^2 - \omega^2/c^2} (-\,\mathrm{d}k) \right] = \frac{1}{\mathrm{i}x} \int_{-\infty}^{\infty} \frac{k \,\mathrm{e}^{\mathrm{i}kx}}{k^2 - \omega^2/c^2} \,\mathrm{d}k. \tag{12}$$

We solve Eq. (12) using the Residual theorem (see Eq. (11.20) and the accompanying text in the textbook Computing Practice). Unfortunately, the contour integral is undefined since it goes right through isolated poles on the real axis at $k = \pm \omega/c$. We get around this obstacle by moving the poles slightly off the real axis (in the "vertical sense", i.e., along the imaginary axis) by adding a "tiny shift" i ϵ in the limit $\epsilon \to 0$. Let's modify the last term of Eq. (12) to

$$\frac{1}{\mathrm{i}x}\int_{-\infty}^{\infty}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{k^2 - (\omega/c + \mathrm{i}\epsilon)^2}\,\mathrm{d}k = \frac{1}{\mathrm{i}x}\int_{-\infty}^{\infty}\frac{k\,\mathrm{e}^{\mathrm{i}kx}}{(k - \omega/c - \mathrm{i}\epsilon)\,(k + \omega/c + \mathrm{i}\epsilon)}\,\mathrm{d}k.\tag{13}$$

Both the poles are first-order, so according to the Residual theorem, we get the solution of the contour integral in the form

$$2\pi i \operatorname{Res}_{-(\omega/c+i\epsilon)} = \frac{2\pi}{x} \lim_{k \to -(\omega/c+i\epsilon)} \frac{k \operatorname{e}^{\mathrm{i}kx}}{(k - \omega/c - i\epsilon)} = \frac{2\pi}{x} \frac{(\omega/c + i\epsilon) \operatorname{e}^{-\mathrm{i}(\omega/c+i\epsilon)x}}{2(\omega/c + i\epsilon)}$$
(14)

and, since $\epsilon \to 0$, it is

$$2\pi i \operatorname{Res}_{-(\omega/c+i\epsilon)} = \frac{\pi}{x} e^{-i\omega x/c}.$$
(15)

Inserting everything into Eq. (9) (with the term 2π from the ϕ integration), we arrive at a form strikingly similar to the inverse Fourier transform of the delta function in Eq. (4),

$$G(t,x) = \frac{\pi}{x(2\pi)^3} \int_{-\infty}^{\infty} e^{i\omega t - i\omega x/c} d\omega = \frac{\delta(t - x/c)}{4\pi x}.$$
 (16)

Finally, we express the latter in the full 4-dimensional form as

$$G(t, \mathbf{x}, t', \mathbf{x}') = \frac{\delta\left(t - t' - \frac{|\mathbf{x} - \mathbf{x}'|}{c}\right)}{4\pi |\mathbf{x} - \mathbf{x}'|}.$$
(17)

The solution of the original equation (1) is then the convolution (G * f) (see Eq. (10.7) and the explanatory text in the textbook Computing Practice).

2 Solve the 1D driven harmonic oscillator using the Green's function:

As another example, we examine a harmonic oscillator of natural frequency ω driven by an external force f(t). We assume that the force begins to act at time t = 0, when the oscillator was at rest. The differential equation that governs the motion of this oscillator is

$$\ddot{X} + \omega^2 X = f,\tag{18}$$

with X measuring the oscillator's displacement from its equilibrium position. The initial conditions are

$$X(0) = 0, \quad \dot{X}(0) = 0. \tag{19}$$

We now wish to find the solution to Eq. (18) for any force f.

We need a Green's function for Eq (18). We replace the arbitrary f(t) by the specific delta function $\delta(t - t')$, where t' is a reference time. Green's equation becomes

$$\ddot{G}(t,t') + \omega^2 G(t,t') = \delta(t-t'),$$
(20)

and we wish to solve it with the initial conditions

$$G(0,t') = 0, \quad \dot{G}(0,t') = 0.$$
 (21)

With G(t, t') identified, the general solution to Eq. (18) with the initial conditions of Eq. (19) is

$$X(t) = \int_0^\infty G(t, t') f(t') dt'.$$
 (22)

We now specify the displacement X(t):

when $t \neq t'$, that is, t < t' or t > t', the Green's function satisfies the homogeneous equation $\ddot{G} + \omega^2 G = 0$, and the solution to this is a superposition of sine and cosine functions. We therefore write

$$G(t,t') = \begin{cases} G_{-}(t,t') = a_{-}\cos(\omega t) + b_{-}\sin(\omega t) & t < t' \\ G_{+}(t,t') = a_{+}\cos(\omega t) + b_{+}\sin(\omega t) & t > t' \end{cases},$$
(23)

where a_- , b_- , a_+ , and b_+ are constants. The initial conditions are imposed at t = 0, so that t < t' and $G(t, t') = G_-(t, t')$. They imply that $a_- = 0$ and $b_- = 0$, so that $G_-(t, t') = 0$. Therefore, we have at this stage

$$G(t,t') = \begin{cases} G_{-}(t,t') = 0 & t < t' \\ G_{+}(t,t') = a_{+}\cos(\omega t) + b_{+}\sin(\omega t) & t > t' \end{cases},$$
(24)

and we still have a_+ and b_+ left to determine.

Continuity of the Green's function at t = t' (between the regions G_{-} and G_{+}) provides the condition

$$0 = G_{+}(t', t') = a_{+} \cos(\omega t') + b_{+} \sin(\omega t'), \qquad (25)$$

we solve this for a_+ and b_+ by writing $a_+ = -c \sin(\omega t')$ and $b_+ = c \cos(\omega t')$, satisfying simply the condition (25), where c is another arbitrary but common constant. Making the substitution in $G_+(t,t')$ yields

$$G_{+}(t,t') = c \left[\sin(\omega t)\cos(\omega t') - \cos(\omega t)\sin(\omega t')\right] = c \sin\left[\omega(t-t')\right],$$
(26)

where the last form makes it clear that $G_+(t, t')$ goes to zero at t = t'.

To find the constant c, we must account for the delta function in Green's equation, which gives rise to a discontinuity in the first derivative of G(t, t') at t = t'. We can derive this condition by integrating Green's equation from $t = \lim_{\epsilon \to 0} (t' - \epsilon)$ to $t = \lim_{\epsilon \to 0} (t' + \epsilon)$. We have

$$\int_{t'-\epsilon}^{t'+\epsilon} \ddot{G} \,\mathrm{d}t + \omega^2 \int_{t'-\epsilon}^{t'+\epsilon} G \,\mathrm{d}t = \int_{t'-\epsilon}^{t'+\epsilon} \delta(t-t') \,\mathrm{d}t, \tag{27}$$

where we see that we can deal easily with the right-hand side as well as the first term on the left-hand side, but the second term presents a somewhat bigger problem. To solve this, we recall that we are interested in the limit $\epsilon \to 0$ and that the Green's function is continuous at t = t'. This implies that the second term can be approximated by $2\epsilon \omega^2 G(t', t')$, and that this contribution to the equation vanishes when $\epsilon \to 0$. Solving Eq. (27) for the first left-hand side and the first right-hand side terms only, we get

$$\dot{G}_{+}(t',t') - \dot{G}_{-}(t',t') = 1.$$
 (28)

Examining the discontinuity condition, we see that the second term vanishes, the first term from Eq. (26) evaluates to $c\omega$, and we arrive at $c = \omega^{-1}$.

Our final expression for the Green's function is

$$G(t,t') = \begin{cases} G_{-}(t,t') = 0 & t < t' \\ G_{+}(t,t') = \frac{\sin\left[\omega(t-t')\right]}{\omega} & t > t' \end{cases}$$
(29)

Substituting this into Eq. (22) gives the convolution

$$X(t) = \frac{1}{\omega} \int_0^t \sin \left[\omega(t - t') \right] f(t') \, \mathrm{d}t', \tag{30}$$

which applies to any forcing function f.

As an example we may choose a forcing term that grows linearly with time. We write $f(t) = F_0 \omega t$, where F_0 is a constant, and substitute this within Eq. (30). A simple calculation returns

$$X(t) = F_0 \frac{\omega t - \sin(\omega t)}{\omega^2}.$$
(31)

Substituting this result into Eq. (18) and checking this by inserting conditions (19), we can easily verify its correctness.

3 Green's theorem and Green's functions in electrostatics:

Soon after Charles-Augustin de Coulomb formulated the inverse square law of the electric force in 1785, theorists Joseph-Louis Lagrange, Pierre-Simon Laplace, Siméon Denis Poisson and others, used the then modern technique of mathematical calculus to analyse the electric field for arbitrary charge distributions. Experimental verification of the inverse quadrature law and its linearity (expressed in terms of the sum of the fields from different charges) led them to define the function, later named as scalar potential, $\phi(\mathbf{x})$; the local electric field vector (electric field intensity) is then defined as the negative gradient of this function, $\mathbf{E}(\mathbf{x}) = -\nabla \phi(\mathbf{x})$. The function ϕ must satisfy the so-called Poisson's equation,

$$\nabla^2 \phi(\mathbf{x}) = -\frac{\rho(\mathbf{x})}{\epsilon_0},\tag{32}$$

where $\rho(\mathbf{x})$ is the local charge density distribution and ϵ_0 is the vacuum electric permittivity. Poisson's equation thus determines the electric potential, and therefore its gradient as the electric field, from any arbitrary charge distribution (special case of a charge-free region is expressed by the Laplace's equation, $\nabla^2 \phi(\mathbf{x}) = 0$).

To understand the structure of the electrostatic potential and electric field intensity equation, let's start with the solution for the point charge Q in an empty space, whose potential is simply given as

$$\phi(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0} \frac{1}{|\mathbf{x} - \mathbf{x}'|},\tag{33}$$

where \mathbf{x} is the position where the potential is measured ("observer") and \mathbf{x}' is the position of the charge Q. Analogously, for a discrete or continuous distribution of charges over a finite volume V with points \mathbf{x}_i (for discrete distribution of charges) or \mathbf{x}' (for continuous distribution of charges), the electrostatic potential will be

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_{i} \frac{Q_i}{|\mathbf{x} - \mathbf{x}_i|}, \qquad \phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \,\mathrm{d}^3 x', \tag{34}$$

and, according to the negative gradient of the potential, the electric field intensity vector for the continuous charge distribution becomes

$$\boldsymbol{E}(\boldsymbol{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\boldsymbol{x}') \frac{\boldsymbol{x} - \boldsymbol{x}'}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \,\mathrm{d}^3 \boldsymbol{x}'.$$
(35)

Integrating the Laplacian of Eq. (33) over an empty volume that contains the point charge, with use of Eq. (32), we have

$$\int_{V} \nabla^{2} \phi(\mathbf{x}) \, \mathrm{d}V = \frac{Q}{4\pi\epsilon_{0}} \int_{V} \nabla^{2} \frac{1}{|\mathbf{x} - \mathbf{x}'|} \, \mathrm{d}V = -\frac{Q}{\epsilon_{0}}.$$
(36)

Since the integral must be zero everywhere except at the point charge, while the integral must be finite, we use the definition of the delta function to satisfy the equation (36) and obtain the expression (because $\int \delta^3(\mathbf{x} - \mathbf{x}') dV = 1$ in general)

$$\nabla^2 \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -4\pi \delta^3 (\boldsymbol{x} - \boldsymbol{x}').$$
(37)

Now, let's substitute the spatial Green's function $G(\mathbf{x}, \mathbf{x}')$ in the unlimited free space as

$$G(\mathbf{x} - \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|},\tag{38}$$

we can express the discrete or continuous charges' generated potential equations (34) as

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \sum_i Q_i G(\mathbf{x} - \mathbf{x}_i), \qquad \phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int_V \rho(\mathbf{x}') G(\mathbf{x} - \mathbf{x}') \,\mathrm{d}^3 x'. \tag{39}$$

The latter equation (39) is clearly the convolution $\rho * G$.

George Green continued in solving this problem by selecting a vector \boldsymbol{V} in the form

$$\boldsymbol{V}(\boldsymbol{x}) = \phi(\boldsymbol{x}) \boldsymbol{\nabla} \psi(\boldsymbol{x}), \tag{40}$$

where $\phi(\mathbf{x})$ and $\psi(\mathbf{x})$ are arbitrary scalar fields. By simple divergence, we arrive at the gradual conclusion that

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = \boldsymbol{\nabla} \cdot (\phi \boldsymbol{\nabla} \psi) = \phi \nabla^2 \psi + \boldsymbol{\nabla} \phi \cdot \boldsymbol{\nabla} \psi.$$
(41)

Since

$$\phi \nabla \psi \cdot \mathrm{d} \boldsymbol{S} = \phi \frac{\partial \psi}{\partial n} \,\mathrm{d} S,\tag{42}$$

where $\partial \psi / \partial n$ is the outer normal derivative at the closed surface S and dS is the surface element magnitude, that is, $dS = ||d\mathbf{S}||$ (see the directional derivatives in Computing practise textbook, Chap. 5). By applying the Gauss (divergence) theorem to Eq. (41), we get the *Green's first identity*

$$\int_{V} \left(\phi \nabla^{2} \psi + \boldsymbol{\nabla} \phi \cdot \boldsymbol{\nabla} \psi \right) \mathrm{d}V = \int_{S} \phi \frac{\partial \psi}{\partial n} \,\mathrm{d}S. \tag{43}$$

We now use the same identity with ϕ and ψ interchanged and subtract this from the first identity: we may write the analogous *Green's second identity* (Green's theorem)

$$\int_{V} \left(\phi \nabla^{2} \psi - \psi \nabla^{2} \phi \right) dV = \int_{S} \left(\phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right) dS.$$
(44)

To apply this identity to electrostatics, one can investigate for example the electrostatic potential generated by some volume charge distribution $\rho(\mathbf{x}')$ in the presence of conducting closed surfaces S, etc.