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## Reference

## Chapter 1

## Gas Dynamics

### 1.1 Boltzmann Kinetic Equation

(Kurfürst 2015): The Boltzmann kinetic equation (hereafter BKE) for a particle  $\alpha$ , used in the gas (plasma) kinetic theory (see Bittencourt 2004, for details), is

$$\frac{\partial f_{\alpha}}{\partial t} + \mathbf{v} \cdot \nabla f_{\alpha} + \mathbf{a}_{\text{ext}} \cdot \nabla_{\mathbf{v}} f_{\alpha} = \left(\frac{\delta f_{\alpha}}{\delta t}\right)_{\text{coll}}.$$
(1.1)

The distribution function  $f_{\alpha}(\mathbf{r}, \mathbf{v}, t)$  is defined as the density of the particles  $\alpha$  in the phase space,

$$f_{\alpha}(\mathbf{r}, \mathbf{v}, t) \,\mathrm{d}^{3}\mathbf{r} \,\mathrm{d}^{3}\mathbf{v} = \mathrm{d}^{6}N_{\alpha}(\mathbf{r}, \mathbf{v}, t), \qquad (1.2)$$

where the quantity  $d^6 N_\alpha(\mathbf{r}, \mathbf{v}, t)$  denotes the number of the particles  $\alpha$  in the phase space volume  $d^3\mathbf{r} d^3\mathbf{v}$  with coordinates  $(\mathbf{r}, \mathbf{v})$  at instant time t. The quantity  $\mathbf{a}_{ext}$  in Eq. (1.1) is the acceleration due to an external force,  $\nabla_{\mathbf{v}}$  is the velocity gradient  $\partial/\partial \mathbf{v}$  and the collision term on the right-hand side quantifies the rate of change of  $f_\alpha$  due to particle collisions. The average value of a general physical quantity  $\chi(\mathbf{r}, \mathbf{v}, t)$  for the particles  $\alpha$  is given by

$$\langle \chi(\mathbf{r}, \mathbf{v}, t) \rangle_{\alpha} = \frac{1}{n_{\alpha}(\mathbf{r}, t)} \int \chi(\mathbf{r}, \mathbf{v}, t) f_{\alpha}(\mathbf{r}, \mathbf{v}, t) \,\mathrm{d}^{3}\mathbf{v}, \qquad (1.3)$$

where  $n_{\alpha}(\mathbf{r}, t)$  is the number density (number of particles  $\alpha$  per unit volume) at instant time t, defined as integral of  $f_{\alpha}(\mathbf{r}, \mathbf{v}, t)$  over the velocity space,

$$n_{\alpha}(\mathbf{r},t) = \int f_{\alpha}(\mathbf{r},\mathbf{v},t) \,\mathrm{d}^{3}\mathbf{v}.$$
(1.4)

We multiply BKE for a particle  $\alpha$  by a general physical quantity  $\chi(\mathbf{v})$ , independent of time and space (a function of only the particle's velocity) and integrate it over the velocity space as

$$\int \chi \frac{\partial f_{\alpha}}{\partial t} \,\mathrm{d}^{3} \boldsymbol{v} + \int \chi \,\boldsymbol{v} \cdot \boldsymbol{\nabla} f_{\alpha} \,\mathrm{d}^{3} \boldsymbol{v} + \int \chi \,\boldsymbol{a}_{\mathrm{ext}} \cdot \boldsymbol{\nabla}_{\boldsymbol{v}} \,f_{\alpha} \,\mathrm{d}^{3} \boldsymbol{v} = \int \chi \left(\frac{\delta f_{\alpha}}{\delta t}\right)_{\mathrm{coll}} \mathrm{d}^{3} \boldsymbol{v}. \tag{1.5}$$

Since  $\mathbf{r}$ ,  $\mathbf{v}$  and t are independent, the spatial derivatives of the velocity dependent quantities,  $\nabla \cdot \mathbf{v}$  and  $\nabla \chi(\mathbf{v})$ , vanish. The force component  $F_i$  is independent of the velocity  $v_i$ , the velocity gradient of the acceleration,  $\nabla_{\mathbf{v}} \cdot \mathbf{a}_{\text{ext}}$ , also vanishes (this is not true in case of the magnetic force, where  $F_{L,i} = q_{\alpha} \epsilon_{ijk} v_j B_k$ ). The solution of Eq. (1.5) gives,

$$\frac{\partial}{\partial t} \left( n_{\alpha} \langle \chi \rangle_{\alpha} \right) + \boldsymbol{\nabla} \cdot \left( n_{\alpha} \langle \chi \boldsymbol{\nu} \rangle_{\alpha} \right) - n_{\alpha} \langle \boldsymbol{a}_{\text{ext}} \cdot \boldsymbol{\nabla}_{\boldsymbol{\nu}} \chi \rangle_{\alpha} = \left[ \frac{\delta}{\delta t} \left( n_{\alpha} \langle \chi \rangle_{\alpha} \right) \right]_{\text{coll}}, \quad (1.6)$$

where the terms in  $\langle \rangle$  are the average values of the corresponding quantities. We also define the mass density for the particles  $\alpha$  as  $\rho_{\alpha} = n_{\alpha}m_{\alpha}$ .

The velocity  $\boldsymbol{v}$  of the particle  $\alpha$  is the vector sum

$$\boldsymbol{\nu} = \boldsymbol{V}_{\alpha} + \boldsymbol{C}_{\alpha}, \tag{1.7}$$

where  $V_{\alpha}(\mathbf{r},t)$  is the flow (macroscopic drift) velocity of the particles  $\alpha$  at the position  $\mathbf{r}$  at time t,

$$\boldsymbol{V}_{\alpha}(\boldsymbol{r},t) = \frac{1}{n_{\alpha}(\boldsymbol{r},t)} \int \boldsymbol{v} f_{\alpha}(\boldsymbol{r},\boldsymbol{v},t) \, \mathrm{d}^{3}\boldsymbol{v}, \qquad (1.8)$$

and  $C_{\alpha}(\mathbf{r},t)$  is the random or peculiar velocity of thermal motion, relative to  $V_{\alpha}(\mathbf{r},t)$ . The average value of the flow velocity (as a macroscopic collective property) is  $\langle V_{\alpha} \rangle = V_{\alpha}$ , while the average thermal velocity  $\langle C_{\alpha} \rangle = \mathbf{0}$ . The average total velocity of the particle  $\alpha$  therefore is

$$\langle \boldsymbol{v} \rangle_{\alpha} = \langle \boldsymbol{V}_{\alpha} \rangle = \boldsymbol{V}_{\alpha}. \tag{1.9}$$

From the kinetic theory of gases follows the mass and momentum density of the matter,

$$\rho = \sum_{\alpha} \rho_{\alpha}, \quad \rho \mathbf{V} = \sum_{\alpha} \rho_{\alpha} \mathbf{V}_{\alpha}, \tag{1.10}$$

where  $\rho$  and  $\boldsymbol{V}$  are the density and the flow velocity of the whole medium. We introduce the diffusion velocity  $\boldsymbol{w}_{\alpha}$ , defined as the vector subtraction of velocity  $\boldsymbol{V}_{\alpha}$  of particle  $\alpha$  and the flow velocity  $\boldsymbol{V}$  of the medium,

$$\boldsymbol{w}_{\alpha} = \boldsymbol{V}_{\alpha} - \boldsymbol{V}, \tag{1.11}$$

which we regard as the velocity of the particle  $\alpha$  in a co-moving frame of the medium. Since the diffusion velocity  $\boldsymbol{w}_{\alpha}$  is clearly the macroscopic quantity,  $\langle \boldsymbol{w}_{\alpha} \rangle = \boldsymbol{w}_{\alpha}$ . We define also the global thermal velocity  $\boldsymbol{C}_{\alpha 0}$  for particles  $\alpha$  relative to the velocity of the fluid. Equations (1.7) and (1.11) give

$$\boldsymbol{C}_{\alpha 0} = \boldsymbol{v} - \boldsymbol{V}, \text{ that is, } \boldsymbol{C}_{\alpha 0} = \boldsymbol{C}_{\alpha} + \boldsymbol{w}_{\alpha}.$$
 (1.12)

We define the kinetic stress tensor  $\mathcal{T}_{\alpha ij}$  (see Sects. 1.4 and 1.4) for the particle  $\alpha$  (where i, j are spatial components) and the global kinetic stress tensor  $\mathcal{T}_{ij}$  as

$$\mathcal{T}_{\alpha i j} = -\rho_{\alpha} \langle C_{\alpha i} C_{\alpha j} \rangle, \quad \mathcal{T}_{i j} = -\sum_{\alpha} \rho_{\alpha} \langle C_{\alpha 0 i} C_{\alpha 0 j} \rangle.$$
(1.13)

The scalar pressure  $p_{\alpha}$  for the particle  $\alpha$  and the global scalar pressure p are defined as (cf. Sect. 1.5.1)

$$p_{\alpha} = \frac{1}{3} \rho_{\alpha} \langle C_{\alpha}^2 \rangle, \quad p = \frac{1}{3} \sum_{\alpha} \rho_{\alpha} \langle C_{\alpha 0}^2 \rangle.$$
(1.14)

The thermal energy flux  $\boldsymbol{q}_{\alpha}$  for the particle  $\alpha$  and the global thermal energy flux  $\boldsymbol{q}$  are

$$\boldsymbol{q}_{\alpha} = \frac{1}{2} \rho_{\alpha} \langle C_{\alpha}^{2} \boldsymbol{C}_{\alpha} \rangle, \quad \boldsymbol{q} = \frac{1}{2} \sum_{\alpha} \rho_{\alpha} \langle C_{\alpha 0}^{2} \boldsymbol{C}_{\alpha 0} \rangle.$$
(1.15)

## **1.2** Mass Conservation (Continuity) Equation

(We hereafter use R, V for radial distance and velocity in Cartesian and cylindrical coordinates, while r, v are used in the spherical coordinates.)

Substituting the mass  $m_{\alpha}$  of the particle  $\alpha$  for the general quantity  $\chi$  into Eq. (1.6), we obtain the mass conservation (continuity) equation (0-th moment of BKE, the mass conservation law) for the particle  $\alpha$ ,

$$\frac{\partial \rho_{\alpha}}{\partial t} + \nabla \cdot (\rho_{\alpha} \boldsymbol{V}_{\alpha}) = S_{\alpha}, \qquad (1.16)$$

where the collision term  $S_{\alpha}$  is (cf. Eq. (1.5))

$$S_{\alpha} = m_{\alpha} \int \left(\frac{\delta f_{\alpha}}{\delta t}\right)_{\text{coll}} \mathrm{d}^{3} \mathbf{v} = \left(\frac{\delta \rho_{\alpha}}{\delta t}\right)_{\text{coll}}.$$
 (1.17)

The term  $S_{\alpha}$  refers to the rate of production or destruction of particles  $\alpha$  due to particle interactions, i.e., due to ionization, recombination, charge transfer, etc. In case of non-isolated physical system it may also refer to source (or sink) of mass (it is also called the *source term*). The detail form of  $S_{\alpha}$  may be very complex in general. It involves the inelastic collisions that lead to production or loss of a particle type. In case of electrons, for example, the most important interactions are ionization and recombination (neglecting the electron capture ionization, etc.). If  $k_i$  and  $k_r$  denote the ionization and recombination collision rates, respectively, we may express such (simplified) collision term for electrons,  $S_e$ , as (Bittencourt 2004)

$$S_e = m_e \left( k_i n_e - k_r n_e^2 + \dots \right), \tag{1.18}$$

while the inclusion of the radiative rates would lead to equations of statistical equilibrium (see Sect. 10). If we sum Eq. (1.16) over all particle types  $\alpha$  in the isolated system, the collision term  $S_{\alpha}$  vanishes due to the total mass conservation, and we obtain the continuity equation for the whole medium,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \, \boldsymbol{V}) = 0. \tag{1.19}$$

An alternative derivation of the continuity equation is based on assumption of an equilibrium between the number of particles leaving some volume  $\Omega$  through its closed surface A (particle flux) and the time rate of particle number density decrease within the volume  $\Omega$ ,

$$\sum_{\alpha} \int_{A} n_{\alpha} m_{\alpha} \boldsymbol{V}_{\alpha} \cdot d\boldsymbol{A} = -\sum_{\alpha} \frac{\partial}{\partial t} \int_{\Omega} n_{\alpha} m_{\alpha} d\Omega, \qquad (1.20)$$

where  $n_{\alpha}m_{\alpha}$  is the mass density  $\rho_{\alpha}$  of the particle  $\alpha$ . Since Eq. (1.20) must hold for any arbitrary volume  $\Omega$ , by applying the Gauss's theorem and summing over all particles  $\alpha$ , we obtain Eq. (1.19).

#### **1.2.1** Equation of continuity in curvilinear coordinates

The general form of continuity equation is given by Eq. (1.19). To transform the continuity equation into *cylindrical* coordinates, we use Eq. A.44 in Kurfürst (2017), the cylindrical continuity equation is

$$\frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \,\rho V_R \right) + \frac{1}{R} \frac{\partial \left( \rho V_\phi \right)}{\partial \phi} + \frac{\partial \left( \rho V_z \right)}{\partial z} = 0. \tag{1.21}$$

Using Eq. A.70 in Kurfürst (2017), we transform the continuity equation into *spherical* coordinates,

$$\frac{\partial\rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho \, v_R \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \, \rho \, v_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial \left( \rho \, v_\phi \right)}{\partial \phi} = 0. \tag{1.22}$$

In axisymmetric or spherically symmetric case the terms with angular (and vertical) derivatives vanish, reducing Eq. (1.21) to the frequently used axisymmetric radial continuity equation

$$\frac{\partial \rho}{\partial t} + \frac{1}{R} \frac{\partial}{\partial R} \left( R \, \rho V_R \right) = 0, \tag{1.23}$$

or the spherically symmetric continuity equation,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \rho \, v_R \right) = 0. \tag{1.24}$$

## **1.3** Equation of Motion

Substituting the particle's momentum,  $m_{\alpha} \mathbf{v}$ , for the general quantity  $\chi(\mathbf{v})$  into Eq. (1.6), we obtain the equation of motion (momentum equation) (1st moment of BKE, the momentum conservation law) for the particle  $\alpha$ ,

$$\frac{\partial}{\partial t} \left( \rho_{\alpha} \boldsymbol{V}_{\alpha} \right) + \boldsymbol{\nabla} \cdot \left[ \rho_{\alpha} \left( \boldsymbol{V}_{\alpha} \otimes \boldsymbol{V}_{\alpha} + \langle \boldsymbol{C}_{\alpha} \otimes \boldsymbol{C}_{\alpha} \rangle \right) \right] - n_{\alpha} \boldsymbol{F}_{\alpha} = \boldsymbol{A}_{\alpha}, \qquad (1.25)$$

where  $\mathbf{V}_{\alpha} \otimes \mathbf{V}_{\alpha}$  and  $\mathbf{C}_{\alpha} \otimes \mathbf{C}_{\alpha}$  are the tensor (dyadic) products of vectors. The third left-hand side term  $\nabla \cdot (\rho_{\alpha} \langle \mathbf{C}_{\alpha} \otimes \mathbf{C}_{\alpha} \rangle)$  expresses the divergence of the stress tensor  $-\nabla_{j} \mathcal{T}_{\alpha}^{ij}$  (cf. Eq. (1.13)), the fourth term  $-n_{\alpha} \mathbf{F}_{\alpha}$  is the sum of external forces (multiplied by particle number density) acting on the particle  $\alpha$ , i.e., the gravity, radiative force, etc. The collision term  $\mathbf{A}_{\alpha}$  quantifies the momentum change due to collisions, creation, and destruction of particles. It is usually given as the linear approximation for a small difference in velocities (Bittencourt 2004),

$$\boldsymbol{A}_{\alpha} = -\rho_{\alpha} \sum_{\beta} \nu_{\alpha\beta} \left( \boldsymbol{V}_{\alpha} - \boldsymbol{V}_{\beta} \right), \qquad (1.26)$$

where we assume that the force exerted on particles  $\alpha$  by colliding particles  $\beta$ , is proportional to the difference of the velocities  $\mathbf{V}_{\alpha} - \mathbf{V}_{\beta}$  of the particles. The constant of proportionality,  $\nu_{\alpha\beta}$ , is called the collision frequency for transfer of momentum.

If we subtract the continuity equation (1.16), multiplied by  $V_{\alpha}$ , from Eq. (1.25) (noting that  $\mathbf{a}_{\alpha} = \mathrm{d}\mathbf{V}_{\alpha}/\mathrm{d}t = \partial\mathbf{V}_{\alpha}/\partial t + \mathbf{V}_{\alpha} \cdot \nabla\mathbf{V}_{\alpha}$ ), we obtain the momentum equation in the form

$$\rho_{\alpha}a^{i}_{\alpha} = \nabla_{j}\mathcal{T}^{ij}_{\alpha} + F^{i}_{\alpha} + A^{i}_{\alpha}, \qquad (1.27)$$

where i, j are the spatial components. If we sum Eq. (1.27) over all particle types  $\alpha$  in the isolated system, the collision term  $A^i_{\alpha}$  vanishes due to the conservation of the total momentum. Equation (1.27) becomes the momentum equation for the whole fluid,

$$\rho a^i = \nabla_j \mathcal{T}^{ij} + F^i. \tag{1.28}$$

We expand the acceleration term on the left-hand side of Eq. (1.28) as

$$\rho a^{i} = \rho \frac{\mathrm{d}V_{i}}{\mathrm{d}t} = \rho \left( \frac{\partial V_{i}}{\partial t} + V^{j} \nabla_{j} V^{i} \right), \quad \text{so that} \quad \rho \boldsymbol{a} = \rho \left[ \frac{\partial \boldsymbol{V}}{\partial t} + \rho \left( \boldsymbol{V} \cdot \boldsymbol{\nabla} \right) \boldsymbol{V} \right], \tag{1.29}$$

where the term (see the vector identities in Sect. 5.3 in Kurfürst (2017))

$$(\boldsymbol{V}\cdot\boldsymbol{\nabla}) \boldsymbol{V} = \frac{1}{2}\boldsymbol{\nabla}V^2 - \boldsymbol{V} \times (\boldsymbol{\nabla} \times \boldsymbol{V}), \qquad (1.30)$$

splits into two terms, to a separated laminar flow and to a rotational (turbulent) motion, respectively.

An alternative derivation of the momentum equation is based on Newton's second law,  $m\mathbf{a} = \mathbf{F}$ , written as the sum of forces acting on the particle  $\alpha$ ,

$$\sum_{\alpha} \int_{\Omega} \frac{\mathrm{d} \left(\rho_{\alpha} \boldsymbol{V}_{\alpha}\right)}{\mathrm{d}t} \mathrm{d}\Omega = \sum_{\alpha} \left( \int_{\Omega} \boldsymbol{F}_{\alpha}^{\Omega} \mathrm{d}\Omega + \int_{A} \boldsymbol{F}_{\alpha}^{A} \mathrm{d}A \right).$$
(1.31)

Following the notation used in Eq. (1.20), we denote  $\boldsymbol{F}_{\alpha}^{\Omega}$  the volume forces that act throughout the volume  $\Omega$  and  $\boldsymbol{F}_{\alpha}^{A}$  the surface forces that act on the surface A. By applying the divergence theorem on Eq. (1.31), we obtain Eq. (1.28).

#### **1.3.1** Equation of motion in curvilinear coordinates

We omit in this section the viscosity terms in the right-hand side of the equation, since they are described in detail in Sect. 1.4. Following the acceleration term expressed in Eq. (1.29) and the cylindrical gradient described in Sect. A.2.1 in Kurfürst (2017), the radial component of the momentum equation in *cylindrical* coordinates is

$$\frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_\phi}{R} \frac{\partial V_R}{\partial \phi} + V_z \frac{\partial V_R}{\partial z} = \frac{V_\phi^2}{R} - \frac{1}{\rho} \frac{\partial p}{\partial R} + F_R, \qquad (1.32)$$

where the right-hand side terms express the centrifugal force, pressure force, and sum of the external forces (e.g., gravitation). The azimuthal component of the momentum equation is

$$\frac{\partial V_{\phi}}{\partial t} + V_R \frac{\partial V_{\phi}}{\partial R} + \frac{V_{\phi}}{R} \frac{\partial V_{\phi}}{\partial \phi} + V_z \frac{\partial V_{\phi}}{\partial z} = -\frac{V_R V_{\phi}}{R} - \frac{1}{\rho R} \frac{\partial p}{\partial \phi} + F_{\phi}, \qquad (1.33)$$

where the meaning of the right-hand side terms is similar. The vertical component of the momentum equation is

$$\frac{\partial V_z}{\partial t} + V_R \frac{\partial V_z}{\partial R} + \frac{V_\phi}{R} \frac{\partial V_z}{\partial \phi} + V_z \frac{\partial V_z}{\partial z} = -\frac{1}{\rho} \frac{\partial p}{\partial z} + F_z.$$
(1.34)

The most frequent source of external gravity within the external forces  $\mathbf{F}$  is that of a spherical body (astronomical objects), whose gravitational potential  $\Phi = -GM_{\star}/r$  (where  $M_{\star}$  is the mass of a central object and r denotes the spherical radial distance,  $r^2 = R^2 + z^2$  in cylindrical coordinates). Including this expression, the nonzero components of the axisymmetric gravitational acceleration term  $-\nabla \Phi$  in cylindrical coordinates are

$$g_R = -\frac{GM_{\star}R}{(R^2 + z^2)^{3/2}}, \qquad g_z = -\frac{GM_{\star}z}{(R^2 + z^2)^{3/2}}.$$
 (1.35)

In *spherical* coordinates, following the acceleration in Eq. (1.29) and using the spherical gradient described in A.3.1 in Kurfürst (2017), the radial component of momentum equation is

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} = \frac{v_\theta^2 + v_\phi^2}{r} - \frac{1}{\rho} \frac{\partial p}{\partial r} - \frac{\partial \Phi}{\partial r}, \tag{1.36}$$

the polar component is

$$\frac{\partial v_{\theta}}{\partial t} + v_r \frac{\partial v_{\theta}}{\partial r} + \frac{v_{\theta}}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{v_{\phi}}{r \sin \theta} \frac{\partial v_{\theta}}{\partial \phi} = \frac{v_{\phi}^2 \cot \theta}{r} - \frac{v_r v_{\theta}}{r} - \frac{1}{\rho r} \frac{\partial p}{\partial \theta} - \frac{1}{r} \frac{\partial \Phi}{\partial \theta}, \quad (1.37)$$

and the azimuthal component is

$$\frac{\partial v_{\phi}}{\partial t} + v_r \frac{\partial v_{\phi}}{\partial r} + \frac{v_{\theta}}{r} \frac{\partial v_{\phi}}{\partial \theta} + \frac{v_{\phi}}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi} = -\frac{v_{\theta} v_{\phi} \cot \theta}{r} - \frac{v_r v_{\phi}}{r} - \frac{1}{\rho r \sin \theta} \frac{\partial p}{\partial \theta} - \frac{1}{r \sin \theta} \frac{\partial \Phi}{\partial \theta}.$$
 (1.38)

The meaning of the right-hand side terms is analogous to Eqs. (1.32) and (1.33). The external gravity induced by a spherically symmetric body is

$$g_r = -\frac{GM_\star}{r^2}.\tag{1.39}$$

#### 1.4 Stress Tensor

The stress tensor for a Newtonian fluid (cf. Eq. (1.13)) may be written in a general form as

$$\mathcal{T}_{ij} = -p\,\delta_{ij} + \sigma_{ij},\tag{1.40}$$

where p is the scalar pressure and  $\sigma_{ij}$  is a symmetric nondiagonal viscous stress tensor. Since the tensor components must have physical dimension of the pressure, where the *i*-th component of the force  $F_i = d\Pi_i/dt$  is tangential to a surface area  $A_j$ , we may write

$$\sigma_{ij} = \frac{1}{A_j} \frac{\mathrm{d}\Pi_i}{\mathrm{d}t}, \quad \text{where} \quad \frac{\mathrm{d}\Pi_i}{\mathrm{d}t} = \rho \Omega \frac{\mathrm{d}V_i}{\mathrm{d}t} = \rho A_j \ell \frac{\mathrm{d}V_i}{\mathrm{d}t}.$$
 (1.41)

The quantity  $\Pi_i$  in Eq. (1.41) denotes the *i*-th component of momentum of fluid particles,  $\Omega$  is a fluid volume,  $\Omega = A_j \ell$ , where the distance  $\ell$  expresses the mean free path of the particles. We denote the mean random velocity of those particles as  $\tilde{v}$ . Considering one-dimensional planar shear viscous stress (Fig. 1.1), Eq. (1.41) is

$$\sigma_{xz} = \rho \ell \frac{\mathrm{d}V_x}{\mathrm{d}z} \frac{\mathrm{d}z}{\mathrm{d}t} = \rho \ell \,\tilde{v}_z \, \frac{\mathrm{d}V_x}{\mathrm{d}z} = \eta \frac{\mathrm{d}V_x}{\mathrm{d}z}.$$
(1.42)

The factor of proportionality  $\eta$  in Eq. (1.42) is called the coefficient of *dynamic viscosity*. Its physical meaning is  $\rho \ell \tilde{v}_z$  or  $f \rho \ell \tilde{v}$ , where the numerical factor f is approximated as 1/3, which corresponds to an average fraction of fluid particles moving in z direction (its exact value depends on the type of particle interactions (Maeder 2009)). Assuming large deformations, one can write the expression for the viscous stress tensor of the form

$$\sigma_{ij} = \eta \left( \nabla_i V_j + \nabla_j V_i + \nabla_i V_k \nabla_j V_k \right) + \lambda \nabla_k V^k \delta_{ij} = 2\eta E_{ij} + \lambda \nabla_k V^k \delta_{ij}, \tag{1.43}$$

where  $\lambda$  is the *dilatation* or *second viscosity* coefficient. The symbol  $\nabla_i$  represents the *covariant* derivative, defined for orthogonal coordinate systems by Eq. A.43 of Kurfürst (2017). The components of stress and the strain tensor  $E_{ij}$  thus take the elementary form

$$\mathcal{T}_{ij} = -p\delta_{ij} + \eta \left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\right) + (\zeta - \frac{2}{3}\eta)\nabla_k V^k \delta_{ij}, \quad E_{ij} = \frac{1}{2} \left(\frac{\partial V_j}{\partial x_i} + \frac{\partial V_i}{\partial x_j}\right).$$
(1.44)



Figure 1.1: Schema of the planar shear stress with vertically increasing magnitude of velocity V. The reference horizontal plane is denoted  $z_0$ , while  $\ell$  is the mean free path of a particle (regarded as a subject of the stress force) and  $\tilde{v}_z$  is the z-component of a mean velocity of random motion. Adapted from Kurfürst (2015).

In Eq. (1.43) we use the full *Green-Lagrangian strain tensor*,

$$E_{ij} = \frac{1}{2} \left( \nabla_i V_j + \nabla_j V_i + \nabla_i V_k \nabla_j V_k \right).$$
(1.45)

In case of small deformations the nonlinear term in the Green-Lagrangian strain tensor drops, and we obtain the symmetric *Cauchy strain tensor*,

$$E_{ij} = \frac{1}{2} \left( \nabla_i V_j + \nabla_j V_i \right). \tag{1.46}$$

Using Eq. (1.45), the viscous stress tensor can be written in the form

$$\sigma_{ij} = \eta \left( \nabla_i V_j + \nabla_j V_i + \nabla_i V_k \nabla_j V_k - \frac{2}{3} \nabla_k V^k \,\delta_{ij} \right) + \zeta \,\nabla_k V^k \delta_{ij}, \tag{1.47}$$

where  $\zeta = \lambda + \frac{2}{3}\eta$  is the *coefficient of bulk viscosity*. Using  $\eta$  and  $\zeta$ , the stress tensor is

$$\mathcal{T}_{ij} = -p\,\delta_{ij} + \eta\,(\nabla_i V_j + \nabla_j V_i + \nabla_i V_k \nabla_j V_k) + \left(\zeta - \frac{2}{3}\eta\right)\,\nabla_k V^k \delta_{ij},\tag{1.48}$$

or, using the strain tensor,

$$\mathcal{T}_{ij} = -p\,\delta_{ij} + 2\eta\,E_{ij} + \left(\zeta - \frac{2}{3}\eta\right)\,\nabla_k V^k \delta_{ij}.$$
(1.49)

Another characteristic quantities, which describe the viscosity, are the kinematic viscosity  $\nu$ , defined as the ratio of the dynamic viscosity  $\eta$  to the fluid density  $\rho$ , and the characteristic timescale  $t_{\text{visc}}$  of the viscous effects (viscous timescale),

$$\nu = \frac{\eta}{\rho} = \frac{1}{3} \ell \tilde{v}, \qquad t_{\rm visc} \sim \frac{L^2}{\nu}, \qquad (1.50)$$

where L is a typical length scale of the system. In fluid mechanics the influence of viscosity is usually expressed using a dimensionless Reynolds number  $\mathcal{R}e$ , given as

$$\mathcal{R}e = \frac{\rho L V}{\eta} = \frac{L V}{\nu},\tag{1.51}$$

where V is a typical velocity. The Reynolds number expresses the ratio of inertial forces to viscous forces. The extent of viscous effects is scaled by a critical value  $\mathcal{R}e_{\text{crit}}$ , which varies according to the fluid type and the geometry of the system.

#### 1.4.1 Stress tensor in Cartesian coordinates

The components of the Cauchy strain tensor  $E_{ij}$  in Cartesian coordinates are

$$E_{xx} = \frac{\partial V_x}{\partial x}, \qquad E_{xy} = \frac{1}{2} \left( \frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \right), \\ E_{yy} = \frac{\partial V_y}{\partial y}, \qquad E_{xz} = \frac{1}{2} \left( \frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z} \right), \\ E_{zz} = \frac{\partial V_z}{\partial z}, \qquad E_{yz} = \frac{1}{2} \left( \frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z} \right). \qquad (1.52)$$

The components of the stress tensor  $\mathcal{T}_{ij}$  in Cartesian coordinates are

$$\mathcal{T}_{xx} = -p + 2\eta \left(\frac{\partial V_x}{\partial x}\right) + \left(\zeta - \frac{2}{3}\eta\right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{V}\right), \qquad \mathcal{T}_{xy} = \eta \left(\frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y}\right),$$
$$\mathcal{T}_{yy} = -p + 2\eta \left(\frac{\partial V_y}{\partial y}\right) + \left(\zeta - \frac{2}{3}\eta\right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{V}\right), \qquad \mathcal{T}_{xz} = \eta \left(\frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z}\right),$$
$$\mathcal{T}_{zz} = -p + 2\eta \left(\frac{\partial V_z}{\partial z}\right) + \left(\zeta - \frac{2}{3}\eta\right) \left(\boldsymbol{\nabla} \cdot \boldsymbol{V}\right), \qquad \mathcal{T}_{yz} = \eta \left(\frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z}\right). \tag{1.53}$$

The general expression of momentum equation is

$$\rho \frac{\mathrm{d}V_i}{\mathrm{d}t} = F_i + \partial_j \mathcal{T}_{ij}. \tag{1.54}$$

The components of momentum equation in Cartesian coordinates are

$$\rho \frac{\mathrm{d}V_x}{\mathrm{d}t} = F_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x} \left[ 2\eta \left( \frac{\partial V_x}{\partial x} \right) + \left( \zeta - \frac{2}{3} \eta \right) (\boldsymbol{\nabla} \cdot \boldsymbol{V}) \right] \\ + \frac{\partial}{\partial y} \left[ \eta \left( \frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[ \eta \left( \frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z} \right) \right], \tag{1.55}$$

$$\rho \frac{\mathrm{d}V_y}{\mathrm{d}t} = F_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y} \left[ 2\eta \left( \frac{\partial V_y}{\partial y} \right) + \left( \zeta - \frac{2}{3}\eta \right) (\boldsymbol{\nabla} \cdot \boldsymbol{V}) \right] \\ + \frac{\partial}{\partial x} \left[ \eta \left( \frac{\partial V_y}{\partial x} + \frac{\partial V_x}{\partial y} \right) \right] + \frac{\partial}{\partial z} \left[ \eta \left( \frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z} \right) \right], \tag{1.56}$$

$$\rho \frac{\mathrm{d}V_z}{\mathrm{d}t} = F_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z} \left[ 2\eta \left( \frac{\partial V_z}{\partial z} \right) + \left( \zeta - \frac{2}{3}\eta \right) (\boldsymbol{\nabla} \cdot \boldsymbol{V}) \right] \\ + \frac{\partial}{\partial x} \left[ \eta \left( \frac{\partial V_z}{\partial x} + \frac{\partial V_x}{\partial z} \right) \right] + \frac{\partial}{\partial y} \left[ \eta \left( \frac{\partial V_z}{\partial y} + \frac{\partial V_y}{\partial z} \right) \right].$$
(1.57)

Neglecting the bulk viscosity coefficient  $\zeta$  and the mixed derivatives, we write the simplified Eq. (1.54) in the frequently used form usually called the *Navier-Stokes equation*,

$$\rho \frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \boldsymbol{F} - \boldsymbol{\nabla}p + \eta \left[ \boldsymbol{\nabla}^2 \boldsymbol{V} + \frac{1}{3} \boldsymbol{\nabla} \left( \boldsymbol{\nabla} \cdot \boldsymbol{V} \right) \right].$$
(1.58)

#### 1.4.2 Stress tensor in cylindrical polar coordinates

Following Eq. (1.44), all the independent components of symmetric stress tensor in cylindrical coordinates are

$$\mathcal{T}_{RR} = -p + 2\eta \left(\frac{\partial V_R}{\partial R}\right) + \left(\zeta - \frac{2}{3}\eta\right) \nabla \cdot \mathbf{V}, \qquad \mathcal{T}_{R\phi} = \eta \left[\frac{1}{R} \frac{\partial V_R}{\partial \phi} + R \frac{\partial}{\partial R} \left(\frac{V_{\phi}}{R}\right)\right],$$
$$\mathcal{T}_{\phi\phi} = -p + \frac{2\eta}{R} \left(\frac{\partial V_{\phi}}{\partial \phi} + V_R\right) + \left(\zeta - \frac{2}{3}\eta\right) \nabla \cdot \mathbf{V}, \qquad \mathcal{T}_{Rz} = \eta \left(\frac{\partial V_z}{\partial R} + \frac{\partial V_R}{\partial z}\right), \qquad (1.59)$$
$$\mathcal{T}_{zz} = -p + 2\eta \left(\frac{\partial V_z}{\partial z}\right) + \left(\zeta - \frac{2}{3}\eta\right) \nabla \cdot \mathbf{V}, \qquad \mathcal{T}_{\phi z} = \eta \left(\frac{1}{R} \frac{\partial V_z}{\partial \phi} + \frac{\partial V_{\phi}}{\partial z}\right),$$

where p is the diagonal component of stress tensor (scalar pressure) and all the non-diagonal terms form the viscous stress tensor that includes the bulk viscosity terms (containing the velocity divergence) and the shear stress terms. Including the stress tensor, the general form of the momentum equation (Eqs. (1.32)-(1.34)) is

$$\rho a_i = \nabla_j \mathcal{T}_{ij} - \rho \nabla_i \Phi, \qquad (1.60)$$

where the left-hand side represents the advective force, the first term on the right-hand side is the pressure (including viscous) force and the last term on the right-hand side is the external force.

Following Eq. (1.60) and including stress tensor components from Eq. (1.59), the radial component of momentum equation in cylindrical coordinates is

$$\rho a_{R} = F_{R} + \nabla_{R} \mathcal{T}_{RR} + \nabla_{\phi} \mathcal{T}_{R\phi} + \nabla_{z} \mathcal{T}_{Rz} = F_{R} + \partial_{R} \mathcal{T}_{RR} + \frac{1}{R} \partial_{\phi} \mathcal{T}_{R\phi} + \partial_{z} \mathcal{T}_{Rz} + \frac{1}{R} \mathcal{T}_{RR} - \frac{1}{R} \mathcal{T}_{\phi\phi}$$

$$= F_{R} - \frac{\partial p}{\partial R} + \frac{\partial}{\partial R} \left[ 2\eta \left( \frac{\partial V_{R}}{\partial R} \right) + (\zeta - \frac{2}{3}\eta) (\nabla \cdot \mathbf{V}) \right]$$

$$+ \frac{1}{R} \frac{\partial}{\partial \phi} \left\{ \eta \left[ \frac{1}{R} \frac{\partial V_{R}}{\partial \phi} + R \frac{\partial}{\partial R} \left( \frac{V_{\phi}}{R} \right) \right] \right\} + \frac{\partial}{\partial z} \left[ \eta \left( \frac{\partial V_{z}}{\partial R} + \frac{\partial V_{R}}{\partial z} \right) \right]$$

$$+ \frac{2\eta}{R} \left[ R \frac{\partial}{\partial R} \left( \frac{V_{R}}{R} \right) - \frac{1}{R} \frac{\partial V_{\phi}}{\partial \phi} \right], \qquad (1.61)$$

where  $F_R$  is the radial gravitational force  $\rho g_R$  (cf. Eq. (1.35)). Analogously, the azimuthal component of momentum equation is

$$\rho a_{\phi} = F_{\phi} + \nabla_R \mathcal{T}_{\phi R} + \nabla_{\phi} \mathcal{T}_{\phi \phi} + \nabla_z \mathcal{T}_{\phi z} = F_{\phi} + \partial_R \mathcal{T}_{\phi R} + \frac{1}{R} \partial_{\phi} \mathcal{T}_{\phi \phi} + \partial_z \mathcal{T}_{\phi z} + \frac{2}{R} \mathcal{T}_{\phi R}$$

$$= F_{\phi} - \frac{1}{R} \frac{\partial p}{\partial \phi} + \frac{\partial}{\partial R} \left\{ \eta \left[ \frac{1}{R} \frac{\partial V_R}{\partial \phi} + R \frac{\partial}{\partial R} \left( \frac{V_{\phi}}{R} \right) \right] \right\}$$

$$+ \frac{1}{R} \frac{\partial}{\partial \phi} \left[ 2\eta \left( \frac{1}{R} \frac{\partial V_{\phi}}{\partial \phi} + \frac{V_R}{R} \right) + (\zeta - \frac{2}{3}\eta)(\nabla \cdot \mathbf{V}) \right]$$

$$+ \frac{\partial}{\partial z} \left[ \eta \left( \frac{1}{R} \frac{\partial V_z}{\partial \phi} + \frac{\partial V_{\phi}}{\partial z} \right) \right] + \frac{2\eta}{R} \left[ \frac{1}{R} \frac{\partial V_R}{\partial \phi} + R \frac{\partial}{\partial R} \left( \frac{V_{\phi}}{R} \right) \right], \qquad (1.62)$$

and the vertical component of momentum equation is

$$\rho a_{z} = F_{z} + \nabla_{R} \mathcal{T}_{zR} + \nabla_{\phi} \mathcal{T}_{z\phi} + \nabla_{z} \mathcal{T}_{zz} = F_{z} + \partial_{R} \mathcal{T}_{zR} + \frac{1}{R} \partial_{\phi} \mathcal{T}_{z\phi} + \partial_{z} \mathcal{T}_{zz} + \frac{1}{R} \mathcal{T}_{Rz}$$

$$= F_{z} - \frac{\partial p}{\partial z} + \frac{\partial}{\partial R} \left[ \eta \left( \frac{\partial V_{z}}{\partial R} + \frac{\partial V_{R}}{\partial z} \right) \right] + \frac{1}{R} \frac{\partial}{\partial \phi} \left[ \eta \left( \frac{1}{R} \frac{\partial V_{z}}{\partial \phi} + \frac{\partial V_{\phi}}{\partial z} \right) \right]$$

$$+ \frac{\partial}{\partial z} \left[ 2\eta \left( \frac{\partial V_{z}}{\partial z} \right) + (\zeta - \frac{2}{3}\eta) (\boldsymbol{\nabla} \cdot \boldsymbol{V}) \right] + \frac{\eta}{R} \left( \frac{\partial V_{z}}{\partial R} + \frac{\partial V_{R}}{\partial z} \right).$$
(1.63)

#### 1.4.3 Stress tensor in spherical polar coordinates

Similarly to Sect. 1.4.2, we write all independent components of symmetric stress tensor in spherical coordinates,

$$\begin{aligned}
\mathcal{T}_{rr} &= -p + 2\eta \left( \frac{\partial v_r}{\partial r} \right) + (\zeta - \frac{2}{3}\eta) \nabla \cdot \mathbf{v}, \\
\mathcal{T}_{r\theta} &= \eta \left[ \frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{v_{\theta}}{r} \right) \right], \\
\mathcal{T}_{r\phi} &= \eta \left[ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_{\phi}}{r} \right) \right], \\
\mathcal{T}_{\theta\theta} &= -p + 2\eta \left( \frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{v_r}{r} \right) + (\zeta - \frac{2}{3}\eta) \nabla \cdot \mathbf{v}, \\
\mathcal{T}_{\theta\phi} &= \eta \left[ \frac{1}{r \sin \theta} \frac{\partial v_{\theta}}{\partial \phi} + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_{\phi}}{\sin \theta} \right) \right] \\
\mathcal{T}_{\phi\phi} &= -p + 2\eta \left( \frac{1}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi} + \frac{v_r + v_{\theta} \cot \theta}{r} \right) + (\zeta - \frac{2}{3}\eta) \nabla \cdot \mathbf{v}.
\end{aligned} \tag{1.64}$$

Following Eq. (1.60) and including spherical stress tensor components introduced in Eq. (1.64), we write the radial component of momentum equation in spherical coordinates,

$$\rho a_{r} = F_{r} + \nabla_{r} \mathcal{T}_{rr} + \nabla_{\theta} \mathcal{T}_{r\theta} + \nabla_{\phi} \mathcal{T}_{r\phi} 
= F_{r} + \frac{1}{r^{2}} \partial_{r} \left( r^{2} \mathcal{T}_{rr} \right) + \frac{1}{r \sin \theta} \partial_{\theta} \left( \sin \theta \mathcal{T}_{r\theta} \right) + \frac{1}{r \sin \theta} \partial_{\phi} \mathcal{T}_{r\phi} - \frac{1}{r} \left( \mathcal{T}_{\theta\theta} + \mathcal{T}_{\phi\phi} \right) 
= F_{r} - \frac{\partial p}{\partial r} + \frac{\partial}{\partial r} \left[ 2\eta \left( \frac{\partial v_{r}}{\partial r} \right) + \left( \zeta - \frac{2}{3} \eta \right) (\nabla \cdot \mathbf{v}) \right] 
+ \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ \eta \left[ \frac{1}{r} \frac{\partial v_{r}}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{v_{\theta}}{r} \right) \right] \right\} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left\{ \eta \left[ \frac{1}{r \sin \theta} \frac{\partial v_{r}}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_{\phi}}{r} \right) \right] \right\} 
+ \frac{\eta}{r} \left[ 4r \frac{\partial}{\partial r} \left( \frac{v_{r}}{r} \right) - \frac{2}{r \sin \theta} \frac{\partial}{\partial \theta} \left( v_{\theta} \sin \theta \right) 
- \frac{2}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi} + r \cot \theta \frac{\partial}{\partial r} \left( \frac{v_{\theta}}{r} \right) + \frac{\cot \theta}{r} \frac{\partial v_{r}}{\partial \theta} \right],$$
(1.65)

where  $F_r$  is the radial gravitational force  $\rho g_r$  (see Eq. 1.39). The polar component of momentum equation is

$$\rho a_{\theta} = F_{\theta} + \nabla_{r} \mathcal{T}_{\theta r} + \nabla_{\theta} \mathcal{T}_{\theta \theta} + \nabla_{\phi} \mathcal{T}_{\theta \phi} 
= F_{\theta} + \frac{1}{r^{2}} \partial_{r} \left( r^{2} \mathcal{T}_{\theta r} \right) + \frac{1}{r \sin \theta} \partial_{\theta} \left( \sin \theta \mathcal{T}_{\theta \theta} \right) + \frac{1}{r \sin \theta} \partial_{\phi} \mathcal{T}_{\theta \phi} + \frac{1}{r} \left( \mathcal{T}_{\theta r} - \cot \theta \mathcal{T}_{\phi \phi} \right) 
= F_{\theta} - \frac{1}{r} \frac{\partial p}{\partial \theta} + \frac{\partial}{\partial r} \left\{ \eta \left[ \frac{1}{r} \frac{\partial v_{r}}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{v_{\theta}}{r} \right) \right] \right\} + \frac{1}{r} \frac{\partial}{\partial \theta} \left[ 2\eta \left( \frac{1}{r} \frac{\partial v_{\theta}}{\partial \theta} + \frac{v_{r}}{r} \right) + (\zeta - \frac{2}{3} \eta) (\nabla \cdot \mathbf{v}) \right] 
+ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left\{ \eta \left[ \frac{1}{r \sin \theta} \frac{\partial v_{\theta}}{\partial \phi} + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_{\phi}}{\sin \theta} \right) \right] \right\} 
+ \frac{\eta}{r} \left\{ \frac{2 \cot \theta}{r} \left[ \sin \theta \frac{\partial}{\partial \theta} \left( \frac{v_{\theta}}{\sin \theta} \right) - \frac{1}{\sin \theta} \frac{\partial v_{\phi}}{\partial \phi} \right] + 3r \frac{\partial}{\partial r} \left( \frac{v_{\theta}}{r} \right) + \frac{3}{r} \frac{\partial v_{r}}{\partial \theta} \right\}.$$
(1.66)

The azimuthal component of momentum equation is

$$\rho a_{\phi} = F_{\phi} + \nabla_{r} \mathcal{T}_{\phi r} + \nabla_{\theta} \mathcal{T}_{\phi \theta} + \nabla_{\phi} \mathcal{T}_{\phi \phi} \\
= F_{\phi} + \frac{1}{r^{2}} \partial_{r} \left(r^{2} \mathcal{T}_{\phi r}\right) + \frac{1}{r \sin \theta} \partial_{\theta} \left(\sin \theta \mathcal{T}_{\phi \theta}\right) + \frac{1}{r \sin \theta} \partial_{\phi} \mathcal{T}_{\phi \phi} + \frac{1}{r} \left(\mathcal{T}_{\phi r} + \cot \theta \mathcal{T}_{\theta \phi}\right) \\
= F_{\phi} - \frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \frac{\partial}{\partial r} \left\{ \eta \left[ \frac{1}{r \sin \theta} \frac{\partial v_{r}}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_{\phi}}{r} \right) \right] \right\} \\
+ \frac{1}{r} \frac{\partial}{\partial \theta} \left\{ \eta \left[ \frac{1}{r \sin \theta} \frac{\partial v_{\theta}}{\partial \phi} + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_{\phi}}{\sin \theta} \right) \right] \right\} \\
+ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \left[ 2\eta \left( \frac{1}{r \sin \theta} \frac{\partial v_{\phi}}{\partial \phi} + \frac{v_{r}}{r} + \frac{v_{\theta} \cot \theta}{r} \right) + (\zeta - \frac{2}{3}\eta) (\nabla \cdot \mathbf{v}) \right] \\
+ \frac{\eta}{r} \left\{ \frac{2 \cot \theta}{r} \left[ \sin \theta \frac{\partial}{\partial \theta} \left( \frac{v_{\phi}}{\sin \theta} \right) + \frac{1}{\sin \theta} \frac{\partial v_{\theta}}{\partial \phi} \right] + 3r \frac{\partial}{\partial r} \left( \frac{v_{\phi}}{r} \right) + \frac{3}{r \sin \theta} \frac{\partial v_{r}}{\partial \phi} \right\}.$$
(1.67)

## 1.5 Equation of Energy

#### 1.5.1 General form

Substituting the kinetic energy  $\frac{1}{2}m_{\alpha}v^2$  for the general quantity  $\chi(\mathbf{v})$  in Eq. (1.6), we obtain the energy equation for the particle  $\alpha$  (2. moment of BKE, the energy conservation law),

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho_{\alpha} V_{\alpha}^{2} \right) + \frac{\partial}{\partial t} \left( \frac{1}{2} \rho_{\alpha} \left\langle C_{\alpha}^{2} \right\rangle \right) + \nabla \cdot \left( \frac{1}{2} \rho_{\alpha} V_{\alpha}^{2} \boldsymbol{V}_{\alpha} \right) + \nabla \cdot \left( \frac{1}{2} \rho_{\alpha} \left\langle C_{\alpha}^{2} \right\rangle \boldsymbol{V}_{\alpha} \right) + \nabla \cdot \left( \rho_{\alpha} \left\langle \boldsymbol{C}_{\alpha} \otimes \boldsymbol{C}_{\alpha} \right\rangle \boldsymbol{V}_{\alpha} \right) + \nabla \cdot \left( \frac{1}{2} \rho_{\alpha} \left\langle C_{\alpha}^{2} \boldsymbol{C}_{\alpha} \right\rangle \right) - n_{\alpha} \left( \boldsymbol{F} \cdot \boldsymbol{V} \right)_{\alpha} = M_{\alpha}.$$
(1.68)

The term  $n_{\alpha} (\boldsymbol{F} \cdot \boldsymbol{V})_{\alpha}$  is the flux of external forces that act on the particle  $\alpha$  The collision term  $M_{\alpha}$  on the right-hand side represents the rate of energy change due to collisions, creation and destruction of particles (cf. the collision terms  $S_{\alpha}$  in Eq. (1.16) and  $\boldsymbol{A}_{\alpha}$  in Eq. (1.25)),

$$M_{\alpha} = \frac{1}{2} m_{\alpha} \int v^2 \left( \frac{\delta f_{\alpha}}{\delta t} \right)_{\text{coll}} d^3 \mathbf{v} = \left[ \frac{\delta \left( \frac{1}{2} \rho_{\alpha} \left\langle v^2 \right\rangle_{\alpha} \right)}{\delta t} \right]_{\text{coll}}.$$
 (1.69)

If we sum Eq. (1.68) over all particle types  $\alpha$  (cf. Eqs. (1.10)-(1.15)), we obtain the energy equation for the whole medium,

$$\frac{\partial}{\partial t} \left( \frac{1}{2} \rho V^2 + \frac{3p}{2} \right) + \nabla_j \left( \frac{1}{2} \rho V^2 V^j + \frac{3p}{2} V^j - V_i \mathcal{T}^{ij} + q^j \right) - V_i F^i = 0, \qquad (1.70)$$

where, due to the conservation of the total energy in the isolated system, the collision term  $M_{\alpha}$  (taking into account only particle collisions and omitting the radiation) vanishes when summed over all particle species. The terms  $\frac{1}{2}\rho_{\alpha} \langle V_{\alpha}^2 \rangle$  and  $\frac{1}{2}\rho_{\alpha} \langle C_{\alpha}^2 \rangle$  in Eq. (1.68) represent the kinetic and internal (thermal) energy of the particle  $\alpha$ , while the corresponding expression  $\frac{1}{2}\sum_{\alpha}\rho_{\alpha} \langle C_{\alpha 0}^2 \rangle$  (cf. Eq. (1.14)) represents the internal energy  $\rho\epsilon$  of the whole fluid. Following Eq. (1.14), we set

$$\frac{3p}{2} = \rho\epsilon, \tag{1.71}$$

where  $\epsilon$  denotes the specific internal energy (internal energy per unit mass). Equation (1.70) therefore is

$$\frac{\partial}{\partial t} \left( \rho \epsilon + \frac{1}{2} \rho V^2 \right) + \nabla_j \left[ \rho V^j \left( \epsilon + \frac{1}{2} V^2 \right) - V_i \mathcal{T}^{ij} + q^j \right] = V_i F^i.$$
(1.72)

Multiplying the equation of motion (1.28) by the velocity  $V_i$  and subtracting the continuity equation, we obtain the equation of the mechanic (kinetic) energy for the whole medium,

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho V^2\right) + \nabla_j \left(\rho V^j \frac{1}{2} V^2\right) = V_i \nabla_j \mathcal{T}^{ij} + V_i F^i.$$
(1.73)

Subtraction of Eq. (1.73) from the total energy equation (1.72) gives the equation of internal (thermal) energy in the form

$$\frac{\partial \left(\rho\epsilon\right)}{\partial t} + \nabla_{j}\left(\rho\epsilon V^{j}\right) \equiv \frac{\mathrm{d}\left(\rho\epsilon\right)}{\mathrm{d}t} + \rho\epsilon\nabla_{j}V^{j} = \mathcal{T}^{ij}\nabla_{j}V_{i} - \nabla_{j}q^{j}, \qquad (1.74)$$

which can be written with use of Eq. (1.71) also as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3p}{2}\right) + \frac{3p}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) = (\boldsymbol{\mathcal{T}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{V} - \boldsymbol{\nabla}\cdot\boldsymbol{q}.$$
(1.75)

Subtracting equation of continuity (1.19) from Eq. (1.74) and following the identity  $d/dt = \partial/\partial t + V^j \nabla_j$ , we write Eq. (1.74) as

$$\rho \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \mathcal{T}^{ij} \nabla_j V_i - \nabla_j q^j. \tag{1.76}$$

Using expression (1.49) for the stress tensor, we write the first right-hand side term of Eq. (1.76),

$$\mathcal{T}^{ij}\nabla_j V_i = \left[-p\delta^{ij} + 2\eta E^{ij} + \left(\zeta - \frac{2}{3}\eta\right)\nabla_k V^k \delta_{ij}\right]\nabla_j V_i.$$
(1.77)

Equation (1.77) can then be explicitly written in the form

$$\mathcal{T}^{ij}\nabla_j V_i = -p\,\nabla_i V^i + 2\eta E^{ij}\nabla_j V_i + \left(\zeta - \frac{2}{3}\eta\right)\left(\nabla_i V^i\right)^2. \tag{1.78}$$

Using the Cauchy strain tensor formalism, Eq. (1.78) becomes

$$\mathcal{T}^{ij}\nabla_j V_i = -p\,\nabla_i V^i + 2\eta E_{ij} E^{ij} + \left(\zeta - \frac{2}{3}\eta\right) \left(\nabla_i V^i\right)^2 \tag{1.79}$$

(see also Eq. (1.95) in Appendix 1.5.2). The first right-hand side term of Eq. (1.79) is the reversible work done by the expanding matter, while the second and third terms represent the *dissipation function*, i.e., the energy of the viscous dissipation of the gas (Mihalas & Mihalas 1984). The dissipation function is usually written as  $\Phi$ , we denote it here  $\Psi$  (while  $\Phi$  is the gravitational potential),

$$\Psi = 2\eta E_{ij} E^{ij} + \left(\zeta - \frac{2}{3}\eta\right) (\boldsymbol{\nabla} \cdot \boldsymbol{V})^2.$$
(1.80)

The dissipation function is always nonnegative,  $\Psi \ge 0$  (see Mihalas & Mihalas (1984) for the proof). The equation of the internal (thermal) energy (1.76) in the vector notation then becomes

$$\rho \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = -p\boldsymbol{\nabla}\cdot\boldsymbol{V} + \Psi - \boldsymbol{\nabla}\cdot\boldsymbol{q}. \tag{1.81}$$

The reversible work done by pressure forces (the first right-hand side term of Eq. (1.81)) vanishes in case of incompressible fluid ( $\nabla \cdot \mathbf{V} = 0$ ). The other terms contribute to the heat energy - the second right-hand side term of Eq. (1.81) is the (already introduced) energy of dissipation, while the third term is a reversible contribution of the heat conduction and of other energy sources (radiation, chemical reactions, etc.). It corresponds to the divergence of  $\mathbf{q}$  in Eq. (1.15).

The energy equation can be alternatively derived from the first law of thermodynamics, using the continuity equation (1.19) and the equation of motion (1.28). Time derivative of the energy (all the quantities are per unit volume) gives

$$\frac{\partial}{\partial t} \left( \rho \epsilon + \frac{\rho V^2}{2} \right) = \rho \frac{\partial \epsilon}{\partial t} - \left( \epsilon + \frac{V^2}{2} \right) \nabla \cdot \rho \mathbf{V} + \rho \mathbf{V} \frac{\partial \mathbf{V}}{\partial t}, \tag{1.82}$$

where, noting that  $\partial \epsilon = d\epsilon - \mathbf{V} \cdot \nabla \epsilon$ , the first term on the right-hand side of Eq. (1.82) can be written as

$$\rho \frac{\partial \epsilon}{\partial t} = \rho T \frac{\mathrm{d}s}{\mathrm{d}t} - p \boldsymbol{\nabla} \cdot \boldsymbol{V} - \rho \boldsymbol{V} \cdot \boldsymbol{\nabla} \epsilon, \qquad (1.83)$$

where s is the specific entropy (entropy per unit mass). We can expand the third term on the right-hand side of Eq. (1.82), using Eqs. (1.28), (1.29) and the vector identity (1.30),

$$\rho \boldsymbol{V} \,\frac{\partial \boldsymbol{V}}{\partial t} = -\rho \boldsymbol{V} \cdot \boldsymbol{\nabla} \frac{1}{2} V^2 - \rho \boldsymbol{V} \cdot (\boldsymbol{\nabla} \times \boldsymbol{V}) \times \boldsymbol{V} - \boldsymbol{V} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{P}} + \boldsymbol{V} \cdot \boldsymbol{F}, \qquad (1.84)$$

where the second term on the right-hand side vanishes (since  $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{A}) = 0$ ) and where  $\mathcal{P}$  is the pressure tensor  $\mathcal{P}_{ij} = -\mathcal{T}_{ij}$ . In that sense, the pressure tensor represents the same physics as the stress tensor, with the opposite sign. Using Eqs. (1.82), (1.83), and (1.84), we write the equation of the total energy (1.70),

$$\frac{\partial}{\partial t} \left( \rho \epsilon + \frac{\rho V^2}{2} \right) + \boldsymbol{\nabla} \cdot \left[ \rho \boldsymbol{V} \left( \epsilon + \frac{V^2}{2} \right) + p \boldsymbol{V} \right] = \boldsymbol{V} \cdot \boldsymbol{\nabla} \cdot \sigma_p + \rho T \frac{\mathrm{d}s}{\mathrm{d}t} + \boldsymbol{V} \cdot \boldsymbol{F}, \qquad (1.85)$$

where the quantity  $\sigma_p$  denotes the non-diagonal part of the pressure tensor, i.e., the tensor of the viscous pressure. Equation (1.83) can be compactly written as

$$\rho \,\frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \rho T \frac{\mathrm{d}s}{\mathrm{d}t} - p \boldsymbol{\nabla} \cdot \boldsymbol{V},\tag{1.86}$$

and, comparing Eqs. (1.86) and (1.76), we obtain

$$\rho T \frac{\mathrm{d}s}{\mathrm{d}t} = \Psi - \boldsymbol{\nabla} \cdot \boldsymbol{q}. \tag{1.87}$$

From Eq. (1.80) follows the relation  $\Psi = \sigma_p \cdot \nabla V$  (cf. Eqs. (1.77)-(1.79)). Including this into Eq. (1.82), we obtain

$$\frac{\partial}{\partial t} \left( \rho \epsilon + \frac{\rho V^2}{2} \right) + \boldsymbol{\nabla} \cdot \left[ \rho \boldsymbol{V} \left( \epsilon + \frac{V^2}{2} \right) + \mathcal{P} \boldsymbol{V} + \boldsymbol{q} \right] = \boldsymbol{V} \cdot \boldsymbol{F}, \quad (1.88)$$

which is identical to the total energy expression given in Eq. (1.72).

Another frequently used form of the energy equation involves the Fourier's law of heat conduction: The heat flux in any material is proportional to its internal temperature gradient; the heat flows from hotter to cooler regions (Mihalas & Mihalas 1984). The term  $-\nabla \cdot \boldsymbol{q}$  for the heat energy flux can be expanded as

$$-\boldsymbol{\nabla} \cdot \boldsymbol{q} = \boldsymbol{\nabla} \cdot (K \, \boldsymbol{\nabla} T) + q_R, \tag{1.89}$$

where the constant of proportionality K is the material heat conductivity and the term  $q_R$  refers to the heat sources other than conduction, i.e., to radiation, chemical reactions, etc. The structure of the term  $q_R$  can be quite complex, it depends on the detailed physics of internal and external heat sources. We may rewrite Eq. (1.81), using Eq. (1.89) and assuming no macroscopic mass transfer ( $\mathbf{V} = 0$ ), no work done (p = const.,  $d\epsilon = c_p dT$ , where  $c_p$  is the specific heat at constant pressure) and no dissipation ( $\Psi = 0$ ), in the following form,

$$\rho c_p \frac{\mathrm{d}T}{\mathrm{d}t} - \boldsymbol{\nabla} \cdot (K \, \boldsymbol{\nabla}T) = q_R, \quad \text{or} \quad \frac{\mathrm{d}T}{\mathrm{d}t} \approx D \boldsymbol{\nabla}^2 T + q_R, \tag{1.90}$$

where  $D = K/(\rho c_p)$  is the *thermal diffusivity*. Equation (1.90) is thus a inhomogeneous parabolic partial differential equation that describes the distribution of heat (variations in temperature) in a given region over time.

#### 1.5.2 Energy dissipation

Following the formalism of the stress tensor introduced in Sect. 1.4, we now discuss the term  $T^{ij}\nabla_j V_i$  in the internal energy equation (1.76),

$$\rho \frac{\mathrm{d}\epsilon}{\mathrm{d}t} = \mathcal{T}^{ij} \nabla_j V_i - \nabla_j q^j. \tag{1.91}$$

We expand the first term on right-hand side of Eq. (1.91) into

$$\mathcal{T}^{ij} \nabla_j V_i = \left[ -p\delta^{ij} + 2\eta E^{ij} + \left(\zeta - \frac{2}{3}\eta\right) \nabla_k V^k \delta^{ij} \right] \nabla_j V_i$$
$$= -p \nabla_i V^i + 2\eta E^{ij} \nabla_j V_i + \left(\zeta - \frac{2}{3}\eta\right) \left(\nabla_i V^i\right)^2, \qquad (1.92)$$

where in the right-hand side term we contract the indexes using Kronecker deltas. Using the definition of the symmetric strain tensor  $E^{ij}$ , we expand the second term on the right-hand side of Eq. (1.92),

$$E^{ij}\nabla_j V_i = \frac{1}{2} \left( \nabla^j V^i + \nabla^i V^j \right) \nabla_j V_i = \frac{1}{2} \left( \nabla^j V^i \nabla_j V_i + \nabla^i V^j \nabla_j V_i \right).$$
(1.93)

The symmetry of the strain tensor  $E^{ij} = E^{ji}$  in *i*, *j* as well as the orthogonality of the coordinates, and implying  $E^{ij}E_{ij} = E_{ij}E^{ij}$ , gives the identity

$$E^{ij}E_{ij} = \frac{1}{4} \left( \nabla^j V^i + \nabla^i V^j \right) \left( \nabla_j V_i + \nabla_i V_j \right)$$
  
$$= \frac{1}{4} \left( \nabla^j V^i \nabla_j V_i + \nabla^j V^i \nabla_i V_j + \nabla^i V^j \nabla_j V_i + \nabla^i V^j \nabla_i V_j \right)$$
  
$$= \frac{1}{2} \left( \nabla^j V^i \nabla_j V_i + \nabla^i V^j \nabla_j V_i \right) = E_{ij} E^{ij}.$$
(1.94)

Equation (1.92) thus becomes the form of Eq. (1.79),

$$\mathcal{T}^{ij}\nabla_j V_i = -p\nabla_i V^i + 2\eta E_{ij} E^{ij} + \left(\zeta - \frac{2}{3}\eta\right) \left(\nabla_i V^i\right)^2.$$
(1.95)

Using Eq. (1.44), the explicit form of the dissipation function in *Cartesian* coordinates is

$$\Psi = 2\eta \left[ \left( \frac{\partial V_x}{\partial x} \right)^2 + \left( \frac{\partial V_y}{\partial y} \right)^2 + \left( \frac{\partial V_z}{\partial z} \right)^2 + \frac{1}{2} \left( \frac{\partial V_x}{\partial y} + \frac{\partial V_y}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial V_x}{\partial z} + \frac{\partial V_z}{\partial x} \right)^2 + \frac{1}{2} \left( \frac{\partial V_y}{\partial z} + \frac{\partial V_z}{\partial y} \right)^2 \right] + \left( \zeta - \frac{2}{3} \eta \right) (\boldsymbol{\nabla} \cdot \boldsymbol{V})^2.$$
(1.96)

In cylindrical coordinates the dissipation function is

$$\Psi = 2\eta \left\{ \left( \frac{\partial V_R}{\partial R} \right)^2 + \left( \frac{1}{R} \frac{\partial V_{\phi}}{\partial \phi} + \frac{V_R}{R} \right)^2 + \left( \frac{\partial V_z}{\partial z} \right)^2 + \frac{1}{2} \left[ \frac{1}{R} \frac{\partial V_R}{\partial \phi} + R \frac{\partial}{\partial R} \left( \frac{V_{\phi}}{R} \right) \right]^2 + \frac{1}{2} \left( \frac{\partial V_z}{\partial R} + \frac{\partial V_R}{\partial z} \right)^2 + \frac{1}{2} \left( \frac{1}{R} \frac{\partial V_z}{\partial \phi} + \frac{\partial V_{\phi}}{\partial z} \right)^2 \right\} + \left( \zeta - \frac{2}{3}\eta \right) \left[ \frac{1}{R} \frac{\partial}{\partial R} (RV_R) + \frac{1}{R} \frac{\partial V_{\phi}}{\partial \phi} + \frac{\partial V_z}{\partial z} \right]^2.$$
(1.97)

In spherical coordinates the dissipation function is

$$\Psi = 2\eta \left\{ \left( \frac{\partial v_r}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r} + \frac{v_\theta \cot \theta}{r} \right)^2 \right. \\ \left. + \frac{1}{2} \left[ \frac{1}{r} \frac{\partial v_r}{\partial \theta} + r \frac{\partial}{\partial r} \left( \frac{v_\theta}{r} \right) \right]^2 + \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left( \frac{v_\phi}{r} \right) \right]^2 \\ \left. + \frac{1}{2} \left[ \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left( \frac{v_\phi}{\sin \theta} \right) \right]^2 \right\} \\ \left. + \left( \zeta - \frac{2}{3} \eta \right) \left[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 v_r \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta v_\theta \right) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right]^2.$$
(1.98)

## 1.6 Equation of State of an Ideal Gas

We close the above system of hydrodynamic equations using the equation of state that describes the (thermodynamic) properties of the variables in the fluid. Following the notation introduced in Sect. 1.5.1, we write the first law of thermodynamics in the form

$$\mathrm{d}q = \mathrm{d}\epsilon - \frac{p}{\rho^2} \,\mathrm{d}\rho,\tag{1.99}$$

which in adiabatic case is equal to zero. The adiabatic transformation of a perfect (ideal) gas is given by the relation  $p/\rho^{\gamma} = \text{const.}$ , where the adiabatic exponent  $\gamma$  is defined as the ratio of specific heats at a constant pressure and a constant volume,  $\gamma = c_p/c_V = (d \ln p/d \ln \rho)_{ad}$ . Since  $c_p$ ,  $c_V$  are defined as  $c_p = (dq/dT)_p$  and  $c_V = (dq/dT)_V$ , Eq. (1.99) gives

$$dh = c_p dT$$
, where  $h = \epsilon + p/\rho$  (enthalpy), and  $d\epsilon = c_V dT$ . (1.100)

For adiabatic transformations in a fluid described by general equation of state (in case of nonideal gas), the basic relation is similar,  $p/\rho^{\Gamma_1} = \text{const.}$ , where  $\Gamma_1$  is the general adiabatic exponent (see Maeder 2009, for details).

The law of ideal gas relates the pressure p, volume V, and temperature T by the equation pV/T = const. Assuming V is the volume occupied by one particle,  $V = \mu m_u/\rho$ , where  $\mu$  is the mean molecular weight (see Eq. (1.102)) and  $m_u$  is the atomic mass unit (i.e., 1/12 of the mass of the neutral <sup>12</sup>C atom), the law of ideal gas becomes

$$p = \frac{k}{\mu m_u} \rho T, \tag{1.101}$$

where k is the Boltzmann constant and the term  $\mu m_u$  is the average mass of the particles (electrons, ions, atoms or molecules) in the gas. The mean molecular weight  $\mu$  is defined in a medium with various elements j (see Maeder 2009) as

$$\frac{1}{\mu} = \sum_{j} \frac{X_j}{A_j} \left( 1 + n_{\mathrm{e},j} \right), \qquad (1.102)$$

where  $X_j$  is the mass fraction of element j with atomic mass  $A_j$  and  $n_{e,j}$  is the number of free electrons per 1 atom (ion) of element j. The number density  $n_{\alpha}$  of particles of the type  $\alpha$ , is  $\rho_{\alpha}/(\mu_{\alpha} m_u)$ , where  $\mu_{\alpha}$  is the mean molecular weight of the particle  $\alpha$ . Equation (1.101) in this case gives  $p_{\alpha} = n_{\alpha}kT$ , so that p = nkT when summed over all the particle types.

Integrating the equation for specific internal energy (the latter equation in (1.100)), we obtain  $\epsilon = c_V T$ . For a perfect gas one has  $c_p - c_V = \mathcal{R} = k/(\mu m_u)$ , where  $\mathcal{R}$  is the specific gas constant. The equation of state (1.101) then becomes

$$p = (\gamma - 1)\,\rho\epsilon.\tag{1.103}$$

This corresponds to Eq. (1.71), where we implicitly assume mono-atomic ideal gas, where  $\gamma = 5/3$ .

We can relate the pressure, density, and temperature by the speed of sound a, given by  $a^2 = \partial p / \partial \rho$ . In adiabatic case we obtain the following relations between pressure and density, and the relation between adiabatic speed of sound and temperature, respectively,

$$\nabla p = a^2 \nabla \rho$$
, so that  $p = \frac{a_{ad}^2 \rho}{\gamma}$ ,  $a_{ad}^2 = \frac{\gamma}{\mu m_u} kT$ . (1.104)

In isothermal case the *index of polytrope* is  $\gamma = 1$ . Following Eqs. (1.101) and (1.104) (with T = const.), we obtain the corresponding relations in the form

$$p = a_{\rm iso}^2 \rho, \quad a_{\rm iso}^2 = \frac{kT}{\mu \, m_u}.$$
 (1.105)

The speed of sound is the key gas dynamics characteristics. The flow of matter is basically determined by whether the flow velocity is subsonic or supersonic. In astrophysics, the temperature of the gas is often determined by thermal balance between heat sources and radiative cooling; the isothermal speed of sound may be applied if the cooling time is much shorter than the propagation speed of the sound waves. We use the isothermal speed of sound in situations where the temperature of the gas (plasma) is determined by external processes (by irradiation of external sources, etc.).

## Chapter 2

## Shock wave in a non-uniform gas

## 2.1 Rankine-Hugoniot relations

The 1D basic hydro equations (involving only the principal terms that are important for nonviscous, non self-gravitating gas expansion) take the following Cartesian explicit form (Sedov 1959; Zel'dovich & Raizer 1967)

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \qquad (2.1)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{1}{\rho} \frac{\partial p}{\partial x} - F_0 = 0, \qquad (2.2)$$

$$\frac{\partial(p\rho^{-\gamma})}{\partial t} + u\frac{\partial(p\rho^{-\gamma})}{\partial x} = 0, \qquad (2.3)$$

where  $\rho$  is the density, u is the flow velocity, p is the pressure,  $\gamma$  is the adiabatic constant, and  $F_0$  is the volume force density (gravity). In a spherical case Eqs. (2.1) - (2.3) take the explicit form

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \rho \frac{\partial u}{\partial r} + \frac{2\rho u}{r} = 0, \qquad (2.4)$$

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial r} + \frac{1}{\rho} \frac{\partial p}{\partial r} - F_0 = 0, \qquad (2.5)$$

$$\frac{\partial(P\rho^{-\gamma})}{\partial t} + u\frac{\partial(p\rho^{-\gamma})}{\partial r} = 0.$$
(2.6)

Neglecting the volume force term in Eqs. (2.2) and (2.5) and assuming the coordinate frame that is co-moving with the shock front and the constant specific heat ratio within the system, we can express the adiabatic hydro equations in their stationary form as simple conservation laws,

$$\rho_1 u_1 = \rho_0 u_0, \tag{2.7}$$

$$\rho_1 u_1^2 + p_1 = \rho_0 u_0^2 + p_0, \tag{2.8}$$

$$\frac{\gamma}{\gamma-1}\frac{p_1}{p_1} + \frac{u_1^2}{2} = \frac{\gamma}{\gamma-1}\frac{p_0}{p_0} + \frac{u_0^2}{2},\tag{2.9}$$

where the upstream versus downstream quantities are distinguished using the subscripts 0 and 1 (keeping in mind that we now "live" in the shock co-moving frame). In the lab frame are the

corresponding velocities transformed as  $u_1 \rightarrow D - u_1$  and  $u_0 \rightarrow D - u_0 = D$ , where D is the propagation speed of the shock front.

Eliminating the particular quantities from Eqs. (2.7) - (2.9), we obtain in the shock comoving frame the following relations,

$$u_0^2 = \frac{\rho_1 \left( p_1 - p_0 \right)}{\rho_0 \left( \rho_1 - \rho_0 \right)}, \quad u_1^2 = \frac{\rho_0 \left( p_1 - p_0 \right)}{\rho_1 \left( \rho_1 - \rho_0 \right)}, \tag{2.10}$$

$$\frac{p_1}{p_0} = \frac{\rho_1 \frac{\gamma+1}{\gamma-1} - \rho_0}{\rho_0 \frac{\gamma+1}{\gamma-1} - \rho_1}, \quad \frac{\rho_1}{\rho_0} = \frac{p_1 \frac{\gamma+1}{\gamma-1} + p_0}{p_1 + \frac{\gamma+1}{\gamma-1} p_0}, \quad \frac{u_1}{u_0} = \frac{p_1 + \frac{\gamma+1}{\gamma-1} p_0}{p_1 \frac{\gamma+1}{\gamma-1} + p_0},$$

$$p_1 = \left[\frac{\gamma-1}{\gamma+1} \left(\frac{\rho_1}{\rho_0}\right)^2 - 1\right] \rho_1 u_1^2 + p_0.$$
(2.11)

Assuming now the strong shock, where  $p_1 \gg p_0$ , Eq. (2.11) becomes,

$$\frac{\rho_1}{\rho_0} = \frac{\gamma + 1}{\gamma - 1}, \quad \frac{u_1}{u_0} = \frac{\gamma - 1}{\gamma + 1}, \quad p_1 = \frac{2}{\gamma - 1}\rho_1 u_1^2 = \frac{2}{\gamma + 1}\rho_0 u_0^2. \tag{2.12}$$

Transforming to lab frame, Eq. (2.11) becomes

$$\frac{p_1}{p_0} = \frac{\rho_1 \frac{\gamma+1}{\gamma-1} - \rho_0}{\rho_0 \frac{\gamma+1}{\gamma-1} - \rho_1}, \quad \frac{\rho_1}{\rho_0} = \frac{p_1 \frac{\gamma+1}{\gamma-1} + p_0}{p_1 + \frac{\gamma+1}{\gamma-1} p_0}, \quad u_1 = \frac{\frac{2}{\gamma-1} (p_1 - p_0)}{p_1 \frac{\gamma+1}{\gamma-1} + p_0} D, \quad (2.13)$$
$$p_1 = \left[\frac{\gamma-1}{\gamma+1} \left(\frac{\rho_1}{\rho_0}\right)^2 - 1\right] \rho_1 (D - u_1)^2 + p_0,$$

while the strong shock condition turns Eq. (2.12) into the form

$$\frac{\rho_1}{\rho_0} = \frac{\gamma+1}{\gamma-1}, \quad u_1 = \frac{2}{\gamma+1}D, \quad p_1 = \frac{2}{\gamma-1}\rho_1(D-u_1)^2 = \frac{2}{\gamma+1}\rho_0 D^2.$$
(2.14)

Using Eq. (2.11), we can rewrite Eq. (2.10) into the form

$$u_0^2 = \frac{1}{2\rho_0} \left[ (\gamma + 1) \, p_1 + (\gamma - 1) \, p_0 \right], \quad u_1^2 = \frac{1}{2\rho_0} \frac{\left[ (\gamma - 1) \, p_1 + (\gamma + 1) \, p_0 \right]^2}{\left[ (\gamma + 1) \, p_1 + (\gamma - 1) \, p_0 \right]}.$$
 (2.15)

From the ideal gas law we obtain the temperature ratio

$$\frac{T_1}{T_0} = \frac{p_1 \rho_0}{p_0 \rho_1}.$$
(2.16)

Very important results can be obtained by comparing velocities of gas on both sides of shock with corresponding speeds of sound. In ideal gas with constant heat capacity,  $a^2 = \gamma p/\rho$ , Eq. (2.15) becomes

$$\left(\frac{u_0}{a_0}\right)^2 = \frac{(\gamma+1)\frac{p_1}{p_0} + (\gamma-1)}{2\gamma} = \mathcal{M}_0^2, \quad \left(\frac{u_1}{a_1}\right)^2 = \frac{(\gamma+1)\frac{p_0}{p_1} + (\gamma-1)}{2\gamma} = \mathcal{M}_1^2, \quad (2.17)$$

where  $\mathcal{M}$  is the *Mach number* in the corresponding region. Substituting Eq. (2.17) into Eq. (2.15), we obtain

$$\frac{u_1}{u_0} = \frac{\rho_0}{\rho_1} = \frac{(\gamma+1)\,\mathcal{M}_1^2}{(\gamma-1)\,\mathcal{M}_1^2 + 2}, \quad \frac{p_1}{p_0} = \frac{\gamma+1}{2\gamma\mathcal{M}_1^2 - \gamma + 1},\tag{2.18}$$

$$\frac{T_1}{T_0} = \frac{(\gamma+1)^2 \mathcal{M}_1^2}{\left[2\gamma \mathcal{M}_1^2 - \gamma + 1\right] \left[(\gamma-1) \mathcal{M}_1^2 + 2\right]}, \quad \mathcal{M}_0^2 = \frac{(\gamma-1) \mathcal{M}_1^2 + 2}{2\gamma \mathcal{M}_1^2 - \gamma + 1}.$$
 (2.19)



Figure 2.1: x-axis is directed into the gas perpendicular to the boundary x = 0 where  $\rho_0(x) = 0$  while the shock wave located at x = X(t) is asumed to propagate in the negative x-direction with (lab frame) velocity U. The boundary condition is given as X = 0 at t = 0, hence the time is negative until the shock reaches the edge (Sakurai 1960).

Substituting Eq. (2.1) into Eq. (2.3) in Cartesian case and Eq. (2.4) into Eq. (2.6) in spherical case, we get respectively

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \gamma p \frac{\partial u}{\partial x} = 0, \qquad (2.20)$$

$$\frac{\partial p}{\partial t} + u \frac{\partial p}{\partial r} + \gamma p \left( \frac{\partial u}{\partial r} + \frac{2u}{r} \right) = 0.$$
(2.21)

However, the Rankine-Hugoniot relations (2.7) - (2.9) are in the spherical case same as in Cartesian geometry, due to the fact that the thickness  $d_{\rm sh}$  of the shock wave region is in vicinity of the stellar edge much smaller than the stellar radius,  $d_{\rm sh} \ll R_{\star}$ .

## 2.2 Shock wave arriving at the edge of a gas

Assuming a gas with decreasing density towards the edge of the gas-filled region, the shock wave that drives through that region can be described using the following considerations (Sakurai 1960). Figure 2.1 shows the configuration of the shock wave arriving at the edge of a gas with lab velocity

$$U = \frac{\mathrm{d}X}{\mathrm{d}t}.\tag{2.22}$$

We also assume the following power law relations for density  $\rho_0(x)$  and shock front velocity U(X),

$$\rho_0(x) = k_1 x^n, \tag{2.23}$$

$$U(X) = k_2 X^{-\lambda}, \tag{2.24}$$

where  $k_1, k_2, n$ , and  $\lambda$  are constants. Setting  $F_0 = 0$ , we introduce the similarity relations for the progressing wave type,

$$\rho = \rho_0(x) f(\eta),$$
  

$$u = U(x) g(\eta),$$
  

$$p = \rho_0(x) U^2(x) h(\eta),$$
  
(2.25)

noting that U(x) in Eq. (2.25) satisfies the equality  $U(x) = k_2 x^{-\lambda}$ .

We introduce in Eq. (2.25) the similarity variable  $\eta$  which is defined as

$$\eta = \left(\frac{X}{x}\right)^{\lambda+1} \tag{2.26}$$

Substituting Eq. (2.24) into Eq. (2.22) gives

$$X^{\lambda+1} = k_2(\lambda+1)t, (2.27)$$

$$\eta = k_2(\lambda + 1) x^{-\lambda - 1} t, \qquad (2.28)$$

where, according to the boundary condition X(t = 0) = 0, the integration constant in Eq. (2.27) is zero. The corresponding partial derivatives of  $\eta$  with respect to t and x are

$$\frac{\partial \eta}{\partial t} = k_2(\lambda+1) x^{-\lambda-1} = \frac{\eta}{t}, \qquad (2.29)$$

$$\frac{\partial \eta}{\partial x} = -k_2(\lambda+1)^2 x^{-\lambda-2} t = -(\lambda+1)\frac{\eta}{x}.$$
(2.30)

Following the formalism, we can rewrite Eqs. (2.1), (2.2), and (2.20) with use of Eq. (2.25) (where the prime values mean the derivatives with respect to  $\eta$ ) respectively as

$$(1 - \eta g)\frac{f'}{f} - \eta g' + \frac{n - \lambda}{1 + \lambda}g = 0, \qquad (2.31)$$

$$(1 - \eta g)g' - \eta \frac{h'}{f} - \frac{\lambda}{1 + \lambda}g^2 + \frac{n - 2\lambda}{1 + \lambda}\frac{h}{f} = 0, \quad \left\{ = \frac{F_0 x}{(1 + \lambda)U^2} \right\}, \tag{2.32}$$

$$(1 - \eta g)\frac{h'}{h} - \eta \gamma g' + \frac{n - \lambda(\gamma + 2)}{1 + \lambda} g = 0, \qquad (2.33)$$

where the additional right-hand side in curly brackets in Eq. (2.32) is involved if the volume force  $F_0$  is not zero. In spherical case will Eqs. (2.31) and (2.33) (following the corresponding notation  $x \to r$  and involving Eqs. (2.4), (2.5), and (2.21)) be modified respectively to

$$(1 - \eta g)\frac{f'}{f} - \eta g' + \frac{n - \lambda + 2}{1 + \lambda}g = 0, \qquad (2.34)$$

$$(1 - \eta g)\frac{h'}{h} - \eta \gamma g' + \frac{n - \lambda(\gamma + 2) + 2\gamma}{1 + \lambda} g = 0.$$
 (2.35)

The Rankine-Hugoniot conditions for a strong shock at x = X ( $\eta = 1$ ) reduce to

$$(\rho)_{x=X} = \frac{\gamma+1}{\gamma-1}\rho_0(X), \quad (u)_{x=X} = \frac{2}{\gamma+1}U(X), \quad (p)_{x=X} = \frac{2}{\gamma+1}\rho_0(X)U^2(X). \tag{2.36}$$

Substituting Eq. (2.36) into Eq (2.25), we get boundary conditions for f, g, and h at  $\eta = 1$  in the form (cf. Eqs. (2.12) and (2.14))

$$f(1) = \frac{\gamma + 1}{\gamma - 1}, \quad g(1) = \frac{2}{\gamma + 1}, \quad h(1) = \frac{2}{\gamma + 1}.$$
 (2.37)

We now have a system of nonlinear ordinary differential equations (2.31), (2.32), and (2.33) and boundary conditions (2.37) at  $\eta = 1$ . However, even providing this, a solution in the region  $0 \le \eta \le 1$  for arbitrary  $n, \gamma$ , and  $\lambda$  cannot be found in general, while it is continuous only for

one special value of  $\lambda$  which is thus appropriate for the system. This value has to be found semi-analytically using the principles described in detail as follows.

Subtracting Eq. (2.31) from Eq. (2.33) (or Eq. (2.34) from Eq. (2.35) in spherical case) gives

$$(1-\eta g)\left(\ln\frac{h}{f}\right)' - \eta g'(\gamma-1) - \lambda g \frac{\gamma+1}{1+\lambda} = 0, \quad \left\{=-2g\frac{\gamma-1}{1+\lambda}\right\},\tag{2.38}$$

where, with respect to following considerations, we combine the variables f and h as a fraction h/f. Substituting the expression for h' from Eq. (2.33) (or from Eq. (2.35)) into Eq. (2.32) and omitting the volume force term, we obtain

$$(1-\eta g)g' - \frac{\lambda g^2}{1+\lambda} - \frac{\eta \left[\eta \gamma g' - \frac{n-\lambda(\gamma+2) + \{2\gamma\}}{1+\lambda} g\right]}{1-\eta g} \frac{h}{f} + \frac{n-2\lambda}{1+\lambda} \frac{h}{f} = 0, \qquad (2.39)$$

where again the additional term  $2\gamma$  in curly brackets denotes the spherical geometry. We now define new substitutional variables x and y (not to confuse with original inverse radial x coordinate), entering relations

$$g = \frac{1}{\eta} \left( 1 - \frac{1}{y} \right), \quad \frac{h}{f} = \frac{1}{\gamma \eta^2} \frac{x}{y^2}, \tag{2.40}$$

whose derivatives (noting that  $\eta' = 1$ ) are

$$g' = -\frac{1}{\eta^2} \left( 1 - \frac{1}{y} \right) + \frac{1}{\eta y} \frac{y'}{y}, \quad \left( \ln \frac{h}{f} \right)' = \frac{x'}{x} - 2 \left( \frac{1}{\eta} + \frac{y'}{y} \right).$$
(2.41)

Substituting Eqs. (2.40) and (2.41) to Eqs. (2.38) and (2.39) respectively gives

$$\eta \frac{x'}{x} - \eta(\gamma+1)\frac{y'}{y} = \frac{2\lambda - (\gamma-1)}{1+\lambda}y + \frac{\gamma+1}{1+\lambda} \left\{ +\frac{2(\gamma-1)(1-y)}{1+\lambda} \right\}, \quad (2.42)$$

$$(1-x)\eta \frac{y'}{y} = (1-x)(y-1) - \frac{\lambda}{1+\lambda} \left[ x - (y-1)^2 \right] - \frac{n - \lambda(\gamma+2)}{\gamma(1+\lambda)} xy \qquad (2.43)$$
$$\left\{ -\frac{2x(1-y)}{1+\lambda} \right\},$$

which are two differential equations in  $x' = dx/d\eta$  and  $y' = dy/d\eta$ . By eliminating  $d\eta/\eta$  from both equations and denoting  $R_1$  and  $R_2$  the right-hand sides of Eqs. (2.42) and (2.43), respectively, we get one differential equation in dy/dx in the form

$$x\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{yR_2}{(1-x)R_1 + (\gamma+1)R_2} =$$

$$= \frac{\lambda y^2 + x - 1 - \left[\lambda - 1 + \frac{n+\gamma-2\lambda}{\gamma}x\right]y - \left\{\frac{2x(1-y)}{1+\lambda}y\right\}}{\lambda\left[(\gamma+1)y - (\gamma-1)\right] - \left[n+2 + \frac{n-2\lambda}{\gamma}\right]x + 2 + \left\{\frac{2(1-y)(\gamma-1-4x)}{1+\lambda}\right\}}.$$
(2.44)

Substituting Eq. (2.37) into Eq. (2.40) gives the values of the variables x and y at the shock wave point  $(\eta = 1)$ ,

$$x(1) = \frac{2\gamma}{\gamma - 1}, \quad y(1) = \frac{\gamma + 1}{\gamma - 1}.$$
 (2.45)

We find the correct value of  $\lambda$  starting at x = 1 (singularity point in Eq. (2.43)) which gives the Dirichlet y boundary value,  $y = [n - \lambda(2 - \gamma)]/\gamma\lambda$ . Then integrate eq. (2.44) up to the shock front point where  $x = 2\gamma/(\gamma - 1)$  with a trial value of  $\lambda$  while the correct value of y there should be  $y = (\gamma + 1)/(\gamma - 1)$ . The semi-analytical solution is found using, e.g., the simple Euler method.

For our purpose it is convenient to choose the polytrope index as  $\gamma = 4/3$ . We found the following values of  $\lambda$  for selected values of n:

tesian case:	$\operatorname{Sph}$	Spherical case:		
$\lambda$	n	$\lambda$		
0.194	1.0	0.163		
0.287	1.5	0.241		
0.378	2.0	0.321		
0.468	2.5	0.408		
0.553	3.0	0.479		
	$\begin{array}{c} \lambda \\ \hline 0.194 \\ 0.287 \\ 0.378 \\ 0.468 \\ 0.553 \end{array}$	$\begin{tabular}{ c c c c c c } \hline \lambda & & & & & & \\ \hline \lambda & & & & & & & \\ \hline 0.194 & & & & & & \\ 0.287 & & & & & & \\ 0.378 & & & & & & \\ 0.378 & & & & & & \\ 0.468 & & & & & & \\ 0.553 & & & & & & & \\ 0.553 & & & & & & & \\ \hline \end{tabular}$		

Table 2.1: Table of values of calculated  $\lambda$  parameter for various selected *n* values in planar (Cartesian) and spherical case.

The values of parameter  $\lambda$  in Tab. 2.1 show almost linear relation to the selected density slopes *n*: in Cartesian geometry  $\lambda \approx 0.19 n$  while in spherical geometry  $\lambda \approx 0.16 n$  (where the spherical dependence is even stronger than the Cartesian). From Eqs. (2.23), (2.24), and (2.25) we may conclude that the shock front propagates with velocity  $U \propto (\rho_0)^{-\lambda/n}$  in the lab frame which corresponds to

$$U \propto (\rho_0)^{-0.19}$$
, and  $U \propto (\rho_0)^{-0.16}$  (2.46)

in Cartesian and spherical case, respectively. As the density decreases towards the edge, the velocity fairly increases. To avoid the singularity in velocity at the point x = 0, we need to involve the following considerations regarding the further expansion of a gas into surrounding vacuum.

Employing the values of  $\lambda$  from Tab. 2.1 we can numerically integrate system of Eqs. (2.31) - (2.33) (or Eqs. (2.34) - (2.35) in spherical case) for  $\gamma = 4/3$  in range  $0 \le \eta \le 1$ . Starting from boundary values at  $\eta = 1$  given in Eq. (2.37) (using the Runge-Kutta method) we found the values f(0), g(0), and h(0) for  $\eta = 0$  at time t = 0 (at X = 0) when the shock arrives at the boundary x = 0, which are of special interest for the following chapter. We summarize the values for various parameters n in Tab. 2.2. The calculation of f(0), g(0), and h(0) also provides

Cartesian case:				Spherical case:			
n	f(0)	g(0)	h(0)	n	f(0)	g(0)	h(0)
1.0	19.419	0.774	1.145	1.0	44.639	0.661	0.291
1.5	26.870	0.735	1.254	1.5	36.091	0.606	0.260
2.0	38.137	0.714	1.463	2.0	31.367	0.566	0.196
2.5	47.713	0.688	1.573	2.5	75.979	0.611	0.525
3.0	58.182	0.669	1.703	3.0	116.870	0.620	0.787

Table 2.2: Table of values of f(0), g(0), h(0) at  $\eta = 0$  in Cartesian and spherical case.

a check of correctly found values of  $\lambda$  (which is very sensitive of, in the spherical case even much more) because the graphs in this case show smooth curves without particular singularities.

### 2.3 Gas expanding into vacuum

For the stage of gas expansion into vacuum outside the edge of the initially gas filled region we use the Lagrangian coordinate a which coincides with Eulerian coordinate x(a, t) at t = 0. We denote the profiles of  $\rho$ , u, and p behind the shock front at time t = 0 (following Eq. (2.25) subscripted with 1, setting  $\eta = 0$ ) as

$$\rho_{1} = \rho_{0}(x) f(0) = f(0) k_{1} x^{n}, 
u_{1} = U(x) g(0) = g(0) k_{2} x^{-\lambda}, 
p_{1} = \rho_{0}(x) U^{2}(x) h(0) = h(0) k_{1} k_{2}^{2} x^{n-2\lambda}.$$
(2.47)

Following Eq. (2.47), we write Lagrangian equations for outward expansion at time t = 0 (noting that time of further outward expansion grows negatively) as

$$x_{1} = x(a,0) = a,$$
  

$$\rho_{1} = \rho(a,0) = \rho_{0}(a) f(0) = f(0) k_{1}a^{n},$$
  

$$u_{1} = u(a,0) = U(a) g(0) = g(0) k_{2}a^{-\lambda},$$
  

$$p_{1} = p(a,0) = \rho_{0}(a)U^{2}(a) h(0) = h(0) k_{1}k_{2}^{2}a^{n-2\lambda},$$
  
(2.48)

$$\rho_1 = u_1 = p_1 = 0, \qquad a < 0. \tag{2.49}$$

From correspondence of the continuity equations at x = 0 follows that the already introduced Eulerian density  $\rho_1$  is in the Lagrangian (material) frame defined as

$$\frac{\partial \rho}{\partial t} = -\rho_1 \frac{\partial u}{\partial x} = -\rho \frac{\partial u}{\partial a},\tag{2.50}$$

and using the stationary Rankine-Hugoniot relation for energy (Eq. (2.9)), we transform the equation for the velocity u and the basic hydro equations (2.1) - (2.6) into the form (we introduce hereafter only the Cartesian solution because the spherical approach becomes analytically extremely complicated in this region while it does not significantly affect the results)

$$\frac{\partial x}{\partial t} = u, \tag{2.51}$$

$$\frac{\partial x}{\partial a} = \frac{\rho_1}{\rho},\tag{2.52}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho_1} \frac{\partial p}{\partial a} + F_0, \qquad (2.53)$$

$$p\rho^{-\gamma} = p_1\rho_1^{-\gamma} = k_1^{1-\gamma}k_2^2 f(0)^{-\gamma}h(0) a^{n(1-\gamma)-2\lambda}, \qquad (2.54)$$

where we use (from the practical point of view fully realistic) partial derivative instead of conventional total derivative. We use the quantity  $\rho_1$  in the first term on the right-hand side of Eq. (2.53) due to equality  $\rho_1 da = \rho dx$  which follows from Eq. (2.52).

Assuming  $F_0 = \text{const.}$ , we may express the solution of Eqs. (2.52) - (2.54), with use of initial conditions (2.48), by a similarity solution with functions  $r(\xi)$ ,  $F(\xi)$ ,  $G(\xi)$ , and  $H(\xi)$ , determined as

$$x = a r(\xi) + \frac{1}{2} F_0 t^2,$$
  

$$\rho = \rho_1(a) F(\xi),$$
  

$$u = u_1(a) G(\xi) + F_0 t,$$
  

$$p = p_1(a) H(\xi).$$
  
(2.55)

The similarity variable  $\xi$  is now defined as (where  $U(a) = k_2 a^{-\lambda}$ , cf. Eq. (2.24))

$$\xi = \frac{U(a)t}{a} = k_2 \, a^{-\lambda - 1} t, \qquad (2.56)$$

where  $-\infty \leq \xi \leq 0$  since  $U(a) \leq 0$ . The similarity solution particularly fits the singularity at a = 0 where both u and x must be  $-\infty$ . Comparing Eqs. (2.48) and (2.55), we can easily verify that

$$r(0) = F(0) = G(0) = H(0) = 1.$$
(2.57)

n	$G_{\max}$
1.0	2
1.5	1.88
2.0	1.777
2.5	1.72
3.0	1.68

Table 2.3: Table of asymptotic maximum values of  $G(G_{\text{max}})$  at  $\xi = 1000$ .

Substituting Eqs. (2.55) into Eqs. (2.51) - (2.54), with use of Eqs. (2.48), (2.50), and (2.56), we obtain

$$r' = g(0) G, (2.58)$$

$$-(1+\lambda)\xi r' + r = \frac{1}{F},$$
(2.59)

$$\frac{f(0)g(0)}{h(0)}G' = -(n-2\lambda)H + (1+\lambda)\xi H', \qquad (2.60)$$

$$F^{-\gamma}H = 1, \tag{2.61}$$

where the prime quantities denote their derivatives with respect of  $\xi$ .

Since  $\lambda$ , f(0), g(0), and h(0) are given by the solution of equations in Sect. 2.2, we can integrate numerically the system of Eqs. (2.58) - (2.61) from the starting value  $\xi = 0$  to  $\xi = -\infty$ . However, first we modify Eqs. (2.58) - (2.61) by eliminating r and F into the more convenient system of two equations for G and H,

$$\frac{H^{-(\gamma+1)/\gamma}}{\gamma g(0)} H' = \lambda G + (1+\lambda) \xi G',$$
  
$$\frac{f(0)g(0)}{h(0)} G' = -(n-2\lambda) H + (1+\lambda) \xi H'.$$
 (2.62)

Solution of Eq. (2.62) must fit the boundary conditions

$$G(0) = H(0) = 1, (2.63)$$

while the resulting functions r and F are given by

$$r = 1 + g(0) \int_0^{\xi} G \,\mathrm{d}\xi, \quad F = H^{1/\gamma},$$
 (2.64)



Figure 2.2: Profiles of  $F(\xi)$ ,  $G(\xi)$ , and  $H(\xi)$  up to  $\xi = -10$  in Cartesian geometry for n = 1, 2, 3. The curves approach the asymptotic maxima, the approximate maximum  $G_{\text{max}}$  for the relative velocity profile is given in Tab. 2.3.

where the integral in the last equation is expressed as an antiderivative, hence the integration constant r(0) is evaluated using Eq. (2.57).

Equations (2.62) are integrated numerically (using again Runge-Kutta) with implemented values of  $\lambda$ , f(0), g(0), and h(0) given in Tables 2.1 and 2.2. The functions r and F are subsequently obtained from Eq. (2.64). We show the results up to  $\xi = -10$  in Fig. 2.2 and the asymptotic values of G in Table 2.3. Since the similarity variable  $G(\xi)$  is connected with velocity, Tab. 2.3 proves that the expansion velocity increases almost twice during the outward expansion.

### 2.4 Flow past finite bodies

Simple arguments show that, in supersonic flow past an arbitrary body, a shock wave must be formed in front of the body. For the disturbances in the supersonic flow caused by the presence of the body are propagated only downstream. Hence a uniform supersonic stream incident on the body would be unperturbed as far as the leading end of the body. The normal component of the gas velocity would then be non-zero at the surface there, in contradiction to the necessary boundary condition. The resolution of this difficulty can only be the occurrence of a shock wave, as a result of which the gas flow between it and the leading end of the body becomes subsonic.

Thus a shock wave is formed in front of the body when the incident flow is supersonic; it is called the *bow wave*. When the leading end of the body is blunt, the bow wave does not touch the body. In front of the shock wave, the flow is uniform; behind it, the flow is modified and bends round the body. The surface of the shock wave extends to infinity, and at great distances from the body, where the shock is weak, it intersects the incident streamlines at an angle approaching the Mach angle. A characteristic feature of flow past a blunt-ended body is the existence of a subsonic flow region behind the shock wave at the most forward part of its surface; this region extends to the body itself, and thus lies between the discontinuity surface, the body, and a lateral sonic surface.

The values of quantities in the incident stream will be denoted, as usual, by the suffix 1, and the values behind the shock wave by the suffix 0, while the stagnation point, where the shock first meets the body, we denote by the suffix 2. The values behind the shock wave are determined from formulae (2.17) and (2.18),

$$\frac{u_0}{a_1} = \frac{(\gamma - 1)\mathcal{M}_1^2 + 2}{(\gamma + 1)\mathcal{M}_1}, \quad \frac{\rho_0}{\rho_1} = \frac{(\gamma + 1)\mathcal{M}_1^2}{(\gamma - 1)\mathcal{M}_1^2 + 2}, \quad \frac{p_0}{p_1} = \frac{2\gamma\mathcal{M}_1^2 - \gamma + 1}{\gamma + 1}.$$
 (2.65)

The pressure  $p_2$  at the stagnation point (where the gas velocity u = 0) can now be obtained by means of the formulae which give the variation of quantities along a streamline. Comparing the enthalpy  $h = c_p T = \gamma c_V T$  and the pressure  $p = (\gamma - 1)\rho\epsilon = (\gamma - 1)\rho c_p T/\gamma$  gives

$$h = \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \frac{a^2}{\gamma - 1}.$$
(2.66)

We obtain from Bernoulli's equation a number of general results concerning adiabatic steady flow of a gas. The equation is, for steady flow,  $h + v^2/2 = \text{constant}$  along each streamline; if we have potential flow, then the constant is the same for every streamline, i.e., at every point in the fluid. If there is a point on some streamline at which the gas velocity is zero, then we can write Bernoulli's equation as

$$h + \frac{v^2}{2} = h_2 = \frac{a_2^2}{\gamma - 1},\tag{2.67}$$

where  $h_2$  is the value of the heat function at the point where v = 0. The equation of conservation of entropy for steady flow is  $\mathbf{v} \cdot \nabla s = v \, \mathrm{d}s / \mathrm{d}l = 0$ , i.e., s is constant along each streamline. We can write this in a form analogous to (2.67)

$$s = s_2. \tag{2.68}$$

We see from equation (2.67) that the velocity v is greater at points where the heat function h is smaller. The maximum value of the velocity (on the streamline considered) is found at the point where h is least. For constant entropy, however, we have  $dh = dp/\rho$ ; since p > 0, the differentials dh and dp have equal signs, and so h and p vary in the same sense. We can therefore say that the velocity increases along a streamline when the pressure decreases, and vice versa. The smallest possible values of the pressure and the heat function (in adiabatic flow) are obtained when the absolute temperature T = 0. The corresponding pressure is p = 0, and the value of h for T = 0 can be arbitrarily taken as the zero of energy; then h = 0 for T = 0. We can now deduce from (2.67) that the greatest possible value of the velocity (for given values of the thermodynamic quantities at the point where v = 0) is

$$v_{\max} = \sqrt{2h_2}.\tag{2.69}$$

This velocity can be attained when a gas flows steadily out into a vacuum.

Let us now consider how the mass flux density  $j = \rho v$  varies along a streamline. From Euler's equation  $(\mathbf{v} \cdot \nabla)\mathbf{v} = -(1/\rho)\nabla p$ , we find that the relation  $v \, dv = -dp/\rho$  between the differentials dp and  $d\rho$  holds along a streamline. Putting  $dp = a^2 \, d\rho$ , we have

$$\frac{\mathrm{d}\rho}{\mathrm{d}v} = -\frac{\rho v}{a^2} \tag{2.70}$$

and, substituting in  $d(\rho v) = \rho dv + v d\rho$ , we obtain

$$\frac{\mathrm{d}(\rho v)}{\mathrm{d}v} = \rho \left(1 - \frac{v^2}{a^2}\right). \tag{2.71}$$

We see that, as the velocity increases along a streamline, the mass flux density increases as long as the flow remains subsonic. In the supersonic range, however, the mass flux density diminishes with increasing velocity, and vanishes together with  $\rho$  when  $v = v_{\text{max}}$ . This important difference between subsonic and supersonic steady flows can be simply interpreted as follows. In a subsonic flow, the streamlines approach in the direction of increasing velocity. In a supersonic flow, however, they diverge in that direction.

The flux  $j = \rho v$  has its maximum value  $j_*$  at the point where the gas velocity is equal to the local velocity of sound:

$$j_* = \rho_* a_*,$$
 (2.72)

where the asterisk suffix indicates values corresponding to this point. The velocity  $v_*c_*$  is called the critical velocity. In the general case of an arbitrary gas, the critical values of quantities can be expressed in terms of their values at the point v = 0, by solving the simultaneous equations

$$s_* = s_2, \quad h_* + \frac{a_*^2}{2} = h_2.$$
 (2.73)

It is evident from the previous that, whenever  $\mathcal{M} = v/a < 1$ , we have also  $v/a_* < 1$ , and if  $\mathcal{M} > 1$  then  $v/a_* > 1$ . Hence the ratio  $\mathcal{M}_* = v/a_*$  serves in this case as a criterion analogous to  $\mathcal{M}$ , and is more convenient, since  $a_*$  is a constant, unlike a, which varies along the stream.

In applications of the general equations of gas dynamics, the case of a perfect gas is of particular importance. For a perfect gas we shall always assume (except where otherwise specified) that the specific heat is a constant independent of temperature in the range considered. Such a gas is often called a *polytropic* gas, and we shall use this term in order to emphasize that the assumption made goes much further than that of a perfect gas. The relations between the thermodynamic quantities for a polytropic gas are given by very simple formulae, and this often allows a complete solution of the equations of gas dynamics.

First law of thermodynamics (noting that  $d\epsilon = c_V dT$ ) in case of an ideal gas where  $p/(\rho T) = \mathcal{R}$  gives

$$c_V \,\mathrm{d}T = T \,\mathrm{d}s + \frac{p}{\rho^2} \,\mathrm{d}\rho. \tag{2.74}$$

Dividing this by T and integrating, we obtain

$$s = c_V \ln(p\rho^{-\gamma}) = c_p \ln(p^{1/\gamma}\rho^{-1}).$$
(2.75)

Let us now investigate steady flow, applying the general relations previously obtained to the case of a polytropic gas. Substituting (2.66) in (2.69), we find that the maximum velocity of steady flow is

$$v_{\max} = a_2 \sqrt{\frac{2}{\gamma - 1}}.$$
 (2.76)

For the critical velocity we obtain from the second equation (2.73)

$$\frac{a_*^2}{\gamma - 1} + \frac{a_*^2}{2} = h_2 = \frac{a_2^2}{\gamma - 1},$$
(2.77)

whence

$$a_* = a_2 \sqrt{\frac{2}{\gamma + 1}}.$$
 (2.78)

From Eq. (1.104) we deduce  $T/T_2 = a^2/a_2^2$ , combining this with Eqs. (2.66), (2.67), and (2.78), we obtain the important particular result

$$T = T_2 \left[ 1 - (\gamma - 1) \frac{v^2}{2a_2^2} \right] = T_2 \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \frac{v^2}{a_*^2} \right].$$
 (2.79)

Following the adiabatic and ideal gas prescriptions  $p = p_2 (\rho/\rho_2)^{\gamma}$  and  $\rho = \rho_2 (T/T_2)^{1/(\gamma-1)}$ , we further obtain

$$\rho = \rho_2 \left[ 1 - (\gamma - 1) \frac{v^2}{2a_2^2} \right]^{1/(\gamma - 1)} = \rho_2 \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \frac{v^2}{a_*^2} \right]^{1/(\gamma - 1)}, \qquad (2.80)$$

$$p = p_2 \left[ 1 - (\gamma - 1) \frac{v^2}{2a_2^2} \right]^{\gamma/(\gamma - 1)} = p_2 \left[ 1 - \frac{\gamma - 1}{\gamma + 1} \frac{v^2}{a_*^2} \right]^{\gamma/(\gamma - 1)},$$
(2.81)

It is sometimes convenient to use these relations in a form which gives the velocity and sound speed in terms of other quantities:

$$v^{2} = \frac{2\gamma}{\gamma - 1} \frac{p_{2}}{\rho_{2}} \left[ 1 - \left(\frac{\rho}{\rho_{2}}\right)^{\gamma - 1} \right] = \frac{2\gamma}{\gamma - 1} \frac{p_{2}}{\rho_{2}} \left[ 1 - \left(\frac{p}{p_{2}}\right)^{(\gamma - 1)/\gamma} \right].$$
 (2.82)

Using Eqs. (2.77) and (2.78), we relate the velocity of sound and the velocity v:

$$a^{2} = a_{2}^{2} - (\gamma - 1) \frac{v^{2}}{2} = (\gamma + 1) \frac{a_{*}^{2}}{2} - (\gamma - 1) \frac{v^{2}}{2}.$$
 (2.83)

Hence we find that the numbers  $\mathcal{M}$  and  $\mathcal{M}_*$  are related by

$$\mathcal{M}_*^2 = \frac{\gamma + 1}{\gamma - 1 + 2/\mathcal{M}^2};$$
(2.84)

when  $\mathcal{M}$  varies from 0 to  $\infty$ ,  $\mathcal{M}^2_*$  varies from 0 to  $(\gamma + 1)/(\gamma - 1)$ .

Finally, we may give expressions for the critical temperature, pressure and density: they are obtained by putting  $v = c_*$  in Eqs. (2.79) - (2.81):

$$T_* = \frac{2T_2}{\gamma + 1},\tag{2.85}$$

$$\rho_* = \rho_2 \left(\frac{2}{\gamma+1}\right)^{1/(\gamma-1)},\tag{2.86}$$

$$p_* = p_2 \left(\frac{2}{\gamma+1}\right)^{\gamma/(\gamma-1)}.$$
 (2.87)

In conclusion, it should be emphasized that the results derived above are valid only for flow in which shock waves do not occur. When shock waves are present, equation (2.68) does not hold; the entropy of the gas increases when a streamline passes through a shock wave. We shall see, however, that Bernoulli's equation (2.67) remains valid even when there are shock waves, since  $h + v^2/2$  is a quantity which is conserved across a surface of discontinuity (see Eq. 2.9). Following now Bernoulli's equation (2.67) with use of Eq. (1.102), a simple calculation gives

$$\mathcal{M}^{2}\left[\frac{1}{2}(\gamma-1) + \frac{1}{\mathcal{M}^{2}}\right] = \frac{a_{2}^{2}}{v^{2}}\frac{v_{2}}{a^{2}}, \quad \text{that is} \quad \frac{T_{2}}{T} = 1 + \frac{1}{2}(\gamma-1)\mathcal{M}^{2}.$$
(2.88)

Analogously to Eqs. (2.79) - (2.81) we obtain

$$\frac{\rho_2}{\rho} = \left[1 + \frac{1}{2}(\gamma - 1)\mathcal{M}^2\right]^{1/(\gamma - 1)}, \quad \frac{p_2}{p} = \left[1 + \frac{1}{2}(\gamma - 1)\mathcal{M}^2\right]^{\gamma/(\gamma - 1)}$$
(2.89)

and, since Eq. (2.89) holds in an arbitrary point along a streamline, we may write

$$p_2 = p_0 \left[ 1 + \frac{1}{2} (\gamma - 1) \frac{v_0^2}{a_0^2} \right]^{\gamma/(\gamma - 1)}.$$
(2.90)

A simple combination of Eqs. (2.90) with the last Eq. (2.65) with use of the second Eq. (2.19) gives

$$p_2 = p_1 \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/(\gamma-1)} \frac{\mathcal{M}_1^2}{\left[\gamma - (\gamma-1)/(2\mathcal{M}_1^2)\right]^{1/(\gamma-1)}}.$$
(2.91)

This determines the pressure at the leading end for a supersonic incident flow  $(\mathcal{M}_1 > 1)$ .

For comparison, we give the formula for the pressure at the stagnation point obtained for a continuous adiabatic retardation of the gas, with no shock wave (as would be true for a subsonic incident flow, cf. Eq. (2.89)):

$$p_2 = p_1 \left[ 1 + \frac{1}{2} (\gamma - 1) \mathcal{M}_1^2 \right]^{\gamma/(\gamma - 1)}.$$
 (2.92)

For  $\mathcal{M}_1 = 1$ , the two formulae give the same value of  $p_2$ , but for  $\mathcal{M}_1 > 1$  the pressure given by formula (2.92) is always greater than the true pressure  $p_2$  given by formula (2.91).

In the limit of very large velocities  $(\mathcal{M}_1 \gg 1)$ , formula (2.91) gives

$$p_2 = p_1 \left(\frac{\gamma+1}{2}\right)^{(\gamma+1)/(\gamma-1)} \gamma^{-1/(\gamma-1)} \mathcal{M}_1^2, \qquad (2.93)$$

i.e., the pressure  $p_2$  is proportional to the square of the incident velocity. From this result we can conclude that the total drag force on the body at velocities large compared with that of sound is proportional to the square of the velocity. It should be noticed that this is the same as the law governing the drag force at velocities small compared with that of sound but yet so large that the Reynolds number is large.

## Chapter 3

# Basics of Magnetohydrodynamics (MHD)

### 3.1 Fundamental Equations of Ideal MHD

(Kurfürst 2015): We review the vacuum differential form of Maxwell equations:

$$\boldsymbol{\nabla} \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0},\tag{3.1}$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},\tag{3.2}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{3.3}$$

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J} + \mu_0 \epsilon_0 \frac{\partial \boldsymbol{L}}{\partial t}, \qquad (3.4)$$

where  $\rho$  is the electric charge density,  $\boldsymbol{E}$  is the electric field intensity,  $\boldsymbol{V}$  is the flow velocity of the matter (ionized gas),  $\boldsymbol{B}$  is the magnetic induction,  $\epsilon_0$  and  $\mu_0$  are the vacuum electric permittivity and magnetic permeability, respectively, and  $\boldsymbol{J}$  is the electric current density,  $\boldsymbol{J} = \rho \boldsymbol{V}$ . The general expression for the electromagnetic Lorentz force is

$$\boldsymbol{F}_L = \rho(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}). \tag{3.5}$$

This equation we may further expand by the vacuum form of the "fourth" Maxwell equation (Maxwell-Ampère's law) (3.4) Neglecting the term  $\mu_0 \epsilon_0 \partial \boldsymbol{E} / \partial t$  we obtain the Ampère's law

$$\boldsymbol{\nabla} \times \boldsymbol{B} = \mu_0 \boldsymbol{J},\tag{3.6}$$

which is often used in MHD calculations.

We involve the generalized Ohm's law,

$$\boldsymbol{J} = \sigma(\boldsymbol{E} + \boldsymbol{V} \times \boldsymbol{B}), \tag{3.7}$$

where  $\sigma$  is the material-dependent conductivity, which for most of the fluids is typically greater than Siemens per meter, S/m. The dimensional analysis of Eq. (3.4) shows that  $\epsilon_0 E/\tau \sim \sigma E$  in case of extremely small characteristic time  $\tau$  for changes in the electric field (i.e., that the term  $\partial \boldsymbol{E}/\partial t$  cannot be neglected only in case of  $\tau$  is of order  $10^{-11}$  s or less) and we may simplify Eq. (3.7) as  $J = \sigma E$ .

Now we use analogous dimensional analysis with the "first" Maxwell equation (Gauss's law) in the vacuum form  $\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0 \approx E/\ell$  (where  $\ell$  is the characteristic length scale of the

system), using Eq. (3.4) in the described approximation  $\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J} \approx B/\ell$ . Combining the two approximations and the ideal Ohm's law for a perfect conductor given by

$$\boldsymbol{E} = -\boldsymbol{V} \times \boldsymbol{B} \tag{3.8}$$

(whose dimension is -BV and noting that  $\epsilon_0\mu_0 = c^{-2}$ ), we obtain the approximate ratio  $\rho E/(\mathbf{J} \times \mathbf{B}) \approx (V/c)^2 \ll 1$  for the non-relativistic ideal MHD. We may therefore neglect the electrostatic force term in Eq. (3.5), writing the magnetic Lorentz force equation in the modified form

$$\boldsymbol{F}_{L} = \boldsymbol{J} \times \boldsymbol{B} = \frac{1}{\mu_{0}} (\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B}.$$
(3.9)

Using the vector identity  $\nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \nabla^2 B$  (with  $\nabla \cdot B = 0$ ), for the right-hand side cross product of vector rotation, Eq. (3.9) becomes

$$\boldsymbol{F}_{L} = \frac{1}{\mu_{0}} (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \boldsymbol{B} - \frac{1}{2} \frac{\boldsymbol{\nabla} B^{2}}{\mu_{0}}, \qquad (3.10)$$

where the first right-hand side term in Eq. (3.10) expresses the advection of the magnetic field and the second term expresses the gradient of the magnetic energy density (Bittencourt 2004).

Basic hydrodynamic equations, including the Lorentz force and the induction equation (3.14), that is, the basic MHD equations, can be written in the following way: the continuity equation (1.23) remains

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \rho \boldsymbol{V} = 0, \qquad (3.11)$$

while the equation of motion (1.29) now is

$$\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \boldsymbol{\nabla})\boldsymbol{V} + \frac{1}{\rho}\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{P}} + \frac{1}{2}\frac{\boldsymbol{\nabla}B^2}{\mu_0\rho} - \frac{1}{\mu_0\rho}(\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} + \boldsymbol{\nabla}\Phi = \boldsymbol{0}, \qquad (3.12)$$

where  $\mathcal{P}$  is the pressure tensor and  $\Phi$  is the gravitational potential. Including the Ohm's law for *ideally* conductive plasma (where the electrical conductivity  $\sigma \to \infty$ ) in the form (3.8), from the "second" Maxwell equation (Faraday's law),

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t},\tag{3.13}$$

we obtain the Maxwell-Faraday equation (usually called the induction equation),

$$\frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) = \boldsymbol{0}. \tag{3.14}$$

We neglect the diffusion term  $\eta \nabla^2 \boldsymbol{B}$  in Eq. (3.14) where  $\eta = 1/(\mu_0 \sigma)$  denotes the magnetic diffusivity (Bittencourt 2004). The diffusion term plays a significant role only in case of very low gas velocity or very small electric conductivity.

To derive the MHD terms, which enter the energy equation, we expand the term  $\mathbf{V} \cdot \mathbf{F}$  on the right-hand side of Eq. (1.85) where we assume the force  $\mathbf{F}$  is the magnetic Lorentz force (3.9). Multiplying Eq. (3.9) by velocity,  $\mathbf{F}_L \cdot \mathbf{V} = (\mathbf{J} \times \mathbf{B}) \cdot \mathbf{V} = -(\mathbf{V} \times \mathbf{B}) \cdot \mathbf{J}$ , and using Eq. (3.8), we obtain  $\mathbf{F}_L \cdot \mathbf{V} = \mathbf{E} \cdot \mathbf{J} = \mathbf{E} \cdot (\nabla \times \mathbf{B})/\mu_0$ , where the last expression comes from the Ampère's law (3.6). We expand the term  $\mathbf{E} \cdot (\nabla \times \mathbf{B})/\mu_0$  as  $[\mathbf{B} \cdot (\nabla \times \mathbf{E}) - \nabla \cdot (\mathbf{E} \times \mathbf{B})]/\mu_0$ , where we
rewrite the first term using the Faraday's law of induction (3.13) and the second term using the Ohm's law for ideally conducting fluid (Eq. (3.8)), into the form

$$\boldsymbol{F}_{L} \cdot \boldsymbol{V} = \frac{1}{\mu_{0}} \left\{ -\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \cdot \left[ (\boldsymbol{V} \times \boldsymbol{B}) \times \boldsymbol{B} \right] \right\}.$$
(3.15)

Using the vector identity for the triple cross product, the term  $F_L \cdot V$  becomes

$$\boldsymbol{F}_{L} \cdot \boldsymbol{V} = -\frac{\partial}{\partial t} \left( \frac{B^{2}}{2\mu_{0}} \right) + \frac{1}{\mu_{0}} \boldsymbol{\nabla} \cdot \left[ \left( \boldsymbol{B} \cdot \boldsymbol{V} \right) \boldsymbol{B} - B^{2} \boldsymbol{V} \right].$$
(3.16)

Inserting Eq. (3.16) into the energy equation (1.88), we obtain the ideal MHD energy equation,

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \left[ (E + \mathcal{P}) \, \boldsymbol{V} + \boldsymbol{q} \right] = \frac{1}{\mu_0} \boldsymbol{\nabla} \cdot \left[ (\boldsymbol{B} \cdot \boldsymbol{V}) \, \boldsymbol{B} - \frac{B^2 \, \boldsymbol{V}}{2} \right] + \rho \boldsymbol{g} \cdot \boldsymbol{V}, \qquad (3.17)$$

where the first right-hand side term in the square bracket is the magnetic tension force that is trying to straighten the magnetic field lines, the second term in the right-hand side's square bracket is the magnetic pressure flux, and  $\mathbf{g}$  denotes the vector of external gravitational acceleration. The explicit form of the total energy density E in Eq. (3.17) is

$$E = \rho \epsilon + \frac{\rho V^2}{2} + \frac{B^2}{2\mu_0},$$
(3.18)

consisting from the densities of internal, kinetic, and magnetic energy, respectively.

# 3.2 Parker Modified Momentum Equation

Bittencourt (2004): In the presence of a strong **B** field the pressure tensor of an inviscid conducting fluid is *anisotropic*. When the cyclotron frequency  $\Omega_c = -(q/m)\mathbf{B}$  much larger than the collision frequency  $\nu_{coll}$ , a charged particle gyrates many times around a magnetic force line during the time between collisions, so that there is equipartition between the particle kinetic energies in the two independent directions normal to **B** but not, in general, in the direction along **B**. If we denote by  $p_{\perp}$  and  $p_{\parallel}$  the scalar pressures in the plane normal to **B** and along **B**, respectively, and consider a *local* coordinate system in which the z-axis is in the direction of **B**, we can write the pressure tensor of an inviscid fluid as

$$\mathcal{P} = \begin{pmatrix} p_{\perp} & 0 & 0\\ 0 & p_{\perp} & 0\\ 0 & 0 & p_{||} \end{pmatrix}.$$
 (3.19)

The parallel and perpendicular pressure indexes do not refer to vector components but indicate the part of the scalar pressures associated with the kinetic energy densities of the particle motions along  $\boldsymbol{B}$  and perpendicular to  $\boldsymbol{B}$ , respectively.

When the magnetic field is not constant, the orientation of the axes of the local coordinate system changes from point to point and this change in direction must be taken into account when evaluating the divergence of the pressure tensor. We express  $\mathcal{P}$ , in (3.19), as the sum of a hydrostatic scalar pressure  $p_{\perp}$  and another tensor referred to the local coordinate system, as

$$\mathcal{P} = p_{\perp} \mathbf{1} + (p_{\parallel} - p_{\perp}) \hat{\boldsymbol{B}} \hat{\boldsymbol{B}}, \qquad (3.20)$$

where **1** is the unit dyad (3×3 unit matrix in this case) and  $\hat{B}\hat{B} = BB/B^2$  is the dyad formed from the unit vector  $\hat{B}$ ,

$$\hat{\boldsymbol{B}}\hat{\boldsymbol{B}} = \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{pmatrix}.$$
(3.21)

The momentum equation, in the form (3.12), must be modified to include the anisotropy of the pressure dyad. To evaluate  $\nabla \cdot \mathcal{P}$  given by (3.12), we note that

$$\boldsymbol{\nabla} \cdot (p_{\perp} \mathbf{1}) = \boldsymbol{\nabla} p_{\perp} \tag{3.22}$$

and using the following identity,

$$\boldsymbol{\nabla} \cdot \left[ (p_{||} - p_{\perp}) \hat{\boldsymbol{B}} \hat{\boldsymbol{B}} \right] = (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \left[ (p_{||} - p_{\perp}) \frac{\boldsymbol{B}}{B^2} \right] + \left[ (p_{||} - p_{\perp}) \frac{\boldsymbol{B}}{B^2} \right] (\boldsymbol{\nabla} \cdot \boldsymbol{B}), \quad (3.23)$$

where the second term in the right-hand side vanishes due to  $\nabla \cdot \boldsymbol{B} = 0$ , we obtain

$$\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{P}} = \boldsymbol{\nabla} p_{\perp} + (\boldsymbol{B} \cdot \boldsymbol{\nabla}) \left[ (p_{\parallel} - p_{\perp}) \frac{\boldsymbol{B}}{B^2} \right].$$
(3.24)

Substituting expressions (3.24) into the momentum equation (3.12), we obtain

$$\rho \frac{\partial \mathbf{V}}{\partial t} + \rho (\mathbf{V} \cdot \nabla) \mathbf{V} + \nabla \left( p_{\perp} + \frac{B^2}{2\mu_0} \right) + (\mathbf{B} \cdot \nabla) \left[ (p_{\parallel} - p_{\perp}) \frac{\mathbf{B}}{B^2} - \frac{\mathbf{B}}{\mu_0} \right] + \rho \nabla \Phi = \mathbf{0}. \quad (3.25)$$

This equation differs from the usual momentum equation for a highly conducting inviscid fluid only through the term  $(p_{||} - p_{\perp})/B^2$ . It is usually referred to as the *Parker modified momentum equation*.

# 3.3 The Double Adiabatic Equations (DAE)

### 3.3.1 Chew, Goldberger, and Low (CGL) solution

(Chew et al. 1956): To complete the momentum equation (3.25), we need equations for the time rate of change of  $p_{\parallel}$  and  $p_{\perp}$ . These equations will take the place of the adiabatic energy equation (1.104), which applies for the isotropic case. From the internal energy equation (1.75), assuming a conducting fluid and omitting heat conduction and external forces, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{3p}{2}\right) + \frac{3p}{2}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) + (\boldsymbol{\mathcal{P}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{V} = 0, \qquad (3.26)$$

where the pressure dyad  $\mathcal{P}$  is given by (3.20) and the scalar pressure p is one-third the trace of  $\mathcal{P}$ ,

$$p = \frac{1}{3} \left( 2p_{\perp} + p_{||} \right). \tag{3.27}$$

Note that 3p/2 represents the *total thermal energy density*. By direct expansion of the last term in the left-hand side of (3.26), using (3.20) for  $\mathcal{P}$ , we find

$$(\mathcal{P} \cdot \boldsymbol{\nabla}) \cdot \boldsymbol{V} = \left[ p_{\perp} \boldsymbol{\nabla} + (p_{\parallel} - p_{\perp}) (\hat{\boldsymbol{B}} \hat{\boldsymbol{B}} \cdot \boldsymbol{\nabla}) \right] \cdot \boldsymbol{V}, \qquad (3.28)$$

and inserting this expression, together with (3.27), into (3.26), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( 2p_{\perp} + p_{\parallel} \right) + \left( 4p_{\perp} + p_{\parallel} \right) \left( \boldsymbol{\nabla} \cdot \boldsymbol{V} \right) + 2(p_{\parallel} - p_{\perp}) (\hat{\boldsymbol{B}} \hat{\boldsymbol{B}} \cdot \boldsymbol{\nabla}) \cdot \boldsymbol{V} = 0.$$
(3.29)

A strong magnetic field constrains the charged particle motion only in the direction transverse to  $\boldsymbol{B}$ , but the particles are still free to move large distances along  $\boldsymbol{B}$ . Thus, it is reasonable to suppose that the contribution to the total thermal energy, arising from the particle motion parallel to  $\boldsymbol{B}$ , also satisfies an energy conservation equation similar to (3.26). This leads to the following equation for the part of the total thermal energy due to the random particle motions along  $\boldsymbol{B}$ :

$$\frac{\mathrm{d}p_{||}}{\mathrm{d}t} + p_{||}\boldsymbol{\nabla}\cdot\boldsymbol{V} + 2p_{||}(\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{V} = 0$$
(3.30)

and, decoupling the parallel and perpendicular motions, the equation for  $p_{\perp}$  becomes

$$\frac{\mathrm{d}p_{\perp}}{\mathrm{d}t} + 2p_{\perp}(\boldsymbol{\nabla}\cdot\boldsymbol{V}) - p_{\perp}(\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{V} = 0.$$
(3.31)

Equations (3.30) and (3.31) enable to calculate  $p_{||}$  and  $p_{\perp}$ . They can be written in a more succinct form, as follows. First we note that if we expand the right-hand side of the induction equation (3.14), using the vector identity  $\nabla \times (\mathbf{V} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{V} - \mathbf{B}(\nabla \cdot \mathbf{V}) - (\mathbf{V} \cdot \nabla)\mathbf{B} + \mathbf{V}(\nabla \cdot \mathbf{B})$ , and noting that  $\nabla \cdot \mathbf{B} = 0$ , we obtain

$$\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = (\boldsymbol{B}\cdot\boldsymbol{\nabla})\boldsymbol{V} - \boldsymbol{B}(\boldsymbol{\nabla}\cdot\boldsymbol{V})$$
(3.32)

If we now take the scalar product of (3.32) with  $\boldsymbol{B}/B^2$ , we obtain

$$\frac{1}{B^2}\frac{\mathrm{d}B^2}{\mathrm{d}t} = \hat{\boldsymbol{B}}\cdot(\hat{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\boldsymbol{V} - \boldsymbol{\nabla}\cdot\boldsymbol{V}, \quad \text{that is,} \quad \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} = (\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{V} - \boldsymbol{\nabla}\cdot\boldsymbol{V}.$$
(3.33)

Equation of continuity (1.19) gives

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = -\frac{1}{\rho_m} \frac{\mathrm{d}\rho_m}{\mathrm{d}t} \tag{3.34}$$

(where  $\rho_m$  is the mass density, to distinguish it from the electric charge density  $\rho$ ) and using Eqs. (3.33) and (3.34), to eliminate the terms  $(\hat{B}\hat{B}\cdot\nabla)\cdot V$  and  $\nabla\cdot V$ , we obtain

$$\frac{\mathrm{d}\ln p_{||}}{\mathrm{d}t} - \frac{3}{\rho_m}\frac{\mathrm{d}\rho_m}{\mathrm{d}t} + \frac{2}{B}\frac{\mathrm{d}B}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}\ln p_{\perp}}{\mathrm{d}t} - \frac{1}{\rho_m}\frac{\mathrm{d}\rho_m}{\mathrm{d}t} - \frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t} = 0, \quad (3.35)$$

which can be written in even compact form as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{p_{||} B^2}{\rho_m^3} \right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{p_{\perp}}{\rho_m B} \right) = 0. \tag{3.36}$$

Equation (3.36) are known as the *double adiabatic equations* for a conducting fluid in a strong magnetic field. They are also known as the Chew, Goldberger, and Low (CGL) equations (Chew et al. 1956). They form the MHD equivalent of the adiabatic energy equation for isotropic plasma:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(p\rho_m^{-\gamma}\right) = 0. \tag{3.37}$$

### 3.3.2 Special Cases of DAE

As a simple application of the double adiabatic equations, consider initially the case in which the only variations are *parallel* to the magnetic field as, for example, in sound waves traveling along the field lines. This situation is usually referred to as *linear compression* parallel to the magnetic field or one-dimensional compression. The magnetic field is assumed to be straight and uniform, and directed along the z-axis. Thus,  $B_x = B_y = 0$  and  $\mathbf{B} = B_z \hat{\mathbf{z}}$ , as well as  $\partial/\partial x = \partial/\partial y = 0$ . In this case, we find

$$(\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{V}=\boldsymbol{\nabla}\cdot\boldsymbol{V}=\frac{\partial V_z}{\partial z}$$
(3.38)

and from Eq. (3.33), we see that B is constant. Equations (3.35), with dB/dt = 0, then yield

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p_{\parallel}}{\rho_m^3}\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p_{\perp}}{\rho_m}\right) = 0. \tag{3.39}$$

If we compare these results with (3.37), we find that we may assign  $\gamma = 3$  along the field lines (one-dimensional compression), and  $\gamma = 1$  across the field lines.

It is useful to introduce a *parallel* and a *perpendicular temperature* through the relations

$$p_{||} = nkT_{||}, \quad p_{\perp} = nkT_{\perp}.$$
 (3.40)

For the case of one-dimensional compression parallel to **B**, noting that  $\rho_m = nm$  in Eq. (3.39), we thus have

$$T_{\parallel} \propto n^2, \quad T_{\perp} = \text{const.},$$
 (3.41)

which shows that this type of compression is isothermal with respect to the perpendicular temperature  $T_{\perp}$ . The perpendicular pressure  $p_{\perp}$  therefore entirely changes due to changes in the number density n, whereas  $p_{\parallel}$  changes due to changes in both n and  $T_{\parallel}$ .

Another special case of interest is the two-dimensional compression perpendicular to the magnetic field, in which all motion is transverse to the field lines. This situation can be pictured as the motion of magnetic flux tubes, identified by the particles contained in them. Assuming straight field lines along the z-axis  $(B_x = B_y = 0, \mathbf{B} = B_z \hat{z})$  and variations only in the transverse direction  $(\partial/\partial z = 0)$ , we find

$$(\hat{\boldsymbol{B}}\hat{\boldsymbol{B}}\cdot\boldsymbol{\nabla})\cdot\boldsymbol{V} = \left(\hat{\boldsymbol{z}}\frac{\partial}{\partial z}\right)\cdot\boldsymbol{V} = 0 \tag{3.42}$$

and Eqs. (3.30) and (3.31) together with the continuity equation (1.19) yield

$$\frac{\mathrm{d}p_{||}}{\mathrm{d}t} - \frac{p_{||}}{\rho_m} \frac{\mathrm{d}\rho_m}{\mathrm{d}t} = 0, \quad \frac{\mathrm{d}p_{\perp}}{\mathrm{d}t} - \frac{2p_{\perp}}{\rho_m} \frac{\mathrm{d}\rho_m}{\mathrm{d}t} = 0.$$
(3.43)

which we write in the compact form as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p_{||}}{\rho_m}\right) = 0, \quad \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p_{\perp}}{\rho_m^2}\right) = 0. \tag{3.44}$$

Comparing with (3.37), we see that  $\gamma = 1$  parallel to the magnetic field and  $\gamma = 2$  transverse to it. From (3.40) we find that for a two-dimensional (cylindrically symmetric) compression perpendicular to **B**,

$$T_{\parallel}$$
 const.,  $T_{\perp} = \propto n,$  (3.45)

so that this type of compression is isothermal with respect to the parallel temperature. The changes in  $p_{||}$  are due entirely to variations in the number density n, whereas those of  $p_{\perp}$  result from variations in n as well as in  $T_{||}$ .

In the case of three-dimensional spherically symmetric compression, we have

$$p_{\perp} = p_{||} = p \tag{3.46}$$

and (3.29) reduces to

$$3\frac{\mathrm{d}p}{\mathrm{d}t} - \frac{5p}{\rho_m}\frac{\mathrm{d}\rho_m}{\mathrm{d}t} = 0, \quad \text{so that} \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho_m^{5/3}}\right) = 0, \tag{3.47}$$

which is again the familiar adiabatic equation (3.37) of gas dynamics, with  $\gamma = 5/3$ . In any of the cases of adiabatic compression, the fluid has to be subjected to a certain system of forces in order to achieve the desired type of adiabatic compression. The required system of forces has to be determined from the momentum equation in conjunction with the conditions appropriate to the analyzed problem.

# 3.4 Magnetic Viscosity and Reynolds Number

The behavior of the magnetic field is important in many MHD problems. To obtain a simple equation for the variations of  $\boldsymbol{B}$ , let us start with the curl of the generalized Ohm's law (3.7),

$$\boldsymbol{\nabla} \times \boldsymbol{J} = \sigma \left[ \boldsymbol{\nabla} \times \boldsymbol{E} + \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) \right], \tag{3.48}$$

where, using Maxwell curl equations (3.2) and (3.4), and the Ampère's law (3.6),

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{B}) = \mu_0 \sigma \left[ -\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) \right].$$
(3.49)

Using the identity  $\nabla \times (\nabla \times B) = \nabla (\nabla \cdot B) - \nabla^2 B$  (with  $\nabla \cdot B = 0$ ), Eq. (3.49) reduces to

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) + \eta_m \boldsymbol{\nabla}^2 \boldsymbol{B}, \qquad (3.50)$$

where  $\eta_m$  is called the magnetic viscosity,

$$\eta_m = \frac{1}{\mu_0 \sigma}.\tag{3.51}$$

This is in fact the extension of the induction equation (3.14), where the first term in the right-hand side of (3.50) is called the *flow term*, while the second term is called the *diffusion term*. To compare the relative magnitude of these two terms, we can use dimensional analysis and approximate,

$$|\boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B})| \simeq \frac{BV}{L}, \quad \eta_m |\boldsymbol{\nabla}^2 \boldsymbol{B}| \simeq \eta_m \frac{B}{L^2},$$
(3.52)

where L denotes some characteristic length for variation of the parameters. The ratio of the flow term to the diffusion term is called the *magnetic Reynolds number* and is given by

$$\mathcal{R}_m = \frac{LV}{\eta_m}.\tag{3.53}$$

In most MHD problems one or the other of these two terms dominates and  $\mathcal{R}_m$  is either very large or very small compared to unity. It is instructive to compare the magnetic viscosity  $\eta_m$ and the magnetic Reynolds number  $\mathcal{R}_m$  with the ordinary hydrodynamic kinematic viscosity  $\nu$  and hydrodynamic Reynolds number  $\mathcal{R}_e = LV/\nu$  (Eq. (1.51)). For this purpose, consider the Navier-Stokes equation of hydrodynamics (1.58), where  $\eta = \rho\nu$  is the dynamic viscosity (kinematic viscosity multiplied by density). Comparing this equation with (3.50) we see that the role played by  $\eta_m$ , in the rate of change of  $\boldsymbol{B}$ , is completely analogous to the role played by  $\nu$ , in the rate of change of the mean fluid velocity  $\boldsymbol{V}$ . The hydrodynamic Reynolds number is defined as the ratio of the inertia term  $(\boldsymbol{V} \cdot \nabla) \boldsymbol{V}$  to the main viscosity term  $\nu \nabla^2 \boldsymbol{V}$  from Eq. (1.58). Using dimensional analysis, we have

$$|(\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}| \simeq \frac{V^2}{L}, \quad \nu|\boldsymbol{\nabla}^2\boldsymbol{V}| \simeq \nu \frac{V}{L^2}, \tag{3.54}$$

which confirms the expression (completely analogous to  $\mathcal{R}_m$ ) for the hydrodynamic Reynolds number,  $\mathcal{R}e = LV/\nu$  (Eq. (1.51)).

# 3.5 Diffusion of Magnetic Field Lines

When  $\mathcal{R}_m \ll 1$ , that is when the diffusion term dominates, Eq. (3.50) becomes approximately,

$$\frac{\partial \boldsymbol{B}}{\partial t} = \eta_m \boldsymbol{\nabla}^2 \boldsymbol{B} \quad (\mathcal{R}_m \ll 1) \,. \tag{3.55}$$

This is the equation of diffusion of a magnetic field in a stationary conductor, resulting in the decay of the magnetic field. It is analogous to the particle diffusion equation studied in Chapter (add). We obtain the characteristic decay time  $\tau_D$  of the magnetic field by dimensional analysis,

$$\left|\frac{\partial \boldsymbol{B}}{\partial t}\right| \simeq \frac{B}{\tau_D}, \quad \eta_m |\boldsymbol{\nabla}^2 \boldsymbol{B}| \simeq \eta_m \frac{B}{L^2}, \tag{3.56}$$

where  $\tau_D$  represents a characteristic time for variation of the plasma parameters. According to (3.55), the magnetic field diffuses away with a characteristic decay time of the order of

$$\tau_D = \frac{L^2}{\eta_m} = L^2 \mu_0 \sigma. \tag{3.57}$$

For ordinary conductors the time of decay is very small. For example, for a copper sphere of radius 1 m, we find that  $\tau_D$  is less than 10 s. For an astronomical body, because of the large dimensions,  $\tau_D$  can be very large. For the Earth's core, considering it to be molten iron, the time of free decay is approximately 10<sup>4</sup> yr, while for the general magnetic field of the sun it is found to be of the order of 10<sup>10</sup> yr.

# 3.6 Freezing of a Magnetic Field

A completely different type of behavior appears when  $\mathcal{R}_m \gg 1$ . In this case, the flow term dominates over the diffusion term and Eq. (3.50) reduces to

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) \quad (\mathcal{R}_m \gg 1).$$
(3.58)

This equation implies that in a highly conducting fluid the magnetic field lines move along exactly with the fluid, rather than simply diffusing out, we say that the magnetic field lines are frozen in the conducting fluid. In effect, the fluid can flow freely along the magnetic field lines, but any motion of the conducting fluid, perpendicular to the field lines, carries them with the fluid. To show this implication of (3.58), we consider initially the concept of magnetic tubes of force that are used to visually describe the direction and magnitude of  $\boldsymbol{B}$  at various points in space. One can think of the space pervaded by a magnetic field as divided into a large number of elementary magnetic tubes of force, all of them enclosing the same magnetic flux  $\Delta \Phi_B$ . If  $\Delta S$  is the local cross-sectional area of an elementary magnetic tube of force (see Fig. 1), then the magnitude of  $\boldsymbol{B}$ , at the local point P, is equal to  $\Delta \Phi_B/\Delta S$ . According to this definition, the magnitude of  $\boldsymbol{B}$  is everywhere inversely proportional to the cross-sectional area of the elementary tube of force.

Let us now consider a closed line whose points move with velocity  $\mathbf{V}$  in a space pervaded by a magnetic field. Assume, for the moment, that  $\mathbf{V}$  is an arbitrary function of position and time (not necessarily equal to the fluid velocity), with the result that the closed curve may change in shape, as well as undergo translational and rotational motion. Let  $C_1$  denote this closed line at time t, bounding the open surface  $\mathbf{S}(t) = \mathbf{S}_1$ . At a time  $\Delta t$  later, let C2 and  $\mathbf{S}(t + \Delta t) = \mathbf{S}_2$  denote the corresponding closed line and open surface (refer to Fig. 2). The flux of the magnetic field through an open surface  $\mathbf{S}$ , at time t, is given by

$$\Phi_B(t) = \int_S \boldsymbol{B}(\boldsymbol{r}, t) \cdot d\boldsymbol{S}.$$
(3.59)

The rate of change of the magnetic flux through an open surface  $\boldsymbol{S}$  can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[ \int_{S} \boldsymbol{B}(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S} \right] = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left[ \int_{S_2} \boldsymbol{B}(\boldsymbol{r},t+\Delta t) \cdot \mathrm{d}\boldsymbol{S} - \int_{S_1} \boldsymbol{B}(\boldsymbol{r},t) \cdot \mathrm{d}\boldsymbol{S} \right].$$
(3.60)

Expanding  $\boldsymbol{B}(\boldsymbol{r},t+\Delta t)$  to the first order about  $B(\boldsymbol{r},t)$ , we obtain

$$B(\mathbf{r}, t + \Delta t) = \mathbf{B}(\mathbf{r}, t) + \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} \Delta t + \dots, \qquad (3.61)$$

so that, in the limit  $\Delta t \to 0$ , the right-hand side of Eq. (3.60) reduces to

$$\lim_{\Delta t \to 0} \left[ \frac{1}{\Delta t} \int_{S_2} B(\boldsymbol{r}, t) \cdot d\boldsymbol{S} + \int_{S_2} \frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t} \cdot d\boldsymbol{S} - \frac{1}{\Delta t} \int_{S_1} \boldsymbol{B}(\boldsymbol{r}, t) \cdot d\boldsymbol{S} \right].$$
(3.62)

To evaluate Eq. (3.62), we use the divergence theorem for any *closed surface*. If we apply this result to the closed surface consisting of  $S_1$ ,  $S_2$ , and the sides of the cylindrical surface of length  $V\Delta t$ , we obtain

$$-\int_{S_1} B(\mathbf{r},t) \cdot \mathrm{d}\mathbf{S} + \int_{S_2} \mathbf{B}(\mathbf{r},t) \cdot \mathrm{d}\mathbf{S} - \oint_{\mathcal{C}_1} \mathbf{B}(\mathbf{r},t) \cdot [\mathbf{V}\Delta t \times \mathrm{d}\mathbf{I}] = 0, \qquad (3.63)$$

where the minus sign in the first term on the left-hand side is due to the fact that the *outwardly* drawn unit vector normal to the surface  $S_1$  is in a direction opposite to that of the surface  $S_2$ , and  $-[\mathbf{V}\Delta t \times d\mathbf{I}]$  is the element of area (pointing outwards) covered by the vector element  $d\mathbf{I}$  of the closed line bounding the surface  $S_1$  (or  $S_2$ ) in the time interval  $\Delta t$ . If (3.63) is substituted into (3.62) and the limit  $\Delta t \to 0$  is evaluated, noting that in this limit  $S_2 = S_1 = S(t)$ , we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[ \int_{S} B(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{S} \right] = \int_{S} \frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t} \cdot \mathrm{d}\boldsymbol{S} + \oint_{\mathcal{C}} \boldsymbol{B}(\boldsymbol{r}, t) \cdot (\boldsymbol{V} \times \mathrm{d}\boldsymbol{I}) \,. \tag{3.64}$$

Using the vector identity  $\boldsymbol{B}(\boldsymbol{r},t) \cdot (\boldsymbol{V} \times d\boldsymbol{l}) = -[\boldsymbol{V} \times \boldsymbol{B}(\boldsymbol{r},t)] \cdot d\boldsymbol{l}$  and the Stokes's theorem, Eq. (3.64) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[ \int_{S} B(\boldsymbol{r}, t) \cdot \mathrm{d}\boldsymbol{S} \right] = \int_{S} \left\{ \frac{\partial \boldsymbol{B}(\boldsymbol{r}, t)}{\partial t} - \boldsymbol{\nabla} \times \left[ \boldsymbol{V} \times \boldsymbol{B}(\boldsymbol{r}, t) \right] \right\} \cdot \mathrm{d}\boldsymbol{S}.$$
(3.65)

Suppose now that the space is filled with a highly conducting fluid so that (3.58), valid for  $\mathcal{R}_m \gg 1$ , applies. If the velocity  $\mathbf{V}$  in (3.65) is the fluid velocity, we conclude, from (3.58) and (3.65), that

$$\frac{\mathrm{d}}{\mathrm{d}t} \left[ \int_{S} B(\boldsymbol{r}, t) \cdot \,\mathrm{d}\boldsymbol{S} \right] = 0, \qquad (3.66)$$

which is a mathematical statement of the fact that the magnetic flux linked by a *closed line* (bounding the open surface S) moving with the fluid velocity V is constant. Note that this conclusion requires that only the velocity component of the *closed line* perpendicular to B be the same as the *fluid velocity* component perpendicular to B, since the velocity component parallel to B does not contribute to the term  $V \times B$ . Thus, (3.58) implies that the lines of magnetic flux are frozen into the highly conducting fluid and are carried by any motion of the fluid *perpendicular* to the field lines. There is no restriction, however, on the motion along the field lines so that the conducting fluid can flow freely in the direction *parallel* to B.

This result is expected on physical grounds since, as the conducting fluid moves across the magnetic field, it induces an electric field that is proportional to the fluid velocity component perpendicular to  $\boldsymbol{B}$ . However, if the fluid *conductivity is infinite*, this perpendicular velocity component must be *infinitesimally small* if the flow of electric current is to remain finite.

In a fluid of *finite conductivity* the result (3.66) is no longer true. Using (3.50) in Eq. (3.65), this yields

$$\frac{\partial \Phi_B}{\partial t} = \eta_m \int_S \boldsymbol{\nabla}^2 \boldsymbol{B} \cdot \,\mathrm{d}\boldsymbol{S},\tag{3.67}$$

where the right-hand side of Eq. (3.67) gives rise to a slipping of magnetic flux through a closed line of the material.

# 3.7 Magnetic Pressure

### 3.7.1 Pressure dyad

The concept of magnetic pressure is very useful in the study of high-temperature plasma confinement. Under steady-state conditions and neglecting the external forces, the MHD equations (3.3), (3.4), and (3.12), reduce to the following closed set of *magnetohydrostatic* equations:

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}, \quad \nabla p = \boldsymbol{J} \times \boldsymbol{B}.$$
 (3.68)

Eliminating  $\boldsymbol{J}$ , we obtain

$$\nabla \cdot \boldsymbol{B} = 0, \quad \nabla p = \frac{1}{\mu_0} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}.$$
 (3.69)

The term in the right-hand side of the first Eq. (3.69) can be written as the divergence of the magnetic part of the electromagnetic stress dyad. Using the vector identity

$$(\boldsymbol{\nabla} \times \boldsymbol{B}) \times \boldsymbol{B} = (\boldsymbol{B} \cdot \boldsymbol{\nabla})\boldsymbol{B} - \frac{1}{2}\boldsymbol{\nabla}(B^2) = \boldsymbol{\nabla} \cdot (\boldsymbol{B}\boldsymbol{B}) - \frac{1}{2}\boldsymbol{\nabla} \cdot (\mathbf{1}B^2)$$
(3.70)

where **1** is the unit dyad, and using the following definition of the magnetic stress dyad,

$$\mathcal{T}^{m} = \frac{1}{\mu_{0}} \left( \boldsymbol{B}\boldsymbol{B} - \frac{1}{2} \mathbf{1}B^{2} \right) = \frac{1}{\mu_{0}} \begin{bmatrix} \left( B_{x}^{2} - B^{2}/2 \right) & B_{x}B_{y} & B_{x}B_{z} \\ B_{y}B_{x} & \left( B_{y}^{2} - B^{2}/2 \right) & B_{y}B_{z} \\ B_{z}B_{x} & B_{z}B_{y} & \left( B_{z}^{2} - B^{2}/2 \right) \end{bmatrix}, \quad (3.71)$$

Eq. (3.69) we write as

$$\nabla p = \boldsymbol{\nabla} \cdot \mathcal{T}^m, \quad \text{or} \quad \boldsymbol{\nabla} \cdot [\mathbf{1}p - \mathcal{T}^m] = 0$$
 (3.72)

Since the stress is considered to be positive if it is tensile, and negative if it is compressive, we may define  $-\mathcal{T}^m$  as the *magnetic pressure dyad*, playing the same role as the fluid pressure dyad.

It is instructive to consider a *local* magnetic coordinate system in which the third axis points along the local direction of **B**. For this local coordinate system, the off-diagonal elements of the magnetic stress dyad vanish, since  $\mathbf{B} = B\hat{\mathbf{z}}$ , so that

$$\mathcal{T}^m = \frac{1}{\mu_0} \begin{pmatrix} -B^2/2 & 0 & 0\\ 0 & -B^2/2 & 0\\ 0 & 0 & B^2/2 \end{pmatrix}.$$
 (3.73)

Therefore, the principal stresses are equivalent to a *tension*  $B^2/(2\mu_0)$  along the magnetic field lines, and a *pressure*  $B^2/(2\mu_0)$  perpendicular to the magnetic field lines, which is similar to a mutual repulsion of the field lines. We express (3.73) alternatively in the form

$$\mathcal{T}^{m} = \frac{1}{\mu_{0}} \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & B^{2} \end{pmatrix} + \frac{1}{\mu_{0}} \begin{pmatrix} -B^{2}/2 & 0 & 0\\ 0 & -B^{2}/2 & 0\\ 0 & 0 & -B^{2}/2 \end{pmatrix},$$
(3.74)

so that the stress caused by the magnetic flux can also be thought of as an *isotropic magnetic* pressure  $B^2/(2\mu_0)$  and a tension  $B^2/\mu_0$  along the magnetic flux lines as if they were elastic cords. The latter representation is very useful, since the isotropic pressure  $B^2/(2\mu_0)$  can always be superposed on the fluid pressure, resulting in a decrease in the pressure exerted by the fluid.

### 3.7.2 Isobaric surfaces

It is convenient to consider hypothetical surfaces, called *isobaric surfaces*, in the plasma, over which the kinetic pressure is constant. At any point, the vector  $\nabla p$  is normal to the isobaric surface passing through the point considered. From the third Eq. (3.68) we see that  $\nabla p$  is normal to the plane containing J and B, that is

$$\boldsymbol{J} \cdot \boldsymbol{\nabla} p = 0, \quad \boldsymbol{B} \cdot \boldsymbol{\nabla} p = 0. \tag{3.75}$$

Therefore, both J and B lie on isobaric surfaces. To illustrate this point, consider the particular case in which the isobaric surfaces are closed concentric cylindrical surfaces, with the kinetic pressure increasing in the direction towards the central axis of the concentric cylindrical surfaces. Thus,  $\nabla p$  is along a radial line directed towards the axis. From Eqs. (3.75) we see that neither B nor J passes through the isobaric surfaces and therefore it follows that the cylindrical isobaric surfaces are formed by a network of magnetic field lines and electric currents. Further, in view of Eq. (3.68), the magnetic field lines and electric currents, lying on the isobaric surfaces, must cross each other in such a way that  $J \times B$  is equal to  $\nabla p$ . The maximum kinetic pressure occurs along the central axis, which also coincides with a magnetic field line. For this reason, this axis is usually called the *magnetic axis* of the magnetoplasma configuration.

# 3.8 Plasma Confinement in a Magnetic Field

The subject of plasma confinement by magnetic fields is of considerable interest in the theory of controlled thermonuclear fusion. Consider, for simplicity, the special case in which the magnetic field is along the z-axis, that is  $\boldsymbol{B} = B_z \hat{\boldsymbol{z}}$ , so that Eq. (3.72) simplifies to

$$\boldsymbol{\nabla} \cdot \begin{pmatrix} p + B^2/(2\mu_0) & 0 & 0\\ 0 & p + B^2/(2\mu_0) & 0\\ 0 & 0 & p - B^2/(2\mu_0) \end{pmatrix} = 0, \quad (3.76)$$

so that, in other words,

$$\frac{\partial}{\partial x}\left(p+\frac{B^2}{2\mu_0}\right) = 0, \quad \frac{\partial}{\partial y}\left(p+\frac{B^2}{2\mu_0}\right) = 0, \quad \frac{\partial}{\partial z}\left(p+\frac{B^2}{2\mu_0}\right) = 0, \quad (3.77)$$

From  $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ , we have

$$\frac{\partial B_z}{\partial z} = 0, \tag{3.78}$$

since, in the local coordinate system, **B** is parallel to the z-axis. Equation (3.78), together with (3.77), imply that both p and **B** do not vary in the z-direction. The solutions of (3.77), combined with this result, give

$$\left(p + \frac{B^2}{2\mu_0}\right) = \text{const.} \tag{3.79}$$

Therefore, in the presence of an externally applied magnetic field, if the plasma is bounded, the plasma kinetic pressure decreases from the axis radially outwards, whereas the magnetic pressure increases in the same direction in such a manner that their sum remains constant at each point, according to (3.79). The plasma kinetic pressure can be forced to vanish on an outer surface if the applied magnetic field is sufficiently strong, with the result that the plasma is confined within this outer surface by the magnetic field.

Let  $B_0$  be the value of the magnetic induction at the plasma boundary. Since the kinetic pressure at the plasma boundary is zero (ideally), we can evaluate the constant in (3.79) from the pressure equilibrium condition at the plasma boundary. Therefore,

$$p + \frac{B^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}.$$
(3.80)

The maximum fluid pressure that can be confined for a given applied field  $B_0$  is,

$$p_{\max} = \frac{B_0^2}{2\mu_0}.$$
(3.81)

A device that can be used to confine a magnetoplasma by straight parallel field lines is called a *theta* ( $\theta$ ) *pinch*, since the effect responsible for the confinement is due to electric currents flowing in the plasma in the azimuthal ( $\theta$ ) direction. The plasma is initially confined inside a hollow cylindrical metal tube, whose side is split in the longitudinal direction in such a way as to form a capacitor. When a high voltage is discharged through the capacitor, the large azimuthal current produced in the metal tube generates a magnetic field in the longitudinal direction inside the plasma. The electric current induced in the plasma is also in the azimuthal direction, but in a sense opposite to that on the metal tube. The resulting  $\mathbf{J} \times \mathbf{B}$  force acting on

the plasma pushes it inwards, towards the axis, until a balance is reached between the kinetic pressure due to the random particle thermal motions and the magnetic pressure that acts to constrict or *pinch* the plasma.

A parameter  $\beta$ , defined as the ratio of the kinetic pressure at a point inside the plasma, to the confining magnetic pressure at the plasma boundary, is usually introduced as a measure of the relative magnitudes of the kinetic and magnetic pressures. It is given by

$$\beta = \frac{p}{B_0^2/(2\mu_0)}.$$
(3.82)

Note that  $\beta$  ranges between 0 and 1, since the field inside the plasma is less than  $B_0$ . From (3.80) we can also express the parameter  $\beta$  as

$$\beta = 1 - \left(\frac{B}{B_0}\right)^2. \tag{3.83}$$

Two special cases of plasma confinement schemes are the so-called *low-* $\beta$  and *high-* $\beta$  devices. In the low- $\beta$  devices, the kinetic pressure is small in comparison to the magnetic pressure at the plasma boundary, whereas in the high- $\beta$  devices they are of an equal order of magnitude ( $\beta \simeq 1$ ).

An important property of a plasma is its *diamagnetic* character. Equation (3.80) implies that the magnetic field inside the plasma is less than its value at the plasma boundary. As the kinetic pressure increases inside the plasma, the magnetic field decreases. Under the action of the externally applied **B** field, the particle motions give rise to internal electric currents that induce a magnetic field opposite to the externally applied field. Consequently, the *resultant* magnetic field inside the plasma is reduced to a value less than that at the plasma boundary. The electric current, induced in the plasma, depends on the number density of the charged particles and on their velocity. As the plasma kinetic pressure increases, the induced electric current and the induced magnetic field also increase, thus enhancing the diamagnetic effect.

# 3.9 MHD Waves

# 3.9.1 Linear perturbations in MHD equations

Consider a static ( $\mathbf{V}_0 = 0$ ), homegenous medium with constant density  $\rho_0$  and pressure  $p_0$  threaded by a uniform magnetic field  $\mathbf{B}_0$ , neglecting other external forces. Assuming small perturbations ( $|q_1| \ll |q_0|$ ), we linearize the perturbed MHD equations (3.11), (3.12), (3.14), and (3.37), respectively, to first order,

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{V}_1 = 0, \qquad (3.84)$$

$$\rho_0 \frac{\partial \mathbf{V}_1}{\partial t} + \mathbf{\nabla} p_1 - \frac{1}{\mu_0} \left( \mathbf{\nabla} \times \mathbf{B}_1 \right) \times \mathbf{B}_0 = 0, \qquad (3.85)$$

$$\frac{\partial \boldsymbol{B}_1}{\partial t} - \boldsymbol{\nabla} \times (\boldsymbol{V}_1 \times \boldsymbol{B}_0) = 0, \qquad (3.86)$$

$$\frac{\partial p_1}{\partial t} + \gamma \rho_0 \boldsymbol{\nabla} \cdot \boldsymbol{V}_1 = 0, \qquad (3.87)$$

where we perform the last equation using Eq. (3.84). Given the linear nature of the system, we can employ a one-dimensional (plane) wave decomposition and write a generic perturbation in

the form  $\propto e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)}$ . Using also the identity  $\nabla \times (\nabla \times \mathbf{A}) = \nabla (\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$  for the triple vector product, the system (3.84) - (3.87) becomes

$$\omega \rho_1 - \rho_0 \boldsymbol{k} \cdot \boldsymbol{V}_1 = 0, \tag{3.88}$$

$$\omega \rho_0 \mathbf{V}_1 - \mathbf{k} p_1 - \frac{1}{\mu_0} \mathbf{k} \left( \mathbf{B}_0 \cdot \mathbf{B}_1 \right) + \frac{1}{\mu_0} \left( \mathbf{k} \cdot \mathbf{B}_0 \right) \mathbf{B}_1 = 0, \qquad (3.89)$$

$$\omega \boldsymbol{B}_1 - (\boldsymbol{k} \cdot \boldsymbol{V}_1) \, \boldsymbol{B}_0 + (\boldsymbol{k} \cdot \boldsymbol{B}_0) \, \boldsymbol{V}_1 = 0, \tag{3.90}$$

$$\omega p_1 - a^2 \rho_0 \mathbf{k} \cdot \mathbf{V}_1 = 0, \tag{3.91}$$

where a is the speed of sound. Note that the first term in the Lorentz force in Eq. (3.89) is related to magnetic pressure while the second one corresponds to magnetic tension.

We subsitute  $\alpha \equiv \mathbf{k} \cdot \mathbf{B}_0$ , multiply Eq. (3.89) by  $\omega$ , express the term  $\omega \mathbf{B}_1$  from Eq. (3.90), and use Eq. (3.91) to eliminate  $p_1$ , obtaining

$$\omega^{2} \rho_{0} \boldsymbol{V}_{1} = a^{2} \rho_{0} \left( \boldsymbol{k} \cdot \boldsymbol{V}_{1} \right) \boldsymbol{k} + \frac{1}{\mu_{0}} \boldsymbol{k} \left\{ \boldsymbol{B}_{0} \cdot \left[ \left( \boldsymbol{k} \cdot \boldsymbol{V}_{1} \right) \boldsymbol{B}_{0} - \alpha \boldsymbol{V}_{1} \right] \right\} - \frac{\alpha}{\mu_{0}} \left[ \left( \boldsymbol{k} \cdot \boldsymbol{V}_{1} \right) \boldsymbol{B}_{0} - \alpha \boldsymbol{V}_{1} \right], \quad (3.92)$$

and we rearrange it into the more illustrative form,

$$\left(\omega^2 \rho_0 - \frac{\alpha^2}{\mu_0}\right) \mathbf{V}_1 = (\mathbf{k} \cdot \mathbf{V}_1) \left[ \left( a^2 \rho_0 + \frac{B_0^2}{\mu_0} \right) \mathbf{k} - \frac{\alpha}{\mu_0} \mathbf{B}_0 \right] - \frac{\alpha}{\mu_0} \left( \mathbf{B}_0 \cdot \mathbf{V}_1 \right) \mathbf{k}.$$
(3.93)

Without loss of generality, we now assume that the equilibrium magnetic field  $B_0$  is directed along the z-axis and that the wave vector  $\mathbf{k}$  lies in the x-z plane  $(k_y = 0)$ . Let  $\theta$  be the angle between  $B_0$  and  $\mathbf{k}$ , so that

$$\boldsymbol{k} = k_x \hat{\boldsymbol{x}} + k_z \hat{\boldsymbol{z}}, \qquad \qquad k_x = k \sin \theta, \qquad \qquad k_z = k \cos \theta \qquad (3.94)$$

$$\boldsymbol{k} \cdot \boldsymbol{V}_1 = k_x V_{1x} + k_z V_{1z}, \qquad \boldsymbol{B}_0 \cdot \boldsymbol{V}_1 = B_0 V_{1z}, \qquad \alpha = k_z B_0, \qquad (3.95)$$

where k is the magnitude of the wavevector  $\mathbf{k}$ . Note also that the *solenoidal condition* in Fourier space becomes  $\mathbf{k} \cdot \mathbf{B}_1 = 0$ , implying that no field perturbation can be developed in the direction of wave propagation.

Dividing Eq. (3.93) by  $\rho_0$  and substituting  $\alpha = \mathbf{k} \cdot \mathbf{B}_0 = k_z B_0 = k_z V_a \sqrt{\mu_0 \rho_0}$  (where  $V_a = |B_0| / \sqrt{\mu_0 \rho}$  is the Alfvén speed), we obtain

$$\left(\omega^{2} - k_{z}^{2}V_{a}^{2}\right)\boldsymbol{V}_{1} - \left(\boldsymbol{k}\cdot\boldsymbol{V}_{1}\right)\left[\left(a^{2} + V_{a}^{2}\right)\boldsymbol{k} - k_{z}V_{a}^{2}\boldsymbol{\hat{z}}\right] + k_{z}V_{a}^{2}V_{1z}\boldsymbol{k} = 0.$$
(3.96)

Equation (3.96) is a linear homogeneous equation in  $V_1$  and, using matrix notation, we rewrite it as

$$\boldsymbol{AV}_1 = 0, \tag{3.97}$$

where the matrix  $\boldsymbol{A}$  is

$$\mathbf{A} = \begin{pmatrix} \omega^2 - k^2 V_{\rm a}^2 - a^2 k_x^2 & 0 & -a^2 k_x k_z \\ 0 & \omega^2 - k_z^2 V_{\rm a}^2 & 0 \\ -a^2 k_z k_x & 0 & \omega^2 - a^2 k_z^2 \end{pmatrix}.$$
 (3.98)

In the first matrix element,  $A_{11}$ , we have used  $-V_{\rm a}^2(k_x^2 + k_z^2) = -k^2 V_{\rm a}^2$ . Equation (3.98) has a non-trivial solution if the determinant of the matrix **A** is zero,

$$\det(\mathbf{A}) = \left(\omega^2 - k_z^2 V_a^2\right) \left[ \left(\omega^2 - k^2 V_a^2 - a^2 k_x^2\right) \left(\omega^2 - a^2 k_z^2\right) - a^4 k_x^2 k_z^2 \right] = 0,$$
(3.99)

which, after simplification, gives the dispersion relation

$$\left[\left(\frac{\omega}{k}\right)^2 - V_{\rm a}^2\cos^2\theta\right] \left[\left(\frac{\omega}{k}\right)^4 - \left(\frac{\omega}{k}\right)^2\left(a^2 + V_{\rm a}^2\right) + a^2V_{\rm a}^2\cos^2\theta\right] = 0.$$
(3.100)

Eq. (3.100) has three independent roots in  $\omega^2$ , corresponding to three different types of waves that can propagate through a magnetized fluid:

### 3.9.2 Alfvén waves

The first root corresponds to the *Alfvén wave* (also known as the *shear Alfvén wave*) and is given by

$$\omega = \pm k V_{\rm a} \cos \theta \equiv \pm k_z V_{\rm a} \equiv \pm \frac{\mathbf{k} \cdot \mathbf{B}_0}{\sqrt{\mu_0 \rho_0}} \tag{3.101}$$

and the corresponding eigenvector from Eq. (3.98) is  $\mathbf{V}_1 = (0, V_{1y}, 0)$ . Thus the velocity perturbation must lie in the y-direction and therefore  $\mathbf{k} \cdot \mathbf{V}_1 = \mathbf{V}_1 \cdot \mathbf{B}_0 = 0$ . By looking at Eqs. (3.88) and (3.91), we see that this wave carries zero perturbation in density and pressure and it has an *incompressible* nature. This does not mean that the plasma is incompressible, but just the Alfvén wave carries oscillations in velocity and magnetic field only. In addition, since  $\mathbf{k} \cdot \mathbf{B}_1 = 0$  always, perturbations in both velocity and magnetic field are always orthogonal to the direction of wave propagation: the Alfvén wave mode is a *transverse* wave. In addition, from Eqs. (3.90), with use of Eq. (3.101), we obtain a relation between the perturbations of magnetic field and velocity:

$$\boldsymbol{B}_1 = -\frac{\boldsymbol{k} \cdot \boldsymbol{B}_0}{\omega} \boldsymbol{V}_1, \quad \text{so that} \quad \frac{\boldsymbol{B}_1}{B_0} = \mp \frac{\boldsymbol{V}_1}{V_a}. \tag{3.102}$$

Taking the square of the previous relation yields

$$\frac{\rho_0 \mathbf{V}_1^2}{\mathbf{B}_1^2/\mu_0} = 1, \tag{3.103}$$

that is, the perturbation carries equal kinetic and magnetic energy contributions. We also note that taking the scalar product of Eq. (3.102) with  $\boldsymbol{B}_0$ , we also have  $\boldsymbol{B}_0 \cdot \boldsymbol{B}_1 = 0$  which means that Alfvén waves are generated by magnetic tension only, thereby strenghtening the analogy between a field line and an elastic string.

Finally, we notice that the Alfvén wave solution found so far, is also an exact solution of the *full* MHD equations without requiring that  $|\boldsymbol{B}_1| \ll |\boldsymbol{B}_0|$  but assuming that

$$|\boldsymbol{B}_0 + \boldsymbol{B}_1| = \text{const.}, \quad \frac{\boldsymbol{B}_1}{B_0} = \pm \frac{\boldsymbol{V}_1}{V_a}.$$
(3.104)

Large perturbations of velocity and magnetic field can be related to nonlinear Alfvén wave modes, as observed in the solar wind.

### 3.9.3 Fast and slow magnetosonic waves

The other two roots are given by

$$\left(\frac{\omega}{k}\right)^2 = \frac{a^2 + V_{\rm a}^2 \pm \sqrt{(a^2 + V_{\rm a}^2)^2 - 4V_{\rm a}^2 a^2 \cos^2\theta}}{2}.$$
(3.105)

The two solutions are always real and correspond to the fast magnetosonic (+) and slow magnetosonic (-) waves. The corresponding eigenvectors, from Eq. (3.98), lie in the x-z plane,  $\mathbf{V}_1 = (V_{1x}, 0, V_{1z})$ . As a consequence, we have  $\mathbf{k} \cdot \mathbf{V}_1 \neq 0$ , and, similarly,  $\mathbf{V}_1 \cdot \mathbf{B}_0 \neq 0$ : these waves are compressive in nature and involve plasma motion in both the parallel and the perpendicular field direction.

Notice that, in general  $(\omega/k)_{\text{slow}} \leq a$  while  $(\omega/k)_{\text{fast}} \geq a$ . The properties of these waves depend on the ratio between  $a^2$  and  $V_a^2$  and on the relative orientation between k and  $B_0$ .

• For parallel propagation  $(\mathbf{k} || \mathbf{B}_0 \text{ or } \theta = 0)$  the matrix  $\mathbf{A}$  in Eq. (3.98) is diagonal and the solutions given by Eq. (3.105) are reduced to

$$\left(\frac{\omega}{k}\right)_{\text{fast}}^{2} = \frac{a^{2} + V_{\text{a}}^{2} + |V_{\text{a}}^{2} - a^{2}|}{2} = \begin{cases} a^{2} & \text{for } a > V_{\text{a}} \\ V_{\text{a}}^{2} & \text{for } a < V_{\text{a}} \end{cases},$$
(3.106)

$$\left(\frac{\omega}{k}\right)_{\text{slow}}^{2} = \frac{a^{2} + V_{\text{a}}^{2} - |V_{\text{a}}^{2} - a^{2}|}{2} = \begin{cases} V_{\text{a}}^{2} & \text{for } a > V_{\text{a}} \\ a^{2} & \text{for } a < V_{\text{a}} \end{cases}.$$
 (3.107)

In a weakly magnetized medium  $(a > V_a)$ , the *fast* magnetosonic wave becomes an *acoustic* (or *sound*) wave whereas the *slow* mode propagates at the Alfvén speed. Conversely, in a strongly magnetized medium where  $V_a > a$ , the fast and slow modes propagate at the Alfvén and sound speed, respectively.

Note that the acoustic mode  $(\omega/k = \pm a)$  is characterized by  $\mathbf{V}_1 = (0, 0, V_{1z}) \propto \mathbf{k}$  and by no field perturbation,  $(\mathbf{B}_1 = 0)$ , the wave is longitudinal. For  $\omega/k = \pm V_a$ , implying  $\mathbf{V}_1 = (V_{1x}, 0, 0)$ , and the wave becomes identical to a transverse Alfvén wave.

• For perpendicular propagation  $(\mathbf{k} \perp \mathbf{B}_0 \text{ or } \theta = \pm \pi/2)$  the matrix  $\mathbf{A}$  in Eq. (3.98) is again diagonal and the solutions of dispersion relation are reduced to

$$\left(\frac{\omega}{k}\right)_{\text{fast}}^2 = \frac{a^2 + V_a^2 + |a^2 + V_a^2|}{2} = a^2 + V_a^2, \qquad (3.108)$$

$$\left(\frac{\omega}{k}\right)_{\text{slow}}^2 = \frac{a^2 + V_a^2 - |a^2 + V_a^2|}{2} = 0.$$
(3.109)

Thus the *fast* mode becomes the magnetoacoustic wave with phase velocity equal to  $\sqrt{a^2 + V_a^2}$ : this is a longitudinal wave ( $\mathbf{k} \propto \mathbf{V}_1$ , as it can be verified from Eq. (3.89), keeping in mind that  $\mathbf{k} \cdot \mathbf{B}_0 = 0$ ) and it is driven by magnetic pressure. Magnetic perturbations develop along the background field (see Eq. (3.90)) and consist of compression and rarefaction of the field without line bending. On the contrary, the slow waves tend to zero.

As a final remark we note that taking the scalar product of Eq. (3.89) with  $\boldsymbol{k}$ , using (3.91) to express  $\boldsymbol{k} \cdot \boldsymbol{V}_1$ , and remembering that  $\boldsymbol{k} \cdot \boldsymbol{B}_1 = 0$ , we obtain

$$p_1\left(\frac{\omega^2}{k^2 a^2} - 1\right) = \frac{1}{\mu_0} \boldsymbol{B}_0 \cdot \boldsymbol{B}_1.$$
(3.110)

The previous equation shows that for *fast* waves,  $(\omega^2/k^2 \ge a^2)$ , pressure and magnetic fluctuations have the same sign and tend to reinforce one another. However, for *slow* waves,  $(\omega^2/k^2 \le a^2)$ , an increase of gas pressure is accompanied by a decrease of magnetic pressure, and vice versa.

# 3.10 Magnetorotational Instability (MRI)

### 3.10.1 Linear analysis of MRI

The problem of the hydrodynamic stability of a fluid, subjected to a magnetic field and rotation, has been studied for a long time. The analysis of MRI is however first systematically described in Balbus & Hawley (1991). Because of the ubiquity of magnetic fields can the turbulences of magnetized rotating matter (the gas is partially or fully ionized) act as a main source of anomalous viscosity. The matter is subjected to very strong shear instabilities caused by weak magnetic field that can be far more destabilizing than a strong one (the strong field would rather enforce the matter to rotate as a rigid body). Arbitrarily small magnetic field cannot therefore be neglected in linear analysis of the disk disturbances.

The basic destabilizing mechanism act as follows: consider a differentially rotating material that is perpendicularly threaded by magnetic field, whose field lines are therefore "vertically" oriented. The motion of the volume element that is displaced in the outward direction from its orbit, is elastically controlled by the magnetic field. The field is trying to eliminate the effects caused by shear friction between radial rotating segments by enforcement of rigid rotation while it simultaneously returns this element back to its original position (and thus eliminates stretching). The second effect is stabilizing, while the first effect acts as the source of the instability. Magnetic field is trying to force the gas element to rotate too fast for its new radial location, the excess of centrifugal force drives the element further outward. At sufficiently long wavelengths (longer than a critical wavelength that corresponds to a critical wavenumber) is the returning force too weak and destabilization wins. The presence of the finite value of the vertical wavenumber of the magnetic field is essential, otherwise no axisymmetric instability occurs. The MRI generates the viscous couple in the rotating matter, caused by an interpenetration of the gas volume elements with higher and lower angular momentum, leading to turbulence.

The dispersion relation is described in Balbus & Hawley (1991) with the following assumptions: the radial component of the magnetic field is in the linear analysis of the axisymmetric case set to zero ( $B_R = 0$ ), while in the more advanced study there is analyzed also the case with the nonzero radial component ( $B_R \neq 0$ ). The behavior of the fluid is subject to the Boussinesq approximation (Boussinesq 1897), which is considered to be valid for the incompressible disturbances of interest. This approximation assumes that the variations of density are negligible (we set  $\rho \rightarrow \rho_0 = \text{const.}$  in the continuity equation and in the advection term of equation of motion). However, the weak density variations are important for buoyancy, so that we retain the density variations in the right-hand side of the equation of motion and in the equation of state, while the pressure perturbations are neglected in the equation of state (Fricke 1969). In fluid flows driven by buoyancy (buoyancy-driven flows) are the density perturbations connected only with the reduced gravity via the equation

$$\delta g = \frac{\delta \rho}{\rho} g$$
 with  $\delta \rho = 0$  otherwise, (3.111)

the fluid is thus essentially incompressible.

The Boussinesq approximation enables us to eliminate acoustic waves, since acoustic waves are the density perturbations. We consider constant angular velocity  $\Omega$  on radial (cylindrical) segments with  $V_R \approx 0$ ,  $V_z \approx 0$  and  $\partial \Omega / \partial z = 0$ . Axisymmetric ( $\partial / \partial \phi = 0$ ) Eulerian space-time dependent perturbation  $\delta \xi$  of a general quantity  $\xi$  with radial and vertical wavenumbers  $k_R$  and  $k_z$  and the angular frequency  $\omega$  of the MRI can be described as

$$\delta\xi = \xi_0 \,\mathrm{e}^{i(k_R R + k_z z - \omega t)}.\tag{3.112}$$

The set of seven basic ideal MHD equations, that is, equation of continuity (3.11), three momentum equations (3.12) and three Maxwell-Faraday (induction) equations (3.14), we expand to first order. Since the density in the continuity equation does not vary, we set  $\nabla \cdot \boldsymbol{V} = \boldsymbol{0}$ , and the equation takes the explicit form (with only the largest terms retained)

$$\frac{1}{R}\frac{\partial}{\partial R}\left[R(V_R+\delta V_r)\right] + \frac{\partial}{\partial z}(V_z+\delta V_z) = 0.$$
(3.113)

We do not take into account the constant unperturbed radial and vertical components of velocity and we also neglect any variations in radius R. After linearization of perturbations according to Eq. (3.112), we write Eq. (3.113) as

$$k_R \delta V_R + k_z \delta V_z = 0. \tag{3.114}$$

The radial component of momentum equation (3.12), where the terms that contain the radial derivative of the radial magnetic field,  $\partial B_R/\partial R$ , cancel, is

$$\frac{\partial V_R}{\partial t} + V_R \frac{\partial V_R}{\partial R} + \frac{V_{\phi}}{R} \frac{\partial V_R}{\partial \phi} + V_z \frac{\partial V_R}{\partial z} - \frac{V_{\phi}^2}{R} + \frac{1}{\rho} \frac{\partial p}{\partial R} + \frac{1}{2\mu_0 \rho} \frac{\partial}{\partial R} \left(B_{\phi}^2 + B_z^2\right) - \frac{1}{\mu_0 \rho} \left(\frac{B_{\phi}}{R} \frac{\partial B_R}{\partial \phi} + B_z \frac{\partial B_R}{\partial z} - \frac{B_{\phi}^2}{R}\right) + g_R = 0.$$
(3.115)

According to the Boussinesq approximation (3.111), the radial component of the density perturbation (assuming the hydrostatic equilibrium) is

$$\delta g_R = -\frac{\delta\rho}{\rho^2} \frac{\partial p}{\partial R}.$$
(3.116)

After linearization of perturbations according to Eq. (3.112), with use of Eq. (3.116), we write

$$-\mathrm{i}\omega\delta V_R - 2\Omega\delta V_\phi + \frac{\mathrm{i}k_R}{\rho}\delta p + \frac{\mathrm{i}k_R}{\mu_0\rho}\left(B_\phi\delta B_\phi + B_z\delta B_z\right) - \frac{\mathrm{i}k_z}{\mu_0\rho}B_z\delta B_R - \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial R} = 0.$$
(3.117)

In a similar way we analyze the vertical component of the momentum equation (3.12). Its explicit form (after cancellation of the terms containing  $\partial B_z/\partial z$ ) is

$$\frac{\partial V_z}{\partial t} + V_R \frac{\partial V_z}{\partial R} + \frac{V_\phi}{R} \frac{\partial V_z}{\partial \phi} + V_z \frac{\partial V_z}{\partial z} + \frac{1}{\rho} \frac{\partial p}{\partial z} + \frac{1}{2\mu_0 \rho} \frac{\partial}{\partial z} \left( B_R^2 + B_\phi^2 \right) - \frac{1}{\mu_0 \rho} \left( B_R \frac{\partial B_z}{\partial R} + \frac{B_\phi}{R} \frac{\partial B_z}{\partial \phi} \right) + g'_z = 0.$$
(3.118)

The same linearization of perturbations, using the analog of Eq. (3.116), leads to

$$-\mathrm{i}\omega\delta V_z + \frac{\mathrm{i}k_z}{\rho}\delta p + \frac{\mathrm{i}k_z}{\mu_0\rho}B_\phi\delta B_\phi - \frac{\delta\rho}{\rho^2}\frac{\partial p}{\partial z} = 0.$$
(3.119)

The explicit form of the azimuthal momentum equation (3.12), omitting the negligible viscous terms (we employ only the scalar pressure, while the terms containing the derivative  $\partial B_{\phi}/\partial \phi$  cancel), is

$$\frac{\partial V_{\phi}}{\partial t} + V_R \frac{\partial V_{\phi}}{\partial R} + \frac{V_{\phi}}{R} \frac{\partial V_{\phi}}{\partial \phi} + V_z \frac{\partial V_{\phi}}{\partial z} + \frac{V_R V_{\phi}}{R} + \frac{1}{\rho} \frac{\partial p}{\partial \phi} + \frac{1}{2\mu_0 \rho} \frac{1}{R} \frac{\partial}{\partial \phi} \left( B_R^2 + B_z^2 \right) - \frac{1}{\mu_0 \rho} \left( B_R \frac{\partial B_{\phi}}{\partial R} + B_z \frac{\partial B_{\phi}}{\partial z} + \frac{B_R B_{\phi}}{R} \right) = 0.$$
(3.120)

Linearization of perturbations in Eq. (3.120), according to Eq. (3.112), leads to

$$-\mathrm{i}\omega\delta V_{\phi} + \delta V_R \left(\frac{\partial V_{\phi}}{\partial R} + \frac{V_{\phi}}{R}\right) - \frac{\mathrm{i}k_z}{\mu_0\rho} B_z \delta B_{\phi} = 0.$$
(3.121)

We rewrite the term in bracket in Eq. (3.121) as  $2\Omega + R d\Omega/dR = \kappa^2/(2\Omega)$ , where  $\kappa$  is the *epicyclic frequency*,  $\kappa^2 = 4\Omega^2 + d\Omega^2/d(\ln R)$ . We derived the epicyclic frequency by considering a small radial displacement of an arbitrary particle (body or fluid parcel), orbiting at the circular trajectory with radius  $R_0$  in a gravitational potential  $\Phi$ . The acceleration of the displacement, described by its only nonzero radial component,  $a_R = \ddot{R} - R\dot{\phi}^2$ . However, since  $a_R$  is the derivative of  $\Phi$ ,  $a_R = -\partial \Phi/\partial R$ , we obtain  $\ddot{R} = -\partial \Phi/\partial R + j^2/R^3$ , where j is the specific angular momentum of the displaced body. Expansion of the radial acceleration  $\ddot{R}$  to first order in  $R_0$  leads to

$$\ddot{R} + \left[ \frac{\partial^2 \Phi}{\partial R^2} \Big|_{R_0} + \frac{3j^2}{R_0^4} \right] (R - R_0) = 0, \qquad (3.122)$$

where the term in bracket represents the square of the epicyclic frequency,  $\kappa^2$ .

Equation (3.121) we rewrite into the simplified form

$$-i\omega\delta V_{\phi} + \frac{\kappa^2}{2\Omega}\delta V_R - \frac{ik_z}{\mu_0\rho}B_z\delta B_{\phi} = 0.$$
(3.123)

We rewrite the radial component of the induction equation (3.14), using the vector identity  $\nabla \times (\mathbf{V} \times \mathbf{B}) = \mathbf{V}(\nabla \cdot \mathbf{B}) + (\mathbf{B} \cdot \nabla)\mathbf{V} - (\mathbf{V} \cdot \nabla)\mathbf{B} - \mathbf{B}(\nabla \cdot \mathbf{V})$ , and involving the Maxwell equation  $\nabla \cdot \mathbf{B} = 0$ ,

$$\frac{\partial B_R}{\partial t} - \frac{B_{\phi}}{R} \frac{\partial V_R}{\partial \phi} - B_z \frac{\partial V_R}{\partial z} + V_R \frac{\partial B_R}{\partial R} + \frac{V_{\phi}}{R} \frac{\partial B_R}{\partial \phi} + V_z \frac{\partial B_R}{\partial z} + B_R \left(\frac{1}{R} \frac{\partial V_{\phi}}{\partial \phi} + \frac{\partial V_z}{\partial z} + \frac{V_R}{R}\right) = 0.$$
(3.124)

Linearization of perturbations in Eq. (3.124), according to Eq. (3.112) and with use of Eq. (3.116), taking into account the above constraints (including the assumption  $B_R = 0$ ), leads to

$$-\mathrm{i}\omega\delta B_R - \mathrm{i}k_z B_z \delta V_R = 0. \tag{3.125}$$

The vertical component of the induction equation (3.14) is

$$\frac{\partial B_z}{\partial t} - B_R \frac{\partial V_z}{\partial R} - \frac{B_\phi}{R} \frac{\partial V_z}{\partial \phi} + V_R \frac{\partial B_z}{\partial R} + \frac{V_\phi}{R} \frac{\partial B_z}{\partial \phi} + V_z \frac{\partial B_z}{\partial z} + B_z \left(\frac{\partial V_R}{\partial R} + \frac{1}{R} \frac{\partial V_\phi}{\partial \phi} + \frac{V_R}{R}\right) = 0.$$
(3.126)

Linearization of perturbations in Eq. (3.126) according to Eq. (3.112) and with use of Eq. (3.116), including the same constraints as in Eq. (3.125), gives

$$-\mathrm{i}\omega\delta B_z + \mathrm{i}k_R B_z \delta V_R = 0, \quad \text{so that (see Eq. (3.114))} \quad -\mathrm{i}\omega\delta B_z - \mathrm{i}k_z B_z \delta V_z = 0, \quad (3.127)$$

while the azimuthal component of the induction equation (3.14) is

$$\frac{\partial B_{\phi}}{\partial t} - B_R \frac{\partial V_{\phi}}{\partial R} - B_z \frac{\partial V_{\phi}}{\partial z} + V_R \frac{\partial B_{\phi}}{\partial R} + \frac{V_{\phi}}{R} \frac{\partial B_{\phi}}{\partial \phi} + V_z \frac{\partial B_{\phi}}{\partial z} + \frac{B_R V_{\phi}}{R} + B_{\phi} \left(\frac{\partial V_R}{\partial R} + \frac{\partial V_z}{\partial z}\right) = 0.$$
(3.128)

Linearization of perturbations in Eq. (3.128), according to Eq. (3.112), with use of Eqs. (3.116) and (3.114), including the same constraints as in Eq. (3.125), gives

$$-\mathrm{i}\omega\delta B_{\phi} - \frac{\partial V_{\phi}}{\partial R}\delta B_R - \mathrm{i}k_z B_z \delta V_{\phi} + \frac{V_{\phi}}{R}\delta B_R = 0, \qquad (3.129)$$

which can be further simplified as

$$-\mathrm{i}\omega\delta B_{\phi} - \frac{d\Omega}{d\ln R}\delta B_R - \mathrm{i}k_z B_z \delta V_{\phi} = 0.$$
(3.130)

The set of Eqs. (3.114), (3.117), (3.119), (3.123), (3.125), (3.127), and (3.130) is closed by conditions of entropy of the adiabatic perturbations (where s is the density of the entropy (1.83)),

$$\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{\partial s}{\partial t} + \boldsymbol{V} \cdot \boldsymbol{\nabla}s = 0.$$
(3.131)

The first law of thermodynamics  $(d\epsilon = c_V dT)$  for the ideal gas gives

$$ds = c_V \frac{dT}{T} - \mathcal{R} \frac{d\rho}{\rho} = c_V \left(\frac{dp}{p} - \gamma \frac{d\rho}{\rho}\right).$$
(3.132)

By integrating Eq. (3.132) in case of monoatomic gas ( $\gamma = 5/3$ ) between two states "0" and "1", we obtain

$$\Delta s = c_V \ln\left[\frac{p_1}{p_0} \left(\frac{\rho_1}{\rho_0}\right)^{-5/3}\right], \quad \text{so that} \quad s = c_V \ln\left(p\rho^{-5/3}\right), \quad (3.133)$$

where s is the entropy of an ideal gas with constant specific heats to within an arbitrary constant of integration (Zel'dovich & Raizer 1967).

Linearization of isentropic perturbations (i.e., of a process that is reversible and adiabatic, the entropy of a considered system therefore does not change) in Eq. (3.133), according to Eq. (3.112) with use of Eq. (3.116), valid for the Boussinesq approximation, gives

$$i\omega \frac{5}{3} \frac{\delta\rho}{\rho} + \delta V_R \frac{\partial \ln \left(p\rho^{-5/3}\right)}{\partial R} + \delta V_z \frac{\partial \ln \left(p\rho^{-5/3}\right)}{\partial z} = 0.$$
(3.134)

Further strategy is to express all perturbations in terms of  $\delta V_z$ , by eliminating all terms  $\delta V_R$ . We can rewrite Eq. (3.134) with use of Eq. (3.114),

$$\frac{\delta\rho}{\rho} = \frac{3}{5i\omega} \delta V_z \left( \frac{k_z}{k_R} \frac{\partial \ln \left( p\rho^{-5/3} \right)}{\partial R} - \frac{\partial \ln \left( p\rho^{-5/3} \right)}{\partial z} \right).$$
(3.135)

Combining Eqs. (3.119) and (3.135), we obtain

$$\frac{\delta p}{\rho} + \frac{B_{\phi} \,\delta B_{\phi}}{\mu_0 \rho} = \frac{\delta V_z}{k_z} \left[ \omega - \frac{3}{5\omega} \frac{1}{\rho} \frac{\partial p}{\partial z} \left( \frac{k_z}{k_R} \frac{\partial \ln \left( p \rho^{-5/3} \right)}{\partial R} - \frac{\partial \ln \left( p \rho^{-5/3} \right)}{\partial z} \right) \right]. \tag{3.136}$$

Using the expression for  $B_{\phi}$  from Eq. (3.130), we write

$$\delta V_{\phi} = \frac{\delta V_z}{\mathrm{i}\omega} \frac{k_z}{k_R} \left( -\frac{\kappa^2}{2\Omega} + \frac{k_z^2 V_{Az}^2}{\omega^2} \frac{d\Omega}{d\ln R} \right) \left( 1 - \frac{k_z^2 V_{Az}^2}{\omega^2} \right)^{-1}, \qquad (3.137)$$

where we introduce the Alfvén speed (Bittencourt 2004),  $V_{Az}^2 = B_z^2/(\mu_0 \rho)$  in SI units. With use of Eq. (3.114) we rewrite Eqs. (3.125) and (3.127) as

$$\delta B_R = \frac{k_z^2}{k_R} B_z \frac{\delta V_z}{\omega}, \quad \delta B_z = -k_z B_z \frac{\delta V_z}{\omega}.$$
(3.138)

By substituting Eq. (3.137) and Eq. (3.138) into Eq. (3.130), we obtain

$$\delta B_{\phi} = 2\Omega \frac{B_z}{\mathrm{i}\omega^2} \frac{k_z^2}{k_R} \left(1 - \frac{k_z^2 V_{Az}^2}{\omega^2}\right)^{-1} \delta V_z. \tag{3.139}$$

Substituting Eqs. (3.135)-(3.138) into Eq. (3.115) and simplifying, we obtain the dispersion relation

$$\tilde{\omega}^{4} + \frac{k_{z}^{2}}{k^{2}} \left[ \frac{3}{5\rho} \left( \frac{k_{R}}{k_{z}} \frac{\partial p}{\partial z} - \frac{\partial p}{\partial R} \right) \left( \frac{k_{R}}{k_{z}} \frac{\partial \ln \left( p\rho^{-5/3} \right)}{\partial z} - \frac{\partial \ln \left( p\rho^{-5/3} \right)}{\partial R} \right) - \kappa^{2} \right] \tilde{\omega}^{2} - 4\Omega^{2} \frac{k_{z}^{4} V_{Az}^{2}}{k^{2}} = 0, \qquad (3.140)$$

where  $\tilde{\omega}^2 = \omega^2 - k_z^2 V_{Az}^2$  and  $k^2 = k_R^2 + k_z^2$ . We can yet simplify the relation by setting

$$\frac{\partial p}{\partial z} \frac{\partial \ln \left( p \rho^{-5/3} \right)}{\partial R} = \frac{\partial p}{\partial R} \frac{\partial \ln \left( p \rho^{-5/3} \right)}{\partial z}, \qquad (3.141)$$

which follows from the assumption of rotation on cylinders, or equivalently, that isobaric and isochoric surfaces coincide (Balbus & Hawley 1991).

The Brunt-Väisälä frequency is defined as a frequency of the fluid parcel with density  $\rho_{\text{int}}$  that oscillates due to small displacement  $\xi' = \xi - \xi_0$  around equilibrium position  $\xi_0$  in surrounding medium with density  $\rho_{\text{ext}}$ , where  $\xi$  is the general coordinate direction. If the fluid parcel is displaced along the coordinate  $\xi$  and the motion is adiabatic without viscous effects, the equation of motion is

$$\rho_{\rm int} \ddot{\xi} = -g \left[ \rho_{\rm int} - \rho_{\rm ext} \right]. \tag{3.142}$$

Expanding the right-hand side of Eq. (3.142) to first order in  $\xi$  around the equilibrium position  $\xi_0$ , we obtain the equation of harmonic oscillator

$$\ddot{\xi}' + \left(\frac{g}{\rho_{\rm int}} \frac{\partial \Delta \rho}{\partial \xi} \bigg|_{\xi_0}\right) \xi' = 0, \qquad (3.143)$$

where  $\Delta \rho = \rho_{\text{int}} - \rho_{\text{ext}}$ . The term in bracket in Eq. (3.143) represents the square of the *Brunt-Väisälä* frequency  $N_{\xi}^2$  that corresponds to oscillations in the direction of the coordinate  $\xi$  (in case of  $\partial(\Delta \rho)/\partial\xi < 0$  we obtain unstable solution, diverging to infinity).

We assume the adiabatic behavior of fluid parcel interior, we also consider the pressure equilibrium of the fluid parcel with surrounding medium,  $p_{\text{int}} = p_{\text{ext}}$  (this approximation is valid only for the subsonic motion). Including the definition of the adiabatic exponent  $\gamma = (d \ln p/d \ln \rho_{\text{int}})_{\text{ad}}$  from Sect. 1.6, we obtain from Eq. (3.143) the adiabatic expression for the Brunt-Väisälä frequency in the form

$$N_{\xi,\,\mathrm{ad}}^2 = g\left(\frac{1}{\gamma p}\frac{\partial p}{\partial \xi} - \frac{1}{\rho}\frac{\partial \rho}{\partial \xi}\right),\tag{3.144}$$

where  $\rho_{\text{ext}}$  we hereafter denote as  $\rho$ . Applying the latter expression for the Brunt-Väisälä frequency for the monoatomic ideal gas in hydrostatic equilibrium, we simplify Eq. (3.141) into the form (for example for the piece  $N_z$ ),

$$-\frac{3}{5\rho}\frac{\partial p}{\partial z}\frac{\partial \ln\left(p\rho^{-5/3}\right)}{\partial z} = N_z^2.$$
(3.145)

We thus obtain the expression for the piece of the Brunt-Väisälä frequency that corresponds to vertical oscillations. Analogously we obtain the expression for the piece of the Brunt-Väisälä frequency that corresponds to radial oscillations,  $N_R$ . The quantities  $N_R$  and  $N_z$  are the pieces of a scalar quantity N, (see Eq. (3.145))

$$N^{2} = -\frac{3}{5\rho} \left( \nabla p \right) \cdot \left[ \nabla \ln \left( p \rho^{-5/3} \right) \right] = N_{R}^{2} + N_{z}^{2}.$$
(3.146)

Using the pieces of Brunt-Väisälä frequency, we express the dispersion relation (3.140) as

$$\frac{k^2}{k_z^2}\tilde{\omega}^4 - \left[\kappa^2 + \left(\frac{k_R}{k_z}N_z - N_R\right)^2\right]\tilde{\omega}^2 - 4\Omega^2 k_z^2 V_{Az}^2 = 0.$$
(3.147)

From Eq. (3.140) follows that only z-component of magnetic field enters the dispersion relation (as a part of term  $V_{Az}$  or as a part of term  $\tilde{\omega}$ ), and that it is always multiplied by the wavenumber  $k_z$ . The importance of arbitrarily small magnetic fields can thus be readily understood: strong magnetic tension forces can be generated at sufficiently small perturbation wavelengths. We also see that by absence of the magnetic field the wavenumbers are not scaled: internal waves propagate with a frequency that depends only on the direction of the wavenumber. The presence of the magnetic field however enables us to establish the inverse length scale for the wavenumbers,  $\Omega/V_{Az}$ . By normalizing the components of the wavenumber  $\mathbf{k}$  with use of the characteristic value  $\Omega/V_{Az}$ , we can completely scale the magnetic field out of the problem. Only the values of the wavenumbers that are relative to the scaled characteristic  $\Omega/V_{Az}$  play a role, not the values of magnetic field induction themselves.

We can also analyze the more general case with nonzero radial component of magnetic field  $(B_R \neq 0)$ . Considering the ideal MHD Faraday's law of electromagnetic induction (3.14), assuming axial symmetry  $(\partial/\partial \phi = 0)$  with  $\Omega = \Omega(R)$ , and neglecting  $V_R$  and  $V_z$  in Eqs. (3.124), (3.126), (3.128), we have the only relevant field freezing equation,

$$\frac{\partial B_{\phi}}{\partial t} = B_R \frac{\partial V_{\phi}}{\partial R} - \frac{B_R V_{\phi}}{R} = B_R \frac{\mathrm{d}\Omega}{\mathrm{d}\ln R}.$$
(3.148)

Since  $B_R$  does not change with time, the solution of the equation (3.148) is

$$B_{\phi}(t) = B_{\phi}(0) \left[ 1 + \frac{B_R}{B_{\phi}(0)} \frac{\mathrm{d}\Omega}{\mathrm{d}\ln R} t \right].$$
 (3.149)

The presence of radial field component leads to a linear growth of  $B_{\phi}$  with time in the unperturbed disk. However, since the azimuthal field component is not present in the dispersion relation (3.147) and the inclusion of radial field component does not change that (the  $\omega$  frequency is also not explicitly time-dependent), no generality is lost by considering only the special case  $B_R = 0$ .

# 3.10.2 Analysis of stability limit of perturbations caused by shears

Since Eq. (3.147) is a quadratic relation for the squared scaled angular MRI frequency  $\tilde{\omega}^2$  (and thus for  $\omega^2$ ), it is always a real and continuous function of its parameters in the dispersion relation. We further investigate the stability of the weakly magnetized disk by conditions in the neighborhood of the values  $\omega^2 = 0$  or  $\tilde{\omega}^2 = -k_z^2 V_{Az}^2$ . In this limit the equation (3.147) is written as

$$k_R^2 \left( k_z^2 V_{Az}^2 + N_z^2 \right) - 2k_R k_z N_R N_z + k_z^2 \left( \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} + N_R^2 + k_z^2 V_{Az}^2 \right) = 0.$$
(3.150)

We regard Eq. (3.150) as a quadratic equation for  $k_R$ , noting that this equation would not allow real solutions for  $k_R$  in case its discriminant D is negative,

$$D \equiv -\left[k_z^4 V_{Az}^4 + k_z^2 V_{Az}^2 \left(N^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R}\right) + N_z^2 \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R}\right],\tag{3.151}$$

and thereby assuring stability, since  $\omega^2$  could not then pass through zero. This requirement of stability we express as

$$-D > 0.$$
 (3.152)

From the assumption  $N_z^2 > 0$ , the inequality (3.152) holds for all non-vanishing  $k_z$  by satisfying

$$\frac{\mathrm{d}\Omega^2}{\mathrm{d}R} \ge 0,\tag{3.153}$$

which we regard as the criterion of stability. The violation of this criterion leads to instability for  $k_z < k_{z, \text{crit}}$ . The value  $k_{z, \text{crit}}$  we obtain from Eq. (3.151) by setting D = 0,

$$(k_z)_{\rm crit}^2 = \frac{1}{2V_{Az}^2} \left\{ \left[ \left( N^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} \right)^2 - 4N_z^2 \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} \right]^{1/2} - \left[ N^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} \right] \right\}.$$
 (3.154)

If  $N_R^2 \ll N_z^2$ , the critical vertical wavenumber becomes

$$\left| \left( k_z \right)_{\text{crit}} \right| = \frac{1}{V_{Az}} \left| \frac{\mathrm{d}\Omega^2}{\mathrm{d}\ln R} \right|^{1/2}.$$
(3.155)

Moreover, if the Brunt-Väisälä frequency  $N^2 = 0$  or if it is quite negligible (noting that the square root of the quadratic term  $(d\Omega^2/d\ln R)^2$  may become negative), the solution of Eq. (3.154) becomes (3.155). If  $N_z^2 = 0$ , the criterion of stability is

$$N_R^2 + \frac{\mathrm{d}\Omega^2}{\mathrm{d}R} \ge 0. \tag{3.156}$$

In case of supersonic rotational velocity (where  $N_R$  becomes negligible), the relation (3.156) equals the criterion expressed in Eq. (3.153).

# 3.11 Rigid Rotators

Force equilibrium for a mass element  $\delta m$  located in the position r in stellar proximity is given by the balance of the components of the gravitational and centrifugal forces tangential to the local magnetic field line (Preuss et al. 2004),

$$(\boldsymbol{F}_G + \boldsymbol{F}_C) \cdot \boldsymbol{B} = \boldsymbol{0}. \tag{3.157}$$

We may write the expressions for centrifugal force in cylindrical frame and in spherical frame, respectively, in the form (where the boldface-typed quantities with "hat" are unit vectors)

$$\boldsymbol{F}_{C} = \delta m \,\Omega^{2} R \hat{\boldsymbol{R}}, \quad \boldsymbol{F}_{C} = \delta m \,\Omega^{2} r \left[ \hat{\boldsymbol{r}} - \hat{\boldsymbol{\Omega}} \left( \hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{\Omega}} \right) \right]. \tag{3.158}$$

Considering the dipole stellar magnetic field, we denote the angle between magnetic and rotation axes as  $\psi$  and the azimuthal angle of the magnetic moment vector  $\boldsymbol{m}$  as  $\phi$ . Magnetic dipole field with a magnetic dipole moment  $\boldsymbol{m} = m\hat{\boldsymbol{m}}$  in spherical coordinates is

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0 m}{4\pi r^3} \left[ 3\left(\boldsymbol{\hat{m}} \cdot \boldsymbol{\hat{r}}\right) \boldsymbol{\hat{r}} - \boldsymbol{\hat{m}} \right].$$
(3.159)

Inserting Eqs. (3.158) (spherical equation) and (3.159) together with the expression for the gravitational force  $\mathbf{F}_G = -\hat{\mathbf{r}} G M_{\star} \delta m/r^2$  in Eq. (3.157) consequently yields (Preuss et al. 2004),

$$\left[\hat{\boldsymbol{r}}\left(1-\frac{GM_{\star}}{\Omega^{2}r^{3}}\right)-\hat{\boldsymbol{\Omega}}\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\Omega}}\right)\right]\cdot\left[3\left(\hat{\boldsymbol{m}}\cdot\hat{\boldsymbol{r}}\right)\hat{\boldsymbol{r}}-\hat{\boldsymbol{m}}\right]=0,$$
(3.160)

where  $(GM_{\star}/\Omega^2)^{1/3} = R_{co}$  denotes the *corotation radius*. We distinguish three possible configurations of distribution of magnetically confined circumstellar matter in oblique rotators:

• Aligned rotator where  $\psi = 0$  ( $\hat{\boldsymbol{m}} = \hat{\boldsymbol{\Omega}}$ ) gives the following equilibrium condition for the confined matter,

$$\left\{2\left[1-\left(\frac{R_{\rm co}}{r}\right)^3\right]-3\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\Omega}}\right)^2+1\right\}\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\Omega}}\right)=0.$$
(3.161)

Equation (3.161) implies two solutions:

- 1.  $\cos \theta \equiv \hat{\mathbf{r}} \cdot \hat{\mathbf{\Omega}} = 0$ : accumulation of matter in (coinciding magnetic and rotational) equatorial plane of the star.
- 2.  $\hat{\mathbf{r}} \cdot \hat{\mathbf{\Omega}} \neq 0$ ,  $\cos^2 \theta = 1 \frac{2}{3} \left(\frac{R_{\rm co}}{r}\right)^3$ : matter accumulation in chimney-shaped surfaces above and below the equatorial plane whose axes coincide with the stellar rotational axis. This solution only exists for  $r \geq (2/3)^{1/3}R_{\rm co}$ , the stability tests however show that the "chimney" solution is unstable while the equatorial solution is stable for  $r > R_{\rm co}$ .
- Perpendicular rotator,  $\psi = \pi/2$  ( $\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{\Omega}} = 0$ ) where Eq. (3.160) gives the following equilibrium,

$$\left\{2\left[1-\left(\frac{R_{\rm co}}{r}\right)^3\right]-3\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\Omega}}\right)^2\right\}\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{m}}\right)=0.$$
(3.162)

Equation (3.162) again implies two solutions:

- 1.  $\cos \theta \equiv \hat{\mathbf{r}} \cdot \hat{\mathbf{m}} = 0$ : the solution corresponds to the equatorial plane, however now with respect to the magnetic axis.
- 2.  $\hat{\boldsymbol{r}} \cdot \hat{\boldsymbol{m}} \neq 0$ ,  $\cos^2 \theta = \frac{2}{3} \left[ 1 \left( \frac{R_{\rm co}}{r} \right)^3 \right]$ : chimney-shaped structure of confined matter that is axisymmetric with respect to the rotation axis of the star. The solution only exists for  $r > R_{\rm co}$ . The stability analysis in this case shows that both solutions are unstable near the star (up to 1-2  $R_{\rm co}$ ) while they are stable further out.
- Oblique rotators where the equilibrium condition from Eq. (3.160) gives

$$\left\{ \left[ 2\left(1 - \left(\frac{R_{\rm co}}{r}\right)^3\right) - 3\left(\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\Omega}}\right)^2 \right] \hat{\boldsymbol{m}} + \cos\psi\,\hat{\boldsymbol{\Omega}} \right\} \cdot \hat{\boldsymbol{r}} = 0.$$
(3.163)

This indicates more complicated structure of the equilibrium regions: in case of small  $\psi$  there forms a disk-like and a chimney- shaped structure (with a disk plane in the magnetic equatorial plane and with the the chimney axis tilted with respect to the rotation axis) while for large  $\psi$  the chimney shapes become more curved and tilted and the disk becomes somewhat warped (see (Preuss et al. 2004) for details).

The study of Preuss et al. (2004) presents a formulation of a strong magnetic field limit based on the condition of the balance of forces that are tangential to the field lines and maps out the complex surfaces on which the circumstellar material can accumulate (Townsend & Owocki 2005).

# 3.11.1 Rigidly rotating magnetosphere (RRM) model

The model of a magnetosphere that is at rest in a corotating reference frame whose basic principles follow the considerations of Townsend & Owocki (2005): if the magnetic field line potential  $\Psi(s)$  exhibits an extremum along the field line (where s is the coordinate direction along the field line), so that  $d\Psi/ds \equiv \Psi' = 0$  at some point, then the plasma parcel remains at rest. Whether the parcel can remain at such an equilibrium point at rest over significant timescales depends however on the nature of the extremum. At a local maximum where  $d^2\Psi/ds^2 \equiv \Psi'' < 0$  the equilibrium is unstable: small displacements away from the extremal point perpetually grow. On the other hand, at a local minimum with  $\Psi'' > 0$ , the equilibrium is stable: any small displacement along the local magnetic field line produces a restoring force directed toward the equilibrium point. Such minima represent the locations for circumstellar matter to accumulate, it forms a magnetosphere that is at rest in a corotating reference frame.

Comparing the potentials that arise from Eq. 3.157: within the Roche limit (where the most of the stellar mass is assumed to be concentrated centrally with a spherically symmetric distribution) the effective potential  $\Psi$  is in spherical coordinates  $(r, \theta, \phi)$  given by

$$\Psi(r,\theta) = -\frac{GM_{\star}}{r(\theta)} - \frac{1}{2}\Omega^2 r^2(\theta)\sin^2\theta.$$
(3.164)

Using the dimensionless coordinate  $\xi = r/R_{co}$ , Eq. (3.164) becomes

$$\Psi(\xi) = \frac{GM_{\star}}{R_{\rm co}} \left( -\frac{1}{\xi} - \frac{1}{2}\xi^2 \sin^2 \theta \right).$$
(3.165)

We introduce the dimensionless potential  $\Xi$ , independent of the angular (rotational) velocity  $\Omega$ :

$$\Xi(\xi) = \frac{R_{\rm co}}{GM_{\star}}\Psi(\xi) = -\frac{1}{\xi} - \frac{1}{2}\xi^2 \sin^2\theta.$$
(3.166)

In Eq. (3.166) we can identify two regimes: if r is much smaller than the corotation radius  $R_{\rm co}$  ( $\xi \ll 1$ ), the potential  $\Xi$  is spherically symmetric and increases outwards. Conversely, if r greatly exceeds  $R_{\rm co}$  ( $\xi \sin \theta \gg 1$ ), the potential  $\Xi$  exhibits the cylindrical symmetry about the same axis and decreases outwards. In the latter regime the field line potential exhibits the minimum near which circumstellar plasma accumulates.

In a corotating spherical frame with aligned axes Eq. (3.159) becomes

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{m}{r^3} \left( 2\cos\theta \,\hat{\boldsymbol{r}} + \sin\theta \,\hat{\boldsymbol{\theta}} \right). \tag{3.167}$$

We obtain the projection  $B_s$  of the field measured along the field line as  $B_s = (B_r^2 + B_\theta^2)^{1/2} = B$ ,

$$B = \frac{\mu_0}{4\pi} \frac{m}{r^3} \sqrt{1 + 3\cos^2\theta}.$$
 (3.168)

By integrating the spherical field-line identity  $dr/B_r = r d\theta/B_\theta$ , where from Eq. (3.167) follows  $B_r/B_\theta = 2\cos\theta/\sin\theta$ , we obtain the parametric equation

$$\xi = \gamma \sin^2 \theta, \tag{3.169}$$

where the parameter  $\gamma$  specifies the maximum radius  $R_{\text{peak}}$  ( $\theta = \pi/2$ ) of the field line in units of the corotation radius  $\gamma = R_{\text{peak}}/R_{\text{co}}$ . Inserting Eq. (3.169) into Eq. (3.166), we obtain the dimensionless potential of a dipole field line in case of *aligned* dipole configuration ( $\hat{\boldsymbol{m}} \cdot \hat{\boldsymbol{\Omega}} = 1$ ),

$$\Xi(\theta) = -\frac{1}{\gamma \sin^2 \theta} - \frac{1}{2} \gamma^2 \sin^6 \theta, \qquad (3.170)$$

By integrating the field-line identity  $ds/B = r d\theta/B_{\theta}$ , from Eqs. (3.167), (3.169) and (3.168) we obtain

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = R_{\mathrm{co}}\gamma\sin\theta\sqrt{1+3\cos^2\theta}.\tag{3.171}$$

For  $\theta = \pi/2$  we use the identity  $\xi = \gamma$ . Differentiation  $\Xi'' = (\theta')^2 (d^2 \Xi/d\theta^2) + \theta''(d\Xi/d\theta)$  gives

$$\frac{\mathrm{d}^2\Xi}{\mathrm{d}s^2} = \Xi'' = \frac{1}{R_{\rm co}^2} \left(-\frac{2}{\xi^3} + 3\right). \tag{3.172}$$

Since  $\Xi''$  must be positive in order to constitute an accumulation surface, the inner truncation radius is thus given by  $\xi_{in} \approx 0.87$  at which  $\Xi''$  changes from positive ( $\xi > \xi_{in}$ ) to negative ( $\xi < \xi_{in}$ ) values. Throughout the region in the equatorial plane between this truncation radius  $\xi_{in}$  and the corotation radius ( $\xi = 1$ ) magnetic tension supports material against the net inward pull caused by gravity that exceeds here the centrifugal force. Beyond this region, when ( $\xi > 1$ ) the centrifugal force surpasses gravity and the effect of magnetic tension holds the material down against the net outward pull (Townsend & Owocki 2005). In Keplerian disks the gravitational and centrifugal force is in exact balance, this is not required in a RRM inasmuch the magnetic tension can absorb any net resultant force perpendicular to field lines.

Hydrostatic stratification along the field line is governed by the equation of hydrostatic balance  $dP/ds = -\rho d\Psi/ds$  where the gas pressure is given by Eq. (1.101). For simplicity we assume the constant temperature T, by integrating the hydrostatic equilibrium condition we obtain the density distribution along the field line,

$$\rho(s) = \rho_m \exp\left[-\mu m_{\rm u} \frac{\Psi(s) - \Psi_m}{kT}\right],\tag{3.173}$$

where the subscript m denotes the value at the potential minimum where  $s = s_m$ . Taylor expansion of the effective potential  $\Psi(s)$  about this minimum gives

$$\Psi(s) = \Psi_m + \frac{1}{2}\Psi''(s - s_m)^2 + \dots, \qquad (3.174)$$

where we have used the fact that by the definition  $\Psi'_m = 0$ . In the neighborhood of the minimum the density distribution Eq. (3.173) may be thus well approximated by

$$\rho(s) \approx \rho_m \exp\left[-\mu m_{\rm u} \frac{\Psi''(s-s_m)^2}{2kT}\right] \approx \rho_m \exp\left[-\frac{(s-s_m)^2}{h_m^2}\right].$$
(3.175)

The density scale height  $h_m$  of the RRM (using Eq. (3.166) and Eq. (3.172) that gives  $\Xi'' = 3/R_{co}^2$  for  $\xi \gg 1$ ) therefore is

$$h_m = \sqrt{\frac{2kT}{\mu m_{\rm u}} \frac{1}{\Psi''}} = \sqrt{\frac{2kT}{\mu m_{\rm u}} \frac{R_{\rm co}}{GM_{\star}}} \sqrt{\frac{1}{\Xi''}}, \qquad h_m = \sqrt{\frac{2kT}{3\mu m_{\rm u}GM_{\star}}} R_{\rm co}^{3/2} \text{ for } r \gg R_{\rm co}.$$
 (3.176)

Similar vertical stratification,  $H \sim R^{3/2}$ , formally applies for Keplerian disks, however, for the RRM this remains constant even far from the origin (it does not produce the flaring disk). By integrating Eq. (3.175) over the Gaussian hydrostatic stratification, we obtain the relation for the local surface density  $\sigma_m$ 

$$\sigma_m = \int_{-\infty}^{\infty} \rho(s) \,\mathrm{d}s \approx \mu_m \,\rho_m \int_{-\infty}^{\infty} \,\exp\left[-\frac{(s-s_m)^2}{h_m^2}\right] \,\mathrm{d}s, \quad \text{i.e.} \quad \sigma_m \approx \mu_m \,\rho_m \sqrt{\pi} h_m, \quad (3.177)$$

where  $\mu_m$  denotes the projection cosine to the surface normal.

Model of the global distribution of the surface density that is proportional to the accumulation rate of material loaded from the star's radiatively driven wind has been proposed by Townsend & Owocki (2005): for a dipole flux-tube bundle intersecting the stellar surface at  $r = R_{\star}$  with a projection cosine  $\mu_{\star}$  and having a cross-sectional area  $dA_{\star}$ , the rate of mass increase is

$$\dot{m} = \frac{2\mu_{\star}\dot{M}}{4\pi R_{\star}^2} \,\mathrm{d}A_{\star},\tag{3.178}$$

where the factor 2 takes into account the mass injection at two distinct footpoints. Considering the simple case with a single minimum at field line coordinate  $s_m$  where the flux-tube area is  $dA_m$  and the projection cosine to the accumulation surface normal is  $\mu_m$ , the corresponding rate of increase of the surface density (where  $\dot{\sigma}_m dA_m = \dot{m}\mu_m$ ) can be written as

$$\dot{\sigma}_m = \mu_m \frac{2\mu_\star \dot{M}}{4\pi R_\star^2} \frac{\mathrm{d}A_\star}{\mathrm{d}A_m}.$$
(3.179)

Due to the divergence free constraint  $\nabla \cdot \boldsymbol{B} = 0$  we have the identity  $dA_{\star}B_{\star} = dA_mB_m$ , whose substitution into Eq. (3.179) gives

$$\dot{\sigma}_m = \mu_m \frac{2\mu_\star M}{4\pi R_\star^2} \frac{B_m}{B_\star}.$$
(3.180)

For a dipole field thus the material feeding rate of the disk obviously declines with radius, according to  $\dot{\sigma}_m \sim B \sim r^{-3}$  (cf. Eq. (3.167), see Townsend & Owocki (2005) for further details).

# 3.12 Dynamo Effect

Dynamo theory describes the process through which a rotating, convecting, and electrically conducting fluid acts to maintain a magnetic field. This theory is used to explain the presence of *anomalously long-lived* magnetic fields in astrophysical bodies. The conductive fluid in the *geodynamo* is liquid iron in the outer core, and in the *solar dynamo* it is an ionized gas in the transition region between the radiative interior and the differentially rotating outer convective zone. Dynamo theory of astrophysical bodies uses magnetohydrodynamic equations to investigate how the fluid can continuously regenerate the magnetic field.

There are three factors necessary for a dynamo to operate:

- An electrically conductive fluid medium
- Kinetic energy provided by rotation of the body
- An internal energy source to drive convective motions within the fluid.

For example, in case of the Earth, the magnetic field is induced and constantly maintained by the convection of liquid iron in the outer core. Rotation in the outer core is supplied by the Coriolis effect caused by the rotation of the Earth. The coriolis force tends to organize fluid motions and electric currents into columns aligned with the rotation axis. Induction or creation of magnetic field is described by the *induction equation* (3.50).

### 3.12.1 Equation of motion in a rotating frame

We now extend the expressions (1.32) - (1.34) derived in Sect. 1.3.1 by adding the terms for non-inertial (fictitious) forces in a rotating frame with a fixed axis of rotation that coincides with the z-axis in a static frame. Denoting the quantities in rotating frame as primed and the quantities in static (inertial) frame as unprimed (due to fixed rotation axis we have  $R' \equiv R$  for the magnitudes of cylindrical position vectors), we obtain the vector of velocity

$$\boldsymbol{V} = \boldsymbol{V}' + \boldsymbol{\Omega} \times \boldsymbol{R}', \tag{3.181}$$

where  $\Omega$  is the angular velocity of the rotating frame (cf. Sect. A.2.3 in Kurfürst (2017)). Substituting the velocity  $\mathbf{V}'$  into the Lagrangian  $\mathcal{L}$  of the free particle in the inertial frame,

$$\mathcal{L} = mV^2/2, \tag{3.182}$$

and differentiating it, we obtain the expression for the acceleration term in the rotating frame,

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} = \frac{\mathrm{d}\boldsymbol{V}'}{\mathrm{d}t} + \boldsymbol{\Omega} \times \left(\boldsymbol{\Omega} \times \boldsymbol{R}'\right) + \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}t} \times \boldsymbol{R}' + 2\boldsymbol{\Omega} \times \boldsymbol{V}'.$$
(3.183)

The second term in the right-hand side of Eq. (3.183) represents the *centrifugal acceleration* that in case of a stationary axisymmetric rotation can be written as

$$\Omega^2 R' = V_{\phi}^2 / R' = V_{\phi}^2 / R, \qquad (3.184)$$

where  $V_{\phi}$  is the azimuthal (rotation) velocity of the rotating frame point in the distance R from the axis. The third term in the right-hand side of Eq. (3.183) is the so-called *Euler acceleration* that in uniformly rotating frame vanishes. The last term is the *Coriolis acceleration* that in case of a stationary axisymmetric rotation is perpendicular to  $\mathbf{V}'$  and can be written as

$$2\Omega V' = 2V\phi V'/R. \tag{3.185}$$

### 3.12.2 Kinematic dynamo theory

In kinematic dynamo theory the velocity field is prescribed, instead of being a dynamic variable. This method cannot provide the time variable behavior of a fully nonlinear chaotic dynamo but it is useful in studying how magnetic field strength varies with the flow structure and rotational speed. Using Maxwell's equations simultaneously with the curl of Ohm's Law, one can derive what is basically the linear eigenvalue equation for magnetic fields  $\boldsymbol{B}$  which can be done when assuming that the magnetic field is independent from the velocity field. One arrives at a critical magnetic field, and below which it decays. The most functional feature of kinematic dynamo theory is that it can be used to test whether a velocity field is or is not capable of dynamo action. By applying a certain velocity field to a small magnetic field, it can be determined through observation whether the magnetic field tends to grow or not in reaction to the applied flow. If the magnetic field does grow, then the system is either capable of dynamo action or is a dynamo, but if the magnetic field does not grow, then it is simply referred to as non-dynamo.

# 3.12.3 Nonlinear dynamo theory

The kinematic approximation becomes invalid when the magnetic field becomes strong enough to affect the fluid motions. In that case the velocity field becomes affected by the Lorentz force, and so the induction equation is no longer linear in the magnetic field. In most cases this leads to a quenching of the amplitude of the dynamo. Such dynamos are sometimes also referred to as *hydromagnetic dynamos*.

Virtually all dynamos in astrophysics and geophysics are hydromagnetic dynamos. In fact, we need numerical approach to simulate fully nonlinear dynamos where a minimum of five following equations are needed: The induction equation in the form (3.50),

$$\frac{\partial \boldsymbol{B}}{\partial t} = \boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B}) + \eta_m \boldsymbol{\nabla}^2 \boldsymbol{B}, \qquad (3.186)$$

Maxwell magnetic field constraint (3.3).

$$\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0, \tag{3.187}$$

We use the simplification given by *Boussinesq approximation* (see Eq. (3.111) and Sect. 3.10), which reduces the continuity equation to

$$\boldsymbol{\nabla} \cdot \boldsymbol{V} = 0, \tag{3.188}$$

in which density variations are ignored except where they are multiplied by the gravitational acceleration  $\boldsymbol{g}$  so that the buoyancy forces can be included. Although this approximation may not be always strictly valid (for example, in the Earth's core the density variations of order 20% can occur), it is nevertheless a useful simplification of equations which are difficult to solve.

Assuming now that we are in the uniformly rotating frame of reference, we write the equation of motion (3.12), using Eq. (3.183) and omitting the prime notation, as

$$\rho \left[ \frac{\mathrm{d} \boldsymbol{V}}{\mathrm{d} t} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{R}) + 2\boldsymbol{\Omega} \times \boldsymbol{V} \right] = -\boldsymbol{\nabla} \cdot \boldsymbol{\mathcal{P}} + \boldsymbol{J} \times \boldsymbol{B} + \rho \boldsymbol{g}, \qquad (3.189)$$

where  $\mathbf{g} = -\nabla \Phi$  is the gravitational acceleration. Finally, we involve a heat transport equation (1.90), noting that it particularly fits the Boussinesq approximation (3.188),

$$\frac{\mathrm{d}T}{\mathrm{d}t} = D\boldsymbol{\nabla}^2 T + q_R,\tag{3.190}$$

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where D is the thermal diffusivity and the term  $q_R$  refers to non-conduction heat sources.

Following the definition of perturbations in Eqs. (3.84) together with the Boussinesq approximation (3.111) and using the simplified Navier-Stokes equation (1.58), the equation of motion (3.189) becomes

$$\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{R}) + 2\boldsymbol{\Omega} \times \boldsymbol{V} \approx \frac{\rho_1}{\rho_0} \boldsymbol{g} - \frac{1}{\rho_0} \boldsymbol{\nabla} p + \nu \boldsymbol{\nabla}^2 \boldsymbol{V} + \frac{1}{\rho_0} \boldsymbol{J} \times \boldsymbol{B}, \quad (3.191)$$

where  $\nu = \eta/\rho_0$  is the kinematic viscosity,  $\rho_1$  is the density perturbation that provides buoyancy, and **J** is the electric current density.

Multiplying Eq. (3.191) by  $\rho_0 \mathbf{V}$  gives the rate of increase of kinetic energy density,  $\rho_0 V^2/2$ , on the left-hand side. The last term on the right-hand side then is  $\mathbf{V} \cdot (\mathbf{J} \times \mathbf{B}) = \mathbf{V} \cdot \mathbf{F}_L$ , which represents the local contribution to the kinetic energy due to Lorentz force. However, following Eqs. (3.15) and (3.16), we obtain

$$\boldsymbol{V} \cdot \boldsymbol{F}_{L} = -\frac{\partial}{\partial t} \left( \frac{B^{2}}{2\mu_{0}} \right), \qquad (3.192)$$

because the last term in the right-hand side of Eq. (3.16) obviously vanishes due to the Maxwell equation (3.187) and due to the Boussinesq approximation (3.188).

The scalar product of the induction equation (3.186) with  $\boldsymbol{B}/\mu_0$  gives the rate of increase of the magnetic energy density,  $B^2/(2\mu_0)$ , on the left-hand side. The first term on the right-hand side of (3.186) is then  $\boldsymbol{B} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B})]/\mu_0$ . Following the discussion in Sect. 3.1 and using Eq. (3.2), we have

$$\frac{1}{\mu_0} \boldsymbol{B} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{V} \times \boldsymbol{B})] = -\frac{1}{\mu_0} \boldsymbol{B} \cdot [\boldsymbol{\nabla} \times \boldsymbol{E}] = \frac{\partial}{\partial t} \left(\frac{B^2}{2\mu_0}\right)$$
(3.193)

Comparing Eqs. (3.192) and (3.193), we see that the term  $-\mathbf{V} \cdot \mathbf{F}_L = -\mathbf{V} \cdot (\mathbf{J} \times \mathbf{B})$  represents the rate of transformation of kinetic energy to magnetic energy. This has to be non-negative at least in part of the volume, for the dynamo to produce magnetic field.

Equation (3.193) for the rate of conversion of kinetic energy to magnetic energy, is equivalent to a rate of work (power) done by a Lorentz force  $\mathbf{F}_L = \mathbf{J} \times \mathbf{B}$  on the matter, whose velocity is  $\mathbf{V}$ . This work is the result of nonconservative, non-inertial, and non-magnetic forces acting on the fluid (particularly of the Coriolis force in case of the geomagnetic field).

A number of non-dimensional parameters can be found from the previous equations. The Ekman number,  $E = \nu/(2\Omega L^2 \cos \theta)$ , where L is a characteristic length scale of a phenomenon and  $\theta$  is the colatitude, measures the relative importance of the viscous force to the Coriolis force. The Prandtl number  $P_r = \nu/D$  and the magnetic Prandtl number  $P_m = \nu/\eta_m$ , where L is a characteristic length scale of a phenomenon and  $\theta$  is the colatitude, measures the relative importance of the viscous force to the Coriolis force. Particularly useful in the geophysics is the Elsasser number,  $\Lambda$ , that represents the ratio of the Lorentz force to the Coriolis force,  $\Lambda = JB/(2\rho\Omega V) = \sigma B^2/(2\rho\Omega)$ , where  $\sigma$  is the conductivity of the fluid, B is the magnetic field induction,  $\rho$  is the mass density, and  $\Omega$  is the angular velocity of rotation of the body.

# Chapter 4

# **Radiative Transfer**

(Rybicki & Lightman 1979; Mihalas & Mihalas 1984): Electromagnetic radiation can be decomposed into a spectrum that corresponds to waves of various wavelengths and frequencies, related by  $\nu\lambda = c$ , where  $\nu$  is the frequency,  $\lambda$  is the wavelength, and  $c \approx 3 \times 10^8 \text{ m s}^{-1}$  is the velocity of light in vacuum (for waves not traveling in a vacuum, c is replaced by the velocity of the wave in the medium). We divide the spectrum into various regions. The frequency dependent energy  $E = h\nu$  and temperature T = E/k, where h is Planck constant  $\approx 6.625 \times 10^{-34} \text{ J s}$ , and k is Boltzmann constant  $\approx 1.38 \times 10^{-23} \text{ J K}^{-1}$ .

# 4.1 Radiative Flux

# 4.1.1 Macroscopic Description of the Propagation of Radiation

When the scale of a system greatly exceeds the wavelength of radiation (e.g., light shining through a keyhole), we can consider radiation to travel in straight lines (rays) in free space or homogeneous media - from this fact a transfer theory can be built. One of the most essential concepts is that of energy flux: consider an element of area dA exposed to radiation for a time dt. The amount of energy passing through the element should be proportional to dA dt, and we quantify it as F dA dt where the energy flux F is measured in J s<sup>-1</sup> m<sup>-2</sup>. Note that F depends on the orientation of the element.

### 4.1.2 Flux from an Isotropic Source - the Inverse Square Law

A source of radiation is called *isotropic* if it emits energy equally in all directions. As an example we take a spherically symmetric, isolated star. If we put imaginary spherical surfaces  $S_1$ , and Sat radii  $r_1$  and r, respectively, about the source, conservation of energy dictates the total energy passing through  $S_1$  must be the same as that passing through S (we assume for simplicity no energy losses or gains between  $S_1$ , and S). Regarding a fixed sphere  $S_1$ ,

$$F(r_1) 4\pi r_1^2 = F(r) 4\pi r^2$$
, so that  $F(r) = \frac{\text{const.}}{r^2}$ . (4.1)



Figure 4.1: Geometry for normally incident rays.

# 4.2 The Specific Intensity and its Moments

# 4.2.1 Specific Intensity

The flux measures an amount of the energy carried by all rays passing through a given area. A more detailed description of radiation is to carry energy by individual rays: Construct an area dA normal to the direction of the given ray and consider all rays passing through dA directed within a solid angle  $d\Omega$  of the given ray (see Fig. 4.1). The energy crossing dA in time dt and in frequency range  $d\nu$  is defined by

$$dE = I_{\nu} \, dA \, dt \, d\Omega \, d\nu, \tag{4.2}$$

where  $I_{\nu}(\nu, \Omega)$  [J s<sup>-1</sup> m<sup>-2</sup> ster<sup>-1</sup> Hz<sup>-1</sup>] is the *specific intensity* or *brightness*. Note that  $I_{\nu}$  depends on position, on direction, and on frequency.

# 4.2.2 Net Flux and Momentum Flux

Suppose a radiation field (rays in all directions) and construct a small area element dA at arbitrary orientation **n** (see Fig. 4.2). The differential amount of flux (reduced according to the lowered effective area  $dA \cos \theta$ ) from the solid angle  $d\Omega$  is

$$\mathrm{d}F_{\nu}\left(\mathrm{J}\,\mathrm{s}^{-1}\,\mathrm{m}^{-2}\,\mathrm{Hz}^{-1}\right) = I_{\nu}\cos\theta\,\mathrm{d}\Omega. \tag{4.3}$$

The net flux  $F_{\nu}(\mathbf{n})$  in the direction  $\mathbf{n}$  is

$$F_{\nu} = \int_{\Omega} I_{\nu} \cos \theta \,\mathrm{d}\Omega. \tag{4.4}$$

If  $I_{\nu}$  is an isotropic (angle independent) radiation field, then  $F_{\nu} = 0$  because  $\int_{\Omega} \cos \theta \, d\Omega = 0$ . In other words, the same amount of energy crosses dA in the **n** direction as in the **-n** direction.

To get the flux of momentum normal to dA (noting that momentum per unit time per unit area = pressure) we involve the photon momentum E/c. Then the differential momentum flux along the ray at angle  $\theta$  is  $dF_{\nu}/c$ . The component of momentum flux normal to dA then is  $dF_{\nu}\cos\theta/c$ . Integration gives the monochromatic radiation pressure

$$p_{\nu} \left( \text{kg s}^{-2} \text{m}^{-1} \text{Hz}^{-1} \right) = \frac{1}{c} \int_{\Omega} I_{\nu} \cos^2 \theta \, \mathrm{d}\Omega.$$
 (4.5)



Figure 4.2: Geometry for obliquely incident rays.

 $F_{\nu}$  and  $p_{\nu}$  are moments of the intensity  $I_{\nu}$  (we multiply by powers of  $\cos \theta$  and integrate over  $d\Omega$ ). Integrating over frequency we obtain the total quantities,

$$F(J s^{-1} m^{-2}) = \int_{\nu} F_{\nu} d\nu, \qquad (4.6)$$

$$p(\text{kg s}^{-2} \text{m}^{-1}) = \int_{\nu} p_{\nu} \,\mathrm{d}\nu,$$
 (4.7)

$$I(J s^{-1} m^{-2} ster^{-1}) = \int_{\nu} I_{\nu} d\nu.$$
(4.8)

# 4.2.3 Radiative Energy Density

The specific energy density  $u_{\nu}$  is defined as the energy per unit volume per unit frequency range. We first consider the energy density per unit solid angle  $u_{\nu}(\Omega)$  by  $dE = u_{\nu}(\Omega) dV d\Omega d\nu$ , where dV is a volume element. Energy in a volume c dA dt of a cylinder about a ray of length ct is

$$dE = u_{\nu}(\Omega) \, dA \, c \, dt \, d\Omega \, d\nu. \tag{4.9}$$

Due to radiation velocity c, all the radiation escapes cylinder in time dt. Equating (4.2) and (4.9) gives

$$u_{\nu}(\Omega) = \frac{I_{\nu}}{c}.\tag{4.10}$$

Integration over solid angle gives the mean intensity  $J_{\nu}$ :

$$u_{\nu} = \int_{\Omega} u_{\nu}(\Omega) \,\mathrm{d}\Omega = \frac{1}{c} \int_{\Omega} I_{\nu} \,\mathrm{d}\Omega = \frac{4\pi}{c} J_{\nu}, \quad J_{\nu} = \frac{1}{4\pi} \int_{\Omega} I_{\nu} \,\mathrm{d}\Omega. \tag{4.11}$$

The total radiation density (J m<sup>-3</sup>) is obtained by integrating  $u_{\nu}$  over all frequencies,

$$u = \int_{\nu} u_{\nu} \, \mathrm{d}\nu = \frac{4\pi}{c} \int_{\nu} J_{\nu} \, \mathrm{d}\nu.$$
 (4.12)

# 4.2.4 Radiation Pressure in an Enclosure Containing an Isotropic Radiation Field

Consider an enclosure with reflecting walls containing an isotropic radiation field. Each photon transfers twice its normal component of momentum on reflecting. We have the relation

$$p_{\nu} = \frac{2}{c} \int_{\Omega} I_{\nu} \cos^2 \theta \,\mathrm{d}\Omega. \tag{4.13}$$

This agrees with our previous formula (4.5), since now we integrate over only  $2\pi$  steradians. From isotropy,  $I_{\nu} = J_{\nu}$ , so

$$p = \frac{2}{c} \int_{\nu} J_{\nu} \,\mathrm{d}\nu \int_{\Omega} \cos^2 \theta \,\mathrm{d}\Omega = u \int_{0}^{\pi/2} \cos^2 \theta \,\sin \theta \,\mathrm{d}\theta = \frac{1}{3}u. \tag{4.14}$$

The radiation pressure of an isotropic field is just one-third of the energy density. This result is essential for the thermodynamics of blackbody radiation.

# 4.2.5 Constancy of Specific Intensity Along Rays in Free Space

Consider a ray L with two points along the ray, and areas  $dA_1$ ,  $dA_2$ , normal to the ray at these points. Due to energy conservation we can express the energy carried by the set of rays passing through both area elements  $dA_1$ ,  $dA_2$ , equivalently as (cf. Eq. (4.2)):

$$dE_1 = I_{\nu_1} dA_1 dt d\Omega_1 d\nu_1 = dE_2 = I_{\nu_2} dA_2 dt d\Omega_2 d\nu_2, \qquad (4.15)$$

where  $d\Omega_1$  is the solid angle subtended by  $dA_2$  at  $dA_1$  and vice versa. Since  $d\Omega_1 = dA_2/R^2$ ,  $d\Omega_2 = dA_1/R^2$ , and  $d\nu_1 = d\nu_2$  in identical ray, we have a constant intensity along a ray,

$$I_{\nu_1} = I_{\nu_2}.$$
 (4.16)

We obtain the above result also from the differential relation  $dI_{\nu}/ds = 0$ , where ds is an element of length along the ray.

### 4.2.6 Proof of the Inverse Square Law for a Uniformly Bright Sphere

To show the connection between the constant specific intensity and the inverse square law, let us calculate the flux at an arbitrary distance from a radiating sphere of uniform brightness B. At an arbitrary point P outside the sphere, the specific intensity is B if the ray intersects the radiating sphere and zero otherwise. The corresponding flux is

$$F = \int_{\Omega} I \cos \theta \, \mathrm{d}\Omega = B \int_{0}^{2\pi} \mathrm{d}\phi \int_{0}^{\theta_{c}} \cos \theta \sin \theta \, \mathrm{d}\theta = \pi B \sin^{2} \theta_{c} = \pi B \left(\frac{R}{r}\right)^{2}, \qquad (4.17)$$

where R is radius of the radiating sphere, r is distance of P to the center of the sphere, and  $\theta_c = \operatorname{asin}(R/r)$  is the angle at which a ray from P is tangent to the sphere. The specific intensity is constant but the solid angle subtended by the given object decreases as the inverse square law. The flux at a surface of uniform brightness B is obtained by setting r = R:

$$F = \pi B. \tag{4.18}$$

# 4.3 Radiative Transfer

If a ray passes through matter, energy may be added or subtracted from it by emission or absorption, and the specific intensity does not remain constant in general. Scattering of photons into and out of the beam also affects the intensity.

### 4.3.1 Emission

The coefficient of spontaneous emission j is defined as the energy emitted per unit time per unit solid angle and per unit volume:

$$dE = j \, dV \, d\Omega \, dt = j_{\nu} \, dV \, d\Omega \, dt \, d\nu, \tag{4.19}$$

where dimension of monochromatic emission coefficient  $j_{\nu}$  is J m<sup>-3</sup> s<sup>-1</sup> ster<sup>-1</sup> Hz<sup>-1</sup>.

The emission coefficient depends in general on the direction of emission. For an isotropic emitter or for a superposition of randomly oriented emitters, we can write

$$j_{\nu} = \frac{1}{4\pi} P_{\nu}, \tag{4.20}$$

where  $P_{\nu}$  is the radiated power per unit volume per unit frequency. Sometimes the spontaneous emission is defined by the (angle integrated) emissivity  $\epsilon_{\nu}$  defined as the energy emitted spontaneously per unit frequency per unit time per unit mass, with units of J kg<sup>-1</sup> s<sup>-1</sup> ster<sup>-1</sup> Hz<sup>-1</sup>. In case of isotropic emission

$$dE = \epsilon_{\nu} \rho \, dV \, dt \, d\nu \frac{d\Omega}{4\pi}, \qquad (4.21)$$

where  $\rho$  is the mass density of the emitting medium. Comparing Eqs. (4.19) and (4.21) gives the relation between  $j_{\nu}$  and  $\epsilon_{\nu}$ :

$$j_{\nu} = \frac{\epsilon_{\nu} \,\rho}{4\pi} \tag{4.22}$$

for isotropic emission. Traveling a distance ds, a beam of cross section dA passes through a volume dV = dA ds, and, combining Eqs. (4.15) and (4.19), the intensity added to the beam by spontaneous emission is:

$$\mathrm{d}I_{\nu} = j_{\nu}\,\mathrm{d}s.\tag{4.23}$$

# 4.3.2 Absorption

We define the absorption coefficient  $\alpha_{\nu}$  (m<sup>-1</sup>) as a measure of loss of intensity in a beam traveling a distance ds ( $\alpha_{\nu}$  is positive if a beam loses energy):

$$\mathrm{d}I_{\nu} = -\alpha_{\nu}I_{\nu}\,\mathrm{d}s.\tag{4.24}$$

This phenomenological law can be understood microscopically for particles with number density n (number of particles per unit volume) where each represents an effective absorbing area, or cross section, of magnitude  $\sigma_{\nu}$  (m<sup>2</sup>). These absorbers are assumed to be randomly distributed. The number of absorbers in the volume element is  $n \, dV = n \, dA \, ds$  and the total absorbing area equals  $n\sigma_{\nu} \, dA \, ds$ . The energy absorbed out of the beam within solid angle  $d\Omega$  is

$$-\mathrm{d}I_{\nu}\,\mathrm{d}A\,\mathrm{d}\Omega\,\mathrm{d}t\,\mathrm{d}\nu = I_{\nu}\left(n\sigma_{\nu}\,\mathrm{d}A\,\mathrm{d}s\right)\,\mathrm{d}\Omega\,\mathrm{d}t\,\mathrm{d}\nu,\quad\mathrm{thus}\quad\mathrm{d}I_{\nu} = -n\sigma_{\nu}I_{\nu}\,\mathrm{d}s.\tag{4.25}$$

This is identical to the phenomenological law (4.24), where

$$\alpha_{\nu} = n\sigma_{\nu}.\tag{4.26}$$

Often  $\alpha_{\nu}$  is introduced as

$$\alpha_{\nu} = \rho \kappa_{\nu},\tag{4.27}$$

where  $\rho$  is the mass density and  $\kappa_{\nu}$  (m<sup>2</sup> kg<sup>-1</sup>) is the mass absorption coefficient or the opacity coefficient.

There are some constraints for validity of the microscopic picture: The most important are

- 1. the linear scale of the cross section must be small in comparison to the mean inter-particle distance d, thus  $\sigma_{\nu}^{1/2} \ll d \sim n^{-1/3}$  and thus  $\alpha_{\nu} d \ll 1$ ,
- 2. the absorbers are independent and randomly distributed.

Fortunately are these conditions almost always met for astrophysical problems.

We consider "absorption" to include both "true absorption" and stimulated emission, because both are proportional to the intensity of the incoming beam (unlike spontaneous emission). Thus the *net absorption* may be positive or negative, depending on whether "true absorption" or stimulated emission dominates. Although this combination may seem artificial, it will prove convenient and obviate the need for a quantum mechanical addition to our classical formulas.

### 4.3.3 The Radiative Transfer Equation (RTE)

We can now include the effects of emission and absorption into a single equation for a specific intensity along a ray,

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}s} = j_{\nu} - \alpha_{\nu}I_{\nu},\tag{4.28}$$

RTE incorporates most of the macroscopic aspects of radiation, relating them to the two coefficients  $\alpha_{\nu}$  and  $j_{\nu}$ . A task is to find forms for these coefficients corresponding to particular physical processes.

Once  $\alpha_{\nu}$  and  $j_{\nu}$  are known, it is relatively easy to solve RTE for the specific intensity  $I_{\nu}$ . Scattering however complicates the solution, because emission into  $d\Omega$  depends on  $I_{\nu}$  in  $d\Omega'$ ; RTE thus becomes an integrodifferential equation solved partly by numerical techniques.

We introduce a formal solution of the complete RTE showing two simple limiting cases:

• Emission only:  $\alpha_{\nu} = 0$ . In this case

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}s} = j_{\nu}, \quad \text{so that} \quad I_{\nu}(s) = I_{\nu}(s_0) + \int_{s_0}^s j_{\nu}(s') \,\mathrm{d}s'. \tag{4.29}$$

The increase in brightness is thus equal to the emission coefficient integrated along the line of sight.

• Absorption only:  $j_{\nu} = 0$ . In this case

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}s} = -\alpha_{\nu}I_{\nu}, \quad \text{so that} \quad I_{\nu}(s) = I_{\nu}(s_0) \exp\left[-\int_{s_0}^s \alpha_{\nu}(s')\,\mathrm{d}s'\right]. \tag{4.30}$$

The brightness decreases along the ray by the exponential of the absorption coefficient integrated along the line of sight.

### 4.3.4 Optical Depth and Source Function

RTE takes a particularly simple form if, instead of s, we use the optical depth  $\tau_{\nu}$ , defined as

$$d\tau_{\nu} = \alpha_{\nu} \, \mathrm{d}s \quad \mathrm{or} \quad \tau_{\nu} = \int_{s_0}^{s} \alpha_{\nu}(s') \, \mathrm{d}s'. \tag{4.31}$$

The optical depth is measured along the path of a traveling ray; occasionally,  $\tau_{\nu}$  is measured backward along the ray and a minus sign appears in (4.31). In plane-parallel media, a standard optical depth is sometimes used to measure distance normal to the surface, so that ds is replaced by dz and  $\tau_{\nu} = \tau_{\nu}(z)$ . The zero point point  $s_0$  (or  $z_0$ ) for the optical depth scale is arbitrary.

A medium is said to be *optically thick* or *opaque* when  $\tau_{\nu}$ , integrated along a typical path, satisfies  $\tau_{\nu} > 1$ . When  $\tau_{\nu} < 1$ , the medium is said to be *optically thin* or *transparent*. In other words, in an optically thin medium the typical photon of frequency  $\nu$  can pass through the medium without being absorbed, whereas in an optically thick medium the photon of frequency  $\nu$  cannot traverse the entire medium without being absorbed.

RTE now can be written, after dividing by  $\alpha_{\nu}$ ,

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}\tau_{\nu}} = S_{\nu} - I_{\nu},\tag{4.32}$$

where the source function  $S_{\nu}$  is defined as:

$$S_{\nu} \equiv \frac{j_{\nu}}{\alpha_{\nu}}.\tag{4.33}$$

The source function  $S_{\nu}$  is often a simpler physical quantity than  $j_{\nu}$ . Also, the optical depth scale  $\tau_{\nu}$  reveals more clearly the important intervals along a ray than  $\alpha_{\nu}$ . The variables  $\tau_{\nu}$  and  $S_{\nu}$  are therefore often used instead of  $\alpha_{\nu}$  and  $j_{\nu}$ .

We now formally solve RTE by regarding all quantities as functions of  $\tau_{\nu}$  instead of s. Solving Eq. (4.32) as a standard non-homogeneous 1st order ODE, we obtain

$$I_{\nu}(\tau_{\nu}) = I_{\nu}(0) e^{-\tau_{\nu}} + \int_{0}^{\tau_{\nu}} e^{-(\tau_{\nu} - \tau_{\nu}')} S_{\nu}(\tau_{\nu}') d\tau_{\nu}'.$$
(4.34)

The above equation is interpreted as the sum of two terms: the initial intensity diminished by absorption plus the integrated source diminished by absorption. As an example consider a constant source function  $S_{\nu}$ , then Eq. (4.34) becomes

$$I_{\nu}(\tau_{\nu}) = I_{\nu}(0) e^{-\tau_{\nu}} + S_{\nu} \left(1 - e^{-\tau_{\nu}}\right) = S_{\nu} + e^{-\tau_{\nu}} \left[I_{\nu}(0) - S_{\nu}\right].$$
(4.35)

As  $\tau_{\nu} \to \infty$ , in Eq. (4.35)  $I_{\nu} \to S_{\nu}$ . But, when scattering is present,  $S_{\nu}$  contains a contribution from  $I_{\nu}$ , so that we cannot specify  $S_{\nu}$  a priori.

We see from RTE that if  $I_{\nu} > S_{\nu}$ , then  $dI_{\nu}/d\tau_{\nu} < 0$  and  $I_{\nu}$  tends to decrease along the ray. If  $I_{\nu} < S_{\nu}$ , then  $I_{\nu}$  tends to increase along the ray, thus  $I_{\nu}$  tries to approach  $S_{\nu}$ . In case of a sufficient optical depth,  $I_{\nu}$  does approach  $S_{\nu}$ , and RTE describes a "relaxation" process.

### 4.3.5 Mean Free Path

describes absorption of radiation in an equivalent way, and is defined as the average distance that a photon can travel without being absorbed. It may be easily related to the absorption coefficient of a homogeneous material. From the absorption law (4.30), the probability of a photon traveling at least an optical depth  $\tau_{\nu}$  is  $e^{-\tau_{\nu}}$ . The mean traveled optical depth is:

$$\langle \tau_{\nu} \rangle = \int_0^\infty \tau_{\nu} \,\mathrm{e}^{-\tau_{\nu}} \,\mathrm{d}\tau_{\nu} = 1. \tag{4.36}$$

The mean distance traveled in a homogeneous medium is defined as the mean free path  $\lambda_{\nu}$  and is determined by  $\langle \tau_{\nu} \rangle = \alpha_{\nu} \lambda_{\nu} = 1$  or

$$\lambda_{\nu} = \alpha_{\nu}^{-1} = \frac{1}{n\sigma_{\nu}}.$$
(4.37)

The mean free path  $\lambda_{\nu}$  is the inverted value of the absorption coefficient for homogeneous material. We also define a *local mean path* at a point in an *inhomogeneous material* as an equivalent to the mean free path of the photon in a homogeneous region of the same properties.

### 4.3.6 Radiation Force

Radiation exerts a force on the absorbing medium, because radiation carries momentum. We define a *radiation flux vector* 

$$\boldsymbol{F}_{\nu} = \int I_{\nu} \, \boldsymbol{n} \, \mathrm{d}\Omega, \qquad (4.38)$$

where  $\boldsymbol{n}$  is a unit vector along the direction of the ray. Since a photon momentum is E/c, the vector of absorbed momentum per unit area per unit time per unit path length is

$$\mathfrak{F} = \frac{1}{c} \int \alpha_{\nu} \, \boldsymbol{F}_{\nu} \, \mathrm{d}\nu. \tag{4.39}$$

Since dA ds = dV,  $\mathfrak{F}$  is the force density exerted on the medium by the radiation field. The specific force (per unit mass) is  $\mathbf{f} = \mathfrak{F}/\rho$  or

$$\boldsymbol{f} = \frac{1}{c} \int \kappa_{\nu} \, \boldsymbol{F}_{\nu} \, \mathrm{d}\nu. \tag{4.40}$$

Equations (4.39) and (4.40) are simplified because they assume the isotropic absorption coefficient. They also assume no momentum imparted by emission; it is true only for isotropic emission.

# 4.4 Thermal Radiation

Thermal radiation is a radiation emitted by matter in thermal equilibrium (TE).

# 4.4.1 Blackbody Radiation

We first investigate the *blackbody radiation*, which is in TE by definition. Consider an enclosure at temperature T and do not let radiation in or out until TE has been achieved. Following general thermodynamic arguments, we can derive several important properties of blackbody radiation.

Since photons are massless, they can be created and destroyed in arbitrary number by the walls of the container (assuming negligible *self-interaction* between photons). There is no
conservation law of photon number (unlike baryon number), we expect that the number of photons will adjust itself in equilibrium at temperature T.

 $I_{\nu}$  is independent of the properties of the enclosure and depends only on the temperature. To prove this, we connect the enclosure to another enclosure of arbitrary shape and place a filter between the two, which passes only a single particular frequency  $\nu$ . If  $I_{\nu} \neq I'_{\nu}$ , energy will flow between the two enclosures. Since these are at the same temperature, this violates the second law of thermodynamics. Therefore, we have the relation

$$I_{\nu} = (\text{universal function of } T \text{ and } \nu) \equiv B_{\nu}(T). \tag{4.41}$$

 $I_{\nu}$  must be independent of the shape of the enclosure, this implies that it is also isotropic,  $I_{\nu} \neq I_{\nu}(\Omega)$ . The function  $B_{\nu}(T)$  is called the Planck function.

#### 4.4.2 Kirchhoff's Law for Thermal Emission

Consider an element of some thermally emitting material at temperature T, its emission depends on its temperature and internal properties. Put this into a blackbody enclosure at the same temperature T. Let the source function of the material be  $S_{\nu}$ . If  $S_{\nu} > B_{\nu}$ , then  $I_{\nu} > B_{\nu}$ , and vice versa (cf. the discussion after Eq. (4.35)). But the new configuration does not affect the radiation, since it also a blackbody enclosure at temperature T. We have

$$S_{\nu} = B_{\nu}(T), \quad j_{\nu} = \alpha_{\nu} B_{\nu}(T).$$
 (4.42)

Equation (4.42) is the Kirchhoff's law. It relates  $\alpha_{\nu}$  and  $j_{\nu}$  to the temperature T of the material. RTE for thermal radiation is (cf. Eq. (4.28)),

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}s} = \alpha_{\nu}B_{\nu}(T) - \alpha_{\nu}I_{\nu} \quad \text{or} \quad \frac{\mathrm{d}I_{\nu}}{\mathrm{d}\tau_{\nu}} = B_{\nu}(T) - I_{\nu}.$$
(4.43)

Since  $S_{\nu} = B_{\nu}$  throughout a blackbody enclosure, we also have  $I_{\nu} = B_{\nu}$ . Blackbody radiation is homogeneous and isotropic, p = u/3.

We note the distinction between blackbody radiation, where  $I_{\nu} = B_{\nu}$ , and thermal radiation, where  $S_{\nu} = B_{\nu}$ . Thermal radiation becomes blackbody only for optically thick media.

#### 4.4.3 Thermodynamics of Blackbody Radiation

Blackbody radiation, like any system in TE, can be treated by thermodynamic methods. Consider a blackbody enclosure with a piston, so that work may be done on, or extracted from the radiation. From the first and second law of thermodynamics we have

$$dQ = dU + p \, dV, \quad dS = \frac{dQ}{T}, \tag{4.44}$$

where Q is heat, U is internal energy, and S is entropy. But U = uV, p = u/3, and u depends only on T, since  $u = (4\pi/c) \int J_{\nu} d_{\nu}$  where  $J_{\nu} = B_{\nu}(T)$ . We have

$$dS = \frac{V}{T}\frac{\mathrm{d}u}{\mathrm{d}T}\,\mathrm{d}T + \frac{u}{T}\,\mathrm{d}V + \frac{1}{3}\frac{u}{T}\,\mathrm{d}V = \frac{V}{T}\frac{\mathrm{d}u}{\mathrm{d}T}\,\mathrm{d}T + \frac{4}{3}\frac{u}{T}\,\mathrm{d}V,\tag{4.45}$$

where dS is a total differential. Eq. (4.45) gives

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{V}{T}\frac{\mathrm{d}u}{\mathrm{d}T} \quad \text{and} \quad \left(\frac{\partial S}{\partial V}\right)_T = \frac{4u}{3T}.$$
 (4.46)

Differentiating again Eq. (4.46), we obtain

$$\frac{\partial^2 S}{\partial T \partial V} = \frac{1}{T} \frac{\mathrm{d}u}{\mathrm{d}T} = \frac{4}{3T} \left( \frac{\mathrm{d}u}{\mathrm{d}T} - \frac{u}{T} \right),\tag{4.47}$$

giving

$$\frac{\mathrm{d}u}{\mathrm{d}T} = \frac{4u}{T}, \quad \frac{\mathrm{d}u}{u} = 4\frac{\mathrm{d}T}{T}, \quad \ln u = \ln\left(aT^4\right), \tag{4.48}$$

where  $\ln a$  is a constant of integration. We obtain the *Stefan-Boltzmann law* 

$$u(T) = aT^4. ag{4.49}$$

This may be related to the Planck function,  $I_{\nu} = J_{\nu}$  for isotropic radiation (cf. Eq. (4.11)),

$$u = \frac{4\pi}{c} \int B_{\nu}(T) \,\mathrm{d}\nu = \frac{4\pi}{c} B(T), \tag{4.50}$$

where the integrated Planck function is

$$B(T) = \frac{ac}{4\pi} T^4.$$
 (4.51)

The emergent flux from an isotropically emitting surface is  $\pi \times$  brightness (see Eq. (4.18)), which gives another form of the Stefan-Boltzmann law,

$$F = \int F_{\nu} \, \mathrm{d}\nu = \pi \int B_{\nu} \, \mathrm{d}\nu = \pi B(T), \quad F = \sigma T^4, \tag{4.52}$$

where  $\sigma \equiv ac/4 \equiv 2\pi^5 k^4/(15c^2h^3) \approx 5.67 \times 10^{-8} \,\mathrm{J \, s^{-1} \, m^{-2} \, K^{-4}}$  is the Stefan-Boltzmann constant and  $a \equiv 4\sigma/c \equiv 8\pi^5 k^4/(15c^3h^3) \approx 7.56 \times 10^{-16} \,\mathrm{J \, m^{-3} \, K^{-4}}$  is the radiation (or radiation density) constant. The constants a and  $\sigma$  cannot be determined by macroscopic thermodynamic arguments, (they are derived below). Combining Eqs. (4.46) and (4.49), the entropy S of blackbody radiation is

$$S = \frac{4}{3}aT^3V \tag{4.53}$$

and the adiabatic expansion law (S = const. and p = u/3) for blackbody radiation is

$$TV^{1/3} = \text{const.}$$
 or  $pV^{4/3} = \text{const.}$  (4.54)

Equations (4.54) give the adiabatic law  $pV^{\gamma} = \text{const.}$  with  $\gamma = 4/3$ .

### 4.4.4 The Planck Spectrum

We now derive the Planck function in two steps: First, we quantify the density of photon states in a blackbody enclosure; second we evaluate the average energy per photon state. Consider a photon of frequency  $\nu$  propagating in direction **n** inside a box. The wave vector of the photon is  $\mathbf{k} = (2\pi/\lambda)\mathbf{n} = (2\pi\nu/c)\mathbf{n}$ . Each dimension of the box,  $L_x, L_y, L_z \gg \lambda$ , the photon can be represented by a standing wave. The number of nodes in the wave in each direction i = x, y, zis  $n_i = k_i L_i/(2\pi)$ . If  $n_i \gg 1$ , the number of node changes in a wave number interval is

$$\Delta n_i = \frac{L_i \Delta k_i}{2\pi}.\tag{4.55}$$

The number of states in the three-dimensional wavevector element  $\Delta k_x \Delta k_y \Delta k_z = d^3 k$  is

$$\Delta N = \Delta n_x \Delta n_y \Delta n_z = \frac{L_x L_y L_z \,\mathrm{d}^3 k}{(2\pi)^3}.$$
(4.56)

Using  $L_x L_y L_z = V$  (the volume of the enclosure) and using two independent photon polarizations (two states per wave vector  $\mathbf{k}$ ), the number of states per unit volume, per unit three-dimensional wavenumber, is  $2/(2\pi)^3$ . In isotropic case

$$d^{3}k = k^{2} dk d\Omega = \frac{(2\pi)^{3} \nu^{2} d\nu d\Omega}{c^{3}}$$
(4.57)

and the density of states (the number of states per solid angle per volume per frequency) is

$$\rho_{\text{states}} = \frac{2\nu^2}{c^3}.\tag{4.58}$$

What is the average energy of each state? Each photon of frequency  $\nu$  has energy  $h\nu$ , so we ask what is the average energy of the state with frequency  $\nu$ . Each state contains n photons of energy  $h\nu$ , the total energy may be  $E_n = nh\nu$ . The probability of a state with  $E_n \sim e^{-\beta E_n}$ , where  $\beta = (kT)^{-1}$ , k is the Boltzmann constant. The average energy is,

$$\langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}} = -\frac{\partial}{\partial \beta} \ln \left( \sum_{n=0}^{\infty} e^{-\beta E_n} \right)$$
(4.59)

and the sum of a geometric series gives,

$$\sum_{n=0}^{\infty} e^{-\beta E_n} = \sum_{n=0}^{\infty} e^{-nh\nu\beta} = \left(1 - e^{-\beta h\nu}\right)^{-1}.$$
(4.60)

We have the result:

$$\langle E \rangle = -\frac{\partial}{\partial\beta} \left[ \ln \left( 1 - e^{-\beta h\nu} \right)^{-1} \right] = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{e^{\beta h\nu} - 1}$$
(4.61)

which states that the average number  $n_{\nu}$  (the "occupation number") is

$$n_{\nu} = \left(e^{\beta h\nu} - 1\right)^{-1}.$$
 (4.62)

Equation (4.61) is the Bose-Einstein statistics with an infinite number of particles (chemical potential  $\mu = 0$ ). The energy per solid angle per volume per frequency is the product of  $\langle E \rangle$  and  $\rho_{\text{states}}$  (Eq. (4.58)). This can also be written in terms of  $u_{\nu}(\Omega)$  (see Eqs. (4.9) and (4.10)). We have

$$u_{\nu}(\Omega) \,\mathrm{d}V \,\mathrm{d}\Omega \,\mathrm{d}\nu = \frac{2\nu^2}{c^3} \frac{h\nu}{\mathrm{e}^{\beta h\nu} - 1} \,\mathrm{d}V \,\mathrm{d}\Omega \,\mathrm{d}\nu, \quad \text{so that} \quad u_{\nu}(\Omega) = \frac{2h}{c^3} \frac{\nu^3}{\mathrm{e}^{\beta h\nu} - 1}. \tag{4.63}$$

Equation (4.10) relates  $u_{\nu}(\Omega)$  and  $I_{\nu}$ . Now we have  $I_{\nu} = B_{\nu}$ , and expressing  $B_{\nu}(T)$  as  $B_{\lambda}(T)$   $(B_{\nu} d\nu = B_{\lambda} d\lambda)$ , we have the *Planck law*,

$$B_{\nu}(T) = \frac{2h}{c^2} \frac{\nu^3}{e^{\beta h\nu} - 1}, \quad B_{\lambda}(T) = \frac{2hc^2/\lambda^5}{e^{\beta hc/\lambda} - 1}.$$
 (4.64)

#### 4.4.5 Properties of the Planck Law

We now describe basic properties of this law:

#### The Rayleigh-Jeans Law, $h\nu \ll kT$ :

We expand the exponential,

$$e^{\beta h\nu} - 1 = \beta h\nu + \dots$$
 so that  $I_{\nu}^{\rm RJ}(T) = \frac{2\nu^2}{c^2}kT.$  (4.65)

This result does not contain Planck constant. The Rayleigh-Jeans law applies at low frequencies (in the radio region). If Eq. (4.65) is applied to all frequencies, the total amount of energy  $\propto \int j_{\nu} d\nu$  would diverge. It is known as the *ultraviolet catastrophe*. This indicates that for  $h\nu \gg kT$ , the quantum nature of photons must be taken into account.

#### The Wien Law, $h\nu \gg kT$ :

In this limit the unity term in the denominator of Planck law can be dropped and we have:

$$I_{\nu}^{\rm W}(T) = \frac{2h\nu^3}{c^2} \,\mathrm{e}^{-\beta h\nu}.$$
 (4.66)

The monochromatic brightness of a blackbody decreases very rapidly with frequency once the maximum is reached.

#### Monotonicity with Temperature:

The blackbody curve with higher temperature lies entirely above the one with lower temperature. To prove this we note that

$$\frac{\partial B_{\nu}(T)}{\partial T} = \frac{2h^2\nu^4}{c^2kT^2} \frac{\mathrm{e}^{\beta h\nu}}{\left(\mathrm{e}^{\beta h\nu} - 1\right)^2} \tag{4.67}$$

is positive. At any frequency the increasing temperature leads to increase of  $B_{\nu}(T)$ . Note that  $B \to 0$  as  $T \to 0$  and  $B \to \infty$  as  $T \to \infty$ .

#### Wien Displacement Law:

The frequency  $\nu_{\text{max}}$  at which the peak of  $B_{\nu}(T)$  occurs is

$$\frac{\partial B_{\nu}(T)}{\partial \nu}\Big|_{\nu=\nu_{\max}} = 0, \quad \text{so that} \quad h\nu_{\max} \approx 2.82kT \quad \text{or} \quad \frac{\nu_{\max}}{T} \approx 5.88 \times 10^{10} \,\text{Hz}\,\text{K}^{-1}. \tag{4.68}$$

The peak frequency of the blackbody law shifts linearly with temperature. Similarly, a wavelength  $\lambda_{\text{max}}$  at which the maximum of  $B_{\lambda}(T)$  occurs can be found by

$$\frac{\partial B_{\lambda}(T)}{\partial \lambda}\Big|_{\lambda=\lambda_{\max}} = 0, \quad \text{so that} \quad \lambda_{\max}T \approx 2.9 \times 10^{-3} \,\text{mK}. \tag{4.69}$$

Eqs. (4.68) and (4.69) are known as the Wien displacement law.

The peaks of  $B_{\nu}$  and  $B_{\lambda}$  given by Eqs. (4.68) and (4.69) however do not occur at the same points in wavelength or frequency,  $\nu_{\max}\lambda_{\max} \neq c$ . For example, if T = 7300 K the peak of  $B_{\nu}$  is at  $\lambda \approx 7000$  Å while the peak of  $B_{\lambda}$  is at  $\lambda \approx 4000$  Å. The Wien displacement characterizes the frequency range for which the Rayleigh-Jeans law is valid,  $\nu \ll \nu_{\max}$ . Similarly for the Wien law,  $\nu \gg \nu_{\max}$ .

#### 4.4.6 Characteristic Temperatures Related to Planck Spectrum

#### **Brightness Temperature:**

We may characterize the specific intensity (brightness) at a certain frequency by setting it equal to the corresponding blackbody temperature, that is, for any value  $I_{\nu}$  we define  $T_b(\nu)$  by

$$I_{\nu} = B_{\nu}(T_b). \tag{4.70}$$

 $T_b$  is the brightness temperature. This way of specifying brightness is closely connected with the physical properties of the emitter, and has the dimension (K) instead of (J m<sup>-2</sup> s<sup>-1</sup> Hz<sup>-1</sup> ster<sup>-1</sup>). This is used mainly in radio astronomy (the Rayleigh-Jeans law domain), where for  $h\nu \ll kT$ :

$$I_{\nu} = \frac{2\nu^2}{c^2} kT_b.$$
(4.71)

RTE for thermal emission takes a particularly simple form in terms of brightness temperature in the Rayleigh-Jeans limit (cf. Eq. (4.43))

$$\frac{\partial T_b}{\partial \tau_\nu} = -T_b + T. \tag{4.72}$$

where T is the temperature of the material. For a constant T we have

$$T_b = T_b(0) e^{-\tau_{\nu}} + T \left( 1 - e^{-\tau_{\nu}} \right), \quad h\nu \ll kT.$$
(4.73)

If  $\tau_{\nu} \gg 1$ ,  $T_b \to T$ . We also note that, in general, the brightness temperature is a function of  $\nu$ , only if the source is blackbody,  $T_b(\nu) = T_b$ . In the Wien region, where  $h\nu \gg kT$ , the concept of brightness temperature is not so useful because of the rapid decrease of  $B_{\nu}$  with  $\nu$  and due to impossibility to formulate RTE linear in the brightness temperature.

#### Color Temperature:

A spectrum has often more or less blackbody profile, but not necessarily the proper absolute value. For example, by measuring  $F_{\nu}$  from an unknown source we cannot quantify  $I_{\nu}$  unless we know the distance to the source and its size. By fitting the measured data to a blackbody curve without regard to vertical scale, we obtain a *color temperature*  $T_c$ . Often the "fitting" procedure is merely an estimate of the peak of the spectrum and applying Wien's displacement law to find a temperature. The color temperature  $T_c$  will correctly give the temperature of a blackbody source of unknown absolute size.  $T_c$  will also give the temperature of a thermal emitter that is optically thin, providing that the optical thickness is fairly constant for frequencies near the peak. In this case the brightness temperature  $T_b$  will be less than the temperature of the emitter, since the blackbody spectrum gives the maximum attainable intensity of a thermal emitter at temperature T.

#### **Effective Temperature:**

The effective temperature  $T_{\text{eff}}$  of a source is derived from the total amount of radiated flux, integrated over all frequencies. We define  $T_{\text{eff}}$  by equating the actual flux F to the flux of a blackbody:

$$F = \int I_{\nu} \cos \theta \, \mathrm{d}\nu \, \mathrm{d}\Omega \equiv \sigma T_{\text{eff}}^4. \tag{4.74}$$

Note that both  $T_{\text{eff}}$  and  $T_b$  depend on the absolute value of the source intensity, but  $T_c$  depends only on the shape of the observed spectrum.

## 4.5 The Einstein Coefficients

#### 4.5.1 Definition of Coefficients

From a Kirchhoff's law,  $j_{\nu} = \alpha_{\nu}B_{\nu}$ , must clearly imply a relationship between emission and absorption at a microscopic level. This relationship was first discovered by Einstein who considered the case of two discrete energy levels: the first of energy E with statistical weight  $g_1$ , the second of energy  $E + h\nu_0$  with statistical weight  $g_2$ . A transition from 1 to 2 occurs due to absorption of a photon of energy  $h\nu_0$ . Similarly, a transition from 2 to 1 occurs when a photon is emitted. We identify three processes:

#### **Spontaneous Emission:**

occurs when the system drops from level 2 to level 1 by emitting a photon; it occurs even in the absence of a radiation field. We define the *Einstein A-coefficient* by

 $A_{21}$  = transition probability per unit time for spontaneous emission (s<sup>-1</sup>). (4.75)

#### Absorption:

A transition from level 1 to level 2 occurs due absorption of a photon of energy  $h\nu_0$ . Since we do not assume any self-interaction within the radiation field, we expect the probability of this process per unit time will be proportional to the density of photons (to the mean intensity) at frequency  $\nu_0$ . However, the energy difference between the two levels is not infinitely sharp but is described by a line-profile function  $\phi(\nu)$ , which is peaked at  $\nu = \nu_0$  and which is normalized by a convention

$$\int_0^\infty \phi(\nu) \,\mathrm{d}\nu = 1. \tag{4.76}$$

This line profile function describes the relative effectiveness of frequencies in the neighborhood of  $\nu_0$  for causing transitions. Following these arguments, we write

$$B_{12}\overline{J}$$
 = transition probability per unit time for absorption, (4.77)

where

$$\bar{J} = \int_0^\infty J_\nu \,\phi(\nu) \,\mathrm{d}\nu = 1.$$
 (4.78)

The constant of proportionality  $B_{12}$  is the *Einstein B-coefficient*.

#### **Stimulated Emission:**

Einstein found that there was yet another process required to derive Planck law, that was proportional to  $\bar{J}$  and caused emission of a photon. It was defined:

$$B_{21}J =$$
 transition probability per unit time for stimulated emission, (4.79)

where  $B_{21}$  is another *Einstein B-coefficient*.

When  $J_{\nu}$  changes slowly over the width  $\Delta \nu$  of the line,  $\phi(\nu)$  behaves like a  $\delta$ -function, and the probabilities per unit time for absorption and stimulated emission become simply  $B_{12}J_{\nu_0}$ , and  $B_{21}J_{\nu_0}$ , respectively. To define the Einstein *B*-coefficients, there is often used the energy density  $u_{\nu}$  instead of  $J_{\nu}$ , which differs in value by  $c/4\pi$  (cf. Eq. (4.11)).

#### 4.5.2 Relations between Einstein Coefficients

In thermodynamic equilibrium (TE) the number of transitions per unit time per unit volume out of state 1 precisely equals the number of transitions per unit time per unit volume into state 1. Let  $n_1$  and  $n_2$  be the number densities of atoms in levels 1 and 2, respectively, then

$$n_1 B_{12} \bar{J} = n_2 A_{21} + n_2 B_{21} \bar{J}. \tag{4.80}$$

Solving for  $\overline{J}$  from Eq. (4.80):

$$\bar{J} = \frac{A_{21}}{B_{21}} \left( \frac{n_1}{n_2} \frac{B_{12}}{B_{21}} - 1 \right)^{-1}.$$
(4.81)

Since the ratio of  $n_1$  to  $n_2$  in TE is

$$\frac{n_1}{n_2} = \frac{g_1 e^{-\beta E}}{g_2 e^{-\beta (E+h\nu_0)}} = \frac{g_1}{g_2} e^{\beta h\nu_0}, \quad \text{then} \quad \bar{J} = \frac{A_{21}}{B_{21}} \left(\frac{g_1}{g_2} \frac{B_{12}}{B_{21}} e^{\beta h\nu_0} - 1\right)^{-1}.$$
(4.82)

In TE  $J_{\nu} = B_{\nu}$ , and since  $B_{\nu}$  varies slowly on the scale of  $\Delta \nu$ , this implies  $\bar{J} = B_{\nu}$ . For Eq. (4.82) to equal the Planck law at all temperatures, we must have the following relations:

$$g_1 B_{12} = g_2 B_{21}, \quad A_{21} = \frac{2h\nu^3}{c^2} B_{21}.$$
 (4.83)

Equations (4.83) connect *atomic properties* and have no reference (unlike Kirhhoff's law) to the temperature T, they must therefore hold whether or not are the atoms in TE. Equations (4.83) represent what is generally known as *detailed balance* relations that connect any microscopic process and its inverse process. These Einstein relations are the extensions of Kirchhoff's law to include the non-thermal emission that occurs when the matter is not in TE. If we determine any one of the coefficients  $A_{21}$ ,  $B_{21}$ , or  $B_{12}$ , these relations allow us to determine the other two.

Einstein included the process of stimulated emission, because without it he could not get Planck law, but only Wien law, which was known to be incorrect. Why we obtain the Wien law when stimulated emission is neglected? The Wien law represents the Planck spectrum when  $h\nu \gg kT$ , but in this case the level 2 is very sparsely populated relative to level 1,  $n_2 \ll n_1$ , and the stimulated emission is unimportant compared to absorption (Eq. (4.80)). A property of stimulated emission that is not clear from the preceding discussion is that the emitted photon has precisely the same direction and frequency (is precisely coherent) as the photon that stimulated the emission.

#### 4.5.3 Absorption and Emission Coefficients in Terms of Einstein Coefficients

To obtain the emission coefficient  $j_{\nu}$ , we must make some assumption about the frequency distribution of the emitted radiation during a spontaneous transition from level 2 to level 1. The simplest assumption is that this emission is distributed in accordance with the same line profile function  $\phi_{\nu}$  that describes absorption (this assumption is very often a good one in astrophysics). The amount of energy emitted in volume dV, solid angle  $d\Omega$ , frequency range  $d\nu$ , and time dt, is  $j_{\nu} dV d\Omega d\nu dt$ . Since the energy contribution of each atom is  $h\nu$  for each transition, distributed over  $4\pi$  solid angle, this may also be expressed as  $(h\nu_0/4\pi)\phi(\nu)n_2A_{21} dV d\Omega d\nu dt$ , so that the emission coefficient is

$$j_{\nu} = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu). \tag{4.84}$$

To obtain the absorption coefficient  $\alpha_{\nu}$ , we first note from Eqs. (4.77) and (4.78) that the total energy absorbed in time dt and volume dV is

$$\mathrm{d}t\,\mathrm{d}V(h\nu_0/4\pi)\,n_1B_{12}\int\,\mathrm{d}\Omega\int\phi(\nu)I_\nu\,\mathrm{d}\nu.\tag{4.85}$$

The energy absorbed out of a beam in a frequency range  $d\nu$ , solid angle  $d\Omega$ , time dt, and volume dV, is

$$dt \, dV \, d\Omega \, d\nu \frac{h\nu_0}{4\pi} n_1 B_{12} \, \phi(\nu) I_{\nu}. \tag{4.86}$$

The volume element dV = dA ds, and using Eqs. (4.2) and (4.24), we have the absorption coefficient (uncorrected for stimulated emission):

$$\alpha_{\nu} = \frac{h\nu}{4\pi} n_1 B_{12} \,\phi(\nu). \tag{4.87}$$

To express the stimulated emission, since it is proportional to the intensity and affects only the photons along the given beam, in close analogy to the process of absorption, we treat stimulated emission as negative absorption and include its effect through the absorption coefficient. These two processes always occur together, analogously to reasons that led to Eq. (4.87) we find the contribution of stimulated emission to the absorption coefficient,

$$\alpha_{\nu} = \frac{h\nu}{4\pi} \phi(\nu) \left( n_1 B_{12} - n_2 B_{21} \right). \tag{4.88}$$

This quantity will always be meant when speaking of the *absorption coefficient*. The form given in Eq. (4.87) will be called the *absorption coefficient uncorrected for stimulated emission*.

It is now possible to write RTE (4.28) in terms of the Einstein coefficients,

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}s} = \frac{h\nu}{4\pi} n_2 A_{21} \phi(\nu) - \frac{h\nu}{4\pi} \phi(\nu) \left(n_1 B_{12} - n_2 B_{21}\right) I_{\nu}.$$
(4.89)

We obtain the source function by dividing Eq. (4.84) by Eq. (4.88),

$$S_{\nu} = \frac{n_2 A_{21}}{n_1 B_{12} - n_2 B_{21}}.$$
(4.90)

Using relations (4.83), we write the absorption coefficient and source function, respectively, as

$$\alpha_{\nu} = \frac{h\nu}{4\pi} n_1 B_{12} \left( 1 - \frac{g_1 n_2}{g_2 n_1} \right) \phi(\nu), \quad S_{\nu} = \frac{2h\nu^3}{c^2} \left( \frac{g_2 n_1}{g_1 n_2} - 1 \right)^{-1}.$$
 (4.91)

Equation (4.91) is a generalized Kirchhoff's law. We identify three qualitative cases:

#### Thermal Emission (LTE):

If the matter is in TE (but not necessarily with the radiation), Eq. (4.82) gives

$$\frac{n_1}{n_2} = \frac{g_1}{g_2} e^{\beta h\nu}, \tag{4.92}$$

the matter is said to be in *local thermodynamic equilibrium* (LTE) with

$$\alpha_{\nu} = \frac{h\nu}{4\pi} n_1 B_{12} \left( 1 - e^{-\beta h\nu} \right) \phi(\nu), \quad S_{\nu} = B_{\nu}(T).$$
(4.93)

The LTE source function is just the Kirchhoff's law. However, the correction factor  $1 - e^{-\beta h\nu}$  in the absorption coefficient involves stimulated emission.

#### **Non-Thermal Emission:**

If the matter is not in TE,

$$\frac{n_1}{n_2} \neq \frac{g_1}{g_2} e^{\beta h \nu}.$$
 (4.94)

This occurs, for example, if the radiating particles in a plasma did not have a Maxwellian velocity distribution or if the atomic populations did not obey the Maxwell-Boltzmann distribution law. Equation (4.94) can also be applied to cases in which scattering is present.

#### **Inverted Populations; Masers:**

If a system is in TE ( $\nu$  and T are positive),

$$\frac{n_2}{n_1}\frac{g_1}{g_2} = e^{-\beta h\nu} < 1 \quad \text{so that} \quad \frac{n_1}{g_1} > \frac{n_2}{g_2}.$$
(4.95)

Even if the material is out of TE, Eq. (4.95) is usually satisfied and we say that there are normal populations. If we put enough atoms in the upper state, we have inverted populations,

$$\frac{n_1}{g_1} < \frac{n_2}{g_2},\tag{4.96}$$

the absorption coefficient is negative,  $\alpha_{\nu} < 0$  (see Eq. (4.91)), and the intensity along a ray increases. Such a system is said to be a *maser* (microwave amplification by stimulated emission of radiation) or *laser* for light.

The amplification involved here can be very large, a negative optical depth  $\tau = -100$ , for example, leads to an amplification of the intensity by a factor of  $10^{43}$  (cf. Eq. (4.30)). We will not discuss here masers in detail, however, maser effect in molecular lines has been observed in many astrophysical sources.

## 4.6 Scattering Effects

#### 4.6.1 Pure Scattering

For a pure thermal emission the amount of radiation emitted by a matter element does not depend on the incident radiation, the source function is always  $B_{\nu}(T)$  and depends only on the local temperature. The element would emit the same whether it is isolated in free space or embedded in a star with the ambient radiation field. This character of thermal radiation makes it particularly easy to treat.

Another emission process is scattering, which completely depends on the amount of radiation incident to the element. Perhaps the most important is electron (Thomson) scattering (see Sect. 6.4). At present we assume isotropic scattering, so that the emission coefficient is directionally independent. We also assume that the total amount of radiation emitted per unit frequency range is just equal to the total amount absorbed in that same frequency range. This is called *coherent, elastic or monochromatic scattering*. Non-relativistic Thomson scattering is fairly coherent, repeated scatterings can build up substantial effects (see Appendix 9). The emission coefficient for coherent, isotropic scattering can be found by equating the power absorbed per unit volume and frequency to the corresponding power emitted,

$$j_{\nu} = \sigma_{\nu} J_{\nu}, \tag{4.97}$$

where  $\sigma_{\nu}$  is the absorption coefficient of the scattering process (*scattering coefficient*). Dividing Eq. (4.97) by  $\sigma_{\nu}$ , we find that the source function for scattering is equal to the mean intensity within the emitting material,

$$S_{\nu} = J_{\nu} = \frac{1}{4\pi} \int I_{\nu} \,\mathrm{d}\Omega.$$
 (4.98)

RTE for pure scattering therefore is

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}s} = \sigma_{\nu} \left(J_{\nu} - I_{\nu}\right). \tag{4.99}$$

We cannot apply here the formal solution (4.34); since the source function is not known a priori and depends on the  $I_{\nu}$ , it becomes an integro-differential equation. An approximate method may be the Eddington approximation (see Sect. 4.7.2). A particularly useful way of treating scattering is by means of random walks. We regard the absorption, emission, and propagation as a probabilistic process of a single photon rather than the phenomenologic average behavior of large ensemble. For example, the probability of a photon traveling an optical depth  $\tau_{\nu}$  before absorption is  $e^{-\tau_{\nu}}$ . Similarly, in case of isotropic scattering, a single photon is scattered with equal probabilities into equal solid angles. We speak of a typical path of a photon, and the measured intensities can be interpreted as statistical averages over photons moving in such paths.

Consider a photon emitted in an infinite, homogeneous scattering region. It travels a displacement  $r_1$  before being scattered, then travels in another over a displacement  $r_2$  before being scattered, and so on. The net displacement of the photon after N free paths is

$$\boldsymbol{R} = \boldsymbol{r}_1 + \boldsymbol{r}_2 + \boldsymbol{r}_3 + \dots + \boldsymbol{r}_N. \tag{4.100}$$

To estimate the distance  $|\mathbf{R}|$  traveled by a typical photon, we square Eq. (4.100) and then average,

$$d^{2} = \langle \mathbf{R}^{2} \rangle = \langle \mathbf{r}_{1}^{2} \rangle + \langle \mathbf{r}_{2}^{2} \rangle + \langle \mathbf{r}_{3}^{2} \rangle + \dots + \langle \mathbf{r}_{N}^{2} \rangle + 2 \langle \mathbf{r}_{1} \cdot \mathbf{r}_{2} \rangle + 2 \langle \mathbf{r}_{1} \cdot \mathbf{r}_{3} \rangle + \dots$$
(4.101)

Each square term in (4.101) averages the square of photon mean free path  $\lambda^2$ . Since the cross terms in (4.101) involve averaging the cosine of the scattering angle and vanish for isotropic scattering as well as for any scattering with front-back symmetry, we obtain

$$d^2 = N\lambda^2$$
, so that  $d = \sqrt{N}\lambda$ . (4.102)

The quantity d is the mean net displacement of the photon.

We use this result to estimate the mean number of scatterings in a finite medium of typical size L. Suppose a photon is generated somewhere in the medium, then it will scatter until it escapes completely. For regions of large optical depth the number of required scatterings is roughly determined by  $d \sim L$ , Eq. (4.102) gives  $N \approx L^2/\lambda^2$ . Since  $\lambda$  is (an order of) mean free path,  $L/\lambda \approx \tau$  of the medium and

$$N \approx \tau^2 \quad (\tau \gg 1). \tag{4.103}$$

For regions of small optical thickness the mean number of scatterings is of order  $1 - e^{-\tau} \approx \tau$ ,

$$N \approx \tau \quad (\tau \ll 1). \tag{4.104}$$

For most estimates it is sufficient to take  $N \approx \tau^2 + \tau$  or  $N \approx \max(\tau, \tau^2)$  for any optical thickness.

#### 4.6.2 Combined Scattering and Absorption

The real emission and absorption of radiation is usually more than one process. Let us assume the material with an absorption coefficient  $\alpha_{\nu}$  describing thermal emission and a scattering coefficient  $\sigma_{\nu}$  describing coherent isotropic scattering. RTE then has two terms on the righthand side,

$$\frac{\mathrm{d}I_{\nu}}{\mathrm{d}s} = \alpha_{\nu} \left(B_{\nu} - I_{\nu}\right) + \sigma_{\nu} \left(J_{\nu} - I_{\nu}\right) = \left(\alpha_{\nu} + \sigma_{\nu}\right) \left(S_{\nu} - I_{\nu}\right).$$
(4.105)

The source function  $S_{\nu}$  in Eq. (4.105) is (cf. Eq. (4.33)),

$$S_{\nu} = \frac{\alpha_{\nu}B_{\nu} + \sigma_{\nu}J_{\nu}}{\alpha_{\nu} + \sigma_{\nu}}.$$
(4.106)

We define the net absorption coefficient  $\chi_{\nu} = \alpha_{\nu} + \sigma_{\nu}$  that is also called the *extinction* coefficient to distinguish it from the "true" absorption coefficient  $\alpha_{\nu}$ . Using this, the optical depth is  $d\tau_{\nu} = \chi_{\nu} ds = (\alpha_{\nu} + \sigma_{\nu}) ds$ 

Consider a matter element deep inside a medium at some constant temperature, we expect the nearly LTE radiation field,  $J_{\nu} = B_{\nu}(\tau)$ . From Eq. (4.106) also  $S_{\nu} = B_{\nu}(\tau)$  in TE. On the other hand, if the element is isolated in free space,  $J_{\nu} = 0$  and the source function  $S_{\nu} =$  $\alpha_{\nu}B_{\nu}/(\alpha_{\nu} + \sigma_{\nu})$  is in general a priori unknown and must be calculated as a part of a selfconsistent solution of the entire radiation field.

We extend the random walk arguments to the case of combined scattering and absorption. The mean free path of a photon before scattering or absorption is

$$\lambda_{\nu} = \chi_{\nu}^{-1}.\tag{4.107}$$

During the random walk process, the probabilities that any free path will by ended by a true absorption or by a scattering event is

$$\epsilon_{\nu} = \frac{\alpha_{\nu}}{\chi_{\nu}}, \quad 1 - \epsilon_{\nu} = \frac{\sigma_{\nu}}{\chi_{\nu}}, \tag{4.108}$$

where the quantity  $1 - \epsilon_{\nu}$  is called the *single-scattering albedo*. The source function (4.106) now is

$$S_{\nu} = (1 - \epsilon_{\nu}) J_{\nu} + \epsilon_{\nu} B_{\nu}. \tag{4.109}$$

Consider an *infinite* homogeneous medium. A random walk starts with thermal emission of a photon (creation) and ends after a number of scatterings with a true absorption (destruction). Since the walk can be terminated with probability  $\epsilon$  at the end of each free path, the mean number of free paths is  $N = \epsilon^{-1}$ . From Eq. (4.102), using Eqs. (4.107) and (4.108), we have

$$d^2 = \frac{\lambda^2}{\epsilon}$$
, so that  $d \approx (\alpha_{\nu} \chi_{\nu})^{-1/2}$ . (4.110)

The length d represents the net displacement between the points of creation and destruction of a typical photon; it is called the *diffusion length*, *thermalization length*, or *effective mean path*, d is generally frequency dependent.

The behavior of a *finite* medium can also be described in terms of random walks, it depends strongly on whether its size L is larger or smaller than the effective free path d. We introduce by convention the *effective optical thickness* of the medium  $\tau_* = L/d$ . Using Eq. (4.110),

$$\tau_* \approx \sqrt{\tau_a(\tau_a + \tau_s)}, \quad \text{where} \quad \tau_a = \alpha_\nu L \text{ and } \tau_s = \sigma_\nu L$$

$$(4.111)$$

are the absorption and scattering optical thickness.

When  $\tau_* \ll 1$ , the medium is said to be *effectively thin* or translucent and most photons will escape out of the medium before being destroyed by a true absorption. The monochromatic luminosity will correspond to total radiation created by thermal emission in the medium and

$$\mathfrak{L}_{\nu} = 4\pi \alpha_{\nu} B_{\nu} V \quad (\tau_* \ll 1), \tag{4.112}$$

where  $\mathfrak{L}_{\nu}$  is the emitted power per unit frequency and V is the volume of the medium.

When  $\tau_* \gg 1$ , the medium is said to be *effectively thick*. Most thermally emitted photons will be destroyed by true absorption before they can escape. The physical conditions at large effective depths approach TE and we expect  $I_{\nu} \rightarrow B_{\nu}$  and  $S_{\nu} \rightarrow B_{\nu}$ . Due to this property is the effective path length d sometimes called the *thermalization length*, since it describes the distance over which radiative TE is established.

We estimate the monochromatic luminosity of an effectively thick medium (to within an order of unity) by substituting the effective emitting volume by the surface area of the medium times the effective path length. This is reasonable, because only the photons emitted within an effective path length of the boundary have a chance to escape before being absorbed. Using Eqs. (4.107) and (4.108), we have

$$\mathfrak{L}_{\nu} \approx 4\pi \alpha_{\nu} B_{\nu} A d \approx 4\pi \sqrt{\epsilon_{\nu}} B_{\nu} A \quad (\tau_* \gg 1).$$
(4.113)

In the limiting case of no scattering,  $\epsilon_{\nu} \to 1$ , the emission will blackbody,  $\mathfrak{L}_{\nu} = \pi B_{\nu} A$ , and the factor  $4\pi$  in Eq. (4.113) should be replaced by  $\pi$ . However, the exact form of the equation depends on  $\epsilon_{\nu}$  and on geometry of a problem in a more complex way, we should take such solution only as an estimate.

#### 4.7 Radiative Diffusion

#### 4.7.1 The Rosseland Approximation

We have used random walk arguments to show that  $S_{\nu}$  approaches  $B_{\nu}$  at large effective optical depths in a homogeneous medium. Real media are rarely homogeneous, however, it is possible to derive a simple expression for the energy flux by relating it to the local temperature gradient. This is called the *Rosseland approximation*.

First let us assume that the material properties  $(T, \chi_{\nu}, \text{etc.})$  depend only on depth in the medium. This is called the plane-parallel approximation, where the intensity depends only on the angle  $\theta$ , which measures the direction of the ray with respect to normal to the planes of constant properties. Using  $\mu = \cos \theta$  and  $ds = dz \mu^{-1}$ , RTE in this case is

$$\mu \frac{\partial I_{\nu}(z,\mu)}{\partial z} = \chi_{\nu} \left( S_{\nu} - I_{\nu} \right), \quad \text{so that} \quad I_{\nu}(z,\mu) = S_{\nu} - \frac{\mu}{\chi_{\nu}} \frac{\partial I_{\nu}}{\partial z}. \tag{4.114}$$

When the studied point is deep in the material, the intensity changes slowly on the scale of a mean free path, the derivative  $\partial I_{\nu}/\partial z$  is small and we write a "zeroth" approximation,

$$I_{\nu}^{(0)}(z,\mu) \approx S_{\nu}^{(0)}(T).$$
 (4.115)

This does not depend on  $\mu$ , the zeroth-order mean intensity  $J_{\nu}^{(0)} = S_{\nu}^{(0)}$ . Eq. (4.106) implies  $I_{\nu}^{(0)} = S_{\nu}^{(0)} = B_{\nu}$ , just as we expected from the random walk arguments. We now get a "first" approximation by using  $I_{\nu}^{(0)} = B_{\nu}$  in the derivative,

$$I_{\nu}^{(1)}(z,\mu) \approx B_{\nu}(T) - \frac{\mu}{\chi_{\nu}} \frac{\partial B_{\nu}}{\partial z}.$$
(4.116)

This is justified, because the derivative term is small, and any approximation there is not critical. The angular dependence of the intensity to this order of approximation is linear in  $\mu$ .

We evaluate the flux  $F_{\nu}(z)$  from Eq. (4.4),

$$F_{\nu}(z) = \int I_{\nu}^{(1)}(z,\mu) \cos\theta \,\mathrm{d}\Omega = 2\pi \int_{-1}^{+1} I_{\nu}^{(1)}(z,\mu) \,\mu \,\mathrm{d}\mu.$$
(4.117)

The angle-independent part of  $I_{\nu}^{(1)}$  (i.e.,  $B_{\nu}$ ) does not contribute to the flux,

$$F_{\nu}(z) = -\frac{2\pi}{\chi_{\nu}} \frac{\partial B_{\nu}}{\partial z} \int_{-1}^{+1} \mu^2 \,\mathrm{d}\mu = -\frac{4\pi}{3\chi_{\nu}} \frac{\partial B_{\nu}(T)}{\partial z} = -\frac{4\pi}{3\chi_{\nu}} \frac{\partial B_{\nu}(T)}{\partial T} \frac{\partial T}{\partial z}$$
(4.118)

for the monochromatic flux.

We obtain the total flux by integration over all frequencies:

$$F(z) = \int_0^\infty F_\nu(z) \,\mathrm{d}\nu = -\frac{4\pi}{3} \frac{\partial T}{\partial z} \int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu(T)}{\partial T} \,\mathrm{d}\nu \tag{4.119}$$

and, using Eqs. (4.51) and (4.52),

$$\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} \,\mathrm{d}\nu = \frac{\partial}{\partial T} \int_0^\infty B_\nu \,\mathrm{d}\nu = \frac{\partial B(T)}{\partial T} = \frac{4\sigma T^3}{\pi} \tag{4.120}$$

Here  $\sigma$  is the Stefan-Boltzmann constant (not to be confused with  $\sigma_{\nu}$ ).

We define the Rosseland mean absorption coefficient  $\chi_R$  by

$$\frac{1}{\chi_R} \equiv \frac{\int_0^\infty \frac{1}{\chi_\nu} \frac{\partial B_\nu(T)}{\partial T} \,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} \,\mathrm{d}\nu},\tag{4.121}$$

which gives

$$F(z) = -\frac{16\sigma T^3}{3\chi_R}\frac{\partial T}{\partial z}.$$
(4.122)

Equation (4.122) is the Rosseland approximation for the energy flux. It is also called the equation of radiative diffusion (although this term is often used for other equations, see Sect. 4.7.2). It shows that radiative energy transport deep in a star is of the same nature as a heat conduction, with an effective heat conductivity  $16\sigma T^3/(3\chi_R)$ . It also shows that the energy flux depends on only one property of the extinction coefficient, on its Rosseland mean  $\chi_R$ . The Rosseland mean involves a weighted average of  $\chi_{\nu}^{-1}$  so that frequencies at which the extinction coefficient is small (the transparent regions) tend to dominate the averaging process. The weighting function  $\partial B_{\nu}/\partial T$  (Eq. (4.67)) has a similar profile to the Planck function, but it now peaks at values of  $h\nu/kT$  of order ~ 3.8 instead of ~ 2.8 as in the Wien's displacement law. Although we used a plane-parallel approximation to prove the Rosseland formula, the result is quite general: the vector of the flux is parallel with the negative temperature gradient and its magnitude is given by Eq. (4.122) (the necessary assumption is that all quantities change slowly on the radiation mean free path scale).

#### 4.7.2 The Eddington Approximation; Two-Stream Approximation:

The basic idea of the Rosseland approximation was that the intensities approach the Planck function at large effective depths in the medium. In the Eddington approximation we only assume that the intensities approach isotropy, and not necessarily their thermal values. Because thermal emission and scattering are isotropic, one expects isotropy of the intensities at depths comparable to an ordinary mean free path. The domain of applicability of the Eddington approximation is therefore potentially much larger than the Rosseland approximation, since the latter requires depths of the order of the effective free path.

With use of appropriate boundary conditions (here introduced via the two-stream approximation), we obtain solutions to scattering problems of reasonable accuracy at all depths. Following the near isotropy assumption, we expand the intensity as power series in  $\mu$ , with terms only up to linear:

$$I_{\nu}(\tau,\mu) = a_{\nu}(\tau) + b_{\nu}(\tau)\mu.$$
(4.123)

We integrate the frequency-dependent variables. Let us take the first three moments of intensity,

$$J \equiv \frac{1}{2} \int_{-1}^{+1} I \,\mathrm{d}\mu = a, \tag{4.124}$$

$$H \equiv \frac{1}{2} \int_{-1}^{+1} \mu I \,\mathrm{d}\mu = \frac{b}{3},\tag{4.125}$$

$$K \equiv \frac{1}{2} \int_{-1}^{+1} \mu^2 I \,\mathrm{d}\mu = \frac{a}{3},\tag{4.126}$$

where J is the mean intensity and H and K are proportional to the flux and radiation pressure, respectively. This gives the *Eddington approximation*,

$$K = \frac{J}{3}.\tag{4.127}$$

Note the equivalence of Eqs. (4.14) and (4.127). The only difference is that Eq. (4.127) is valid even for slightly non-isotropic fields, containing terms linear in  $\mu$ . We define the plane-parallel optical depth

$$\mathrm{d}\tau(z) = -\chi_{\nu}\,\mathrm{d}z,\tag{4.128}$$

and the corresponding RTE (Eq. (4.114)),

$$\mu \frac{\partial I}{\partial \tau} = I - S. \tag{4.129}$$

The source function S, given by Eqs. (4.106) and (4.109), is isotropic. If we multiply Eq. (4.129) by the factor 1/2 and integrate over  $\mu$  from -1 to +1, we obtain

$$\frac{\partial H}{\partial \tau} = J - S. \tag{4.130}$$

Similarly, multiplying by an extra factor  $\mu$  before integrating, we obtain

$$\frac{\partial K}{\partial \tau} = H = \frac{1}{3} \frac{\partial J}{\partial \tau},\tag{4.131}$$

using the Eddington approximation (4.127). The last two equations can be combined (with use of Eq. (4.109)) to the single second-order non-homogeneous equation for J,

$$\frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = J - S, \quad \text{so that} \quad \frac{1}{3}\frac{\partial^2 J}{\partial \tau^2} = \epsilon \left(J - B\right). \tag{4.132}$$

Equation (4.132) is also called the *radiative diffusion equation*. Given the temperature structure of the medium,  $B(\tau)$ , one can solve this equation for J and determine S from Eq. (4.109). The problem is essentially solved, because the full intensity  $I(\tau, \mu)$  can be found by formal solution of Eq. (4.129).

A significant form of Eq. (4.132) reflects the case when  $\epsilon$  does not depend on depth. Let us define (cf. Eq. (4.111)) the new optical depth scale

$$\tau_* = \sqrt{3\epsilon} \tau = \sqrt{3\tau_a(\tau_a + \tau_s)}.$$
(4.133)

The corresponding RTE is

$$\frac{\partial^2 J}{\partial \tau_*^2} = J - B. \tag{4.134}$$

We use Eq. (4.134) to demonstrate the properties of  $\tau_*$  as an effective optical depth.

To solve Eq. (4.132), we yet provide the boundary conditions. This can be done in several ways, we use here the *two-stream approximation*: we assume that the entire radiation field can be represented by rays traveling at just *two* angles,  $\mu = \pm 1/\sqrt{3}$ . Let us denote the outward and inward intensities by  $I^+(\tau) \equiv I(\tau, +1/\sqrt{3})$  and  $I^-(\tau) \equiv I(\tau, -1/\sqrt{3})$ . In terms of  $I^+$  and  $I^-$ , the moments J, H, and K have the representations

$$J = \frac{1}{2} \left( I^{+} + I^{-} \right), \quad H = \frac{1}{2\sqrt{3}} \left( I^{+} - I^{-} \right), \quad K = \frac{1}{6} \left( I^{+} + I^{-} \right) = \frac{J}{3}.$$
 (4.135)

Equation (4.135) is simply the Eddington approximation; in fact, the choice of the angles  $\mu = \pm 1/\sqrt{3}$  is motivated by the requirement this relations to be valid.

We now solve Eq. (4.135) for  $I^+$  and  $I^-$ , using Eq. (4.131):

$$I^{+} = J + \frac{1}{\sqrt{3}} \frac{\partial J}{\partial \tau}, \quad I^{-} = J - \frac{1}{\sqrt{3}} \frac{\partial J}{\partial \tau}.$$
(4.136)

These equations provide the necessary boundary conditions for Eq. (4.132). For example, suppose the medium extending from  $\tau = 0$  to  $\tau = \tau_0$  with no incident radiation. Then  $I^-(0) = 0$ ,  $I^+(\tau_0) = 0$ , and the boundary conditions are

$$\frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} = J \text{ at } \tau = 0, \quad \frac{1}{\sqrt{3}}\frac{\partial J}{\partial \tau} = -J \text{ at } \tau = \tau_0.$$
(4.137)

The two conditions are sufficient to determine the solution of Eq. (4.132).

One has proposed various methods to obtain boundary conditions; they all give equations similar to (4.137), with factors slightly different than  $1/\sqrt{3}$ . It is not worth for our purpose to discuss here all the alternatives in detail.

## Chapter 5

# **Radiation Field**

## 5.1 Maxwell Equations

(Rybicki & Lightman 1979; Mihalas & Mihalas 1984): We first review the theory for non-relativistic particles (in SI units). The definitions of the electric field  $\boldsymbol{E}(\boldsymbol{r},t)$  and the magnetic field  $\boldsymbol{B}(\boldsymbol{r},t)$  are made for a particle of charge q at point  $\boldsymbol{r}$  with velocity  $\boldsymbol{v}$ , using the Lorentz force:

$$\boldsymbol{F} = q \left( \boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} \right). \tag{5.1}$$

The rate of work (power) exerted by the electromagnetic field on a particle is

$$\boldsymbol{v} \cdot \boldsymbol{F} = q \boldsymbol{v} \cdot \boldsymbol{E}, \tag{5.2}$$

v because  $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0$ . Since  $F = m \, \mathrm{d} \mathbf{v} / \mathrm{d} t$  in case of non-relativistic particles, we have

$$q\mathbf{v} \cdot \mathbf{E} = \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left( mv^2 \right).$$
(5.3)

This may be generalized to force density (force on a volume element containing many charges),

$$\boldsymbol{f} = q\boldsymbol{E} + \boldsymbol{j} \times \boldsymbol{B},\tag{5.4}$$

where  $\rho$  and j are charge and current densities, respectively, defined as

$$\rho = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i} q_{i}, \quad \boldsymbol{j} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \sum_{i} q_{i} \boldsymbol{v}_{i}, \tag{5.5}$$

and  $\Delta V$  is the volume element. In Eqs. (5.4) and (5.5)  $\Delta V$  must be much smaller than characteristic scales but much larger than the volume containing a single particle.

The rate of work done by the field per unit volume then is

$$\frac{1}{\Delta V} \sum_{i} q_i \boldsymbol{v}_i \cdot \boldsymbol{E} = \boldsymbol{j} \cdot \boldsymbol{E}.$$
(5.6)

From Eq. (5.3) this is also the rate of change of mechanical energy per unit volume due to the electromagnetic field:

$$\frac{\mathrm{d}U_{\mathrm{mech}}}{\mathrm{d}t} = \boldsymbol{j} \cdot \boldsymbol{E}.$$
(5.7)

Maxwell equations relate  $\boldsymbol{E}, \boldsymbol{B}, \rho$ , and  $\boldsymbol{j}$ . In SI units their basic form is

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = \boldsymbol{\rho} \qquad \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \tag{5.8}$$

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$
  $\nabla \times \boldsymbol{H} = \boldsymbol{j} + \frac{\partial \boldsymbol{D}}{\partial t},$  (5.9)

while in Gaussian units, they are

$$\boldsymbol{\nabla} \cdot \boldsymbol{D} = 4\pi\rho \qquad \qquad \boldsymbol{\nabla} \cdot \boldsymbol{B} = 0 \qquad (5.10)$$

$$\boldsymbol{\nabla} \times \boldsymbol{E} = -\frac{1}{c} \frac{\partial \boldsymbol{B}}{\partial t} \qquad \boldsymbol{\nabla} \times \boldsymbol{H} = \frac{4\pi}{c} \boldsymbol{j} + \frac{1}{c} \frac{\partial \boldsymbol{D}}{\partial t}. \tag{5.11}$$

The fields D and H are in the both systems related to E and B by the same linear relations

$$\boldsymbol{D} = \epsilon \boldsymbol{E}, \quad \boldsymbol{B} = \mu \boldsymbol{H}, \tag{5.12}$$

where  $\epsilon$  and  $\mu$  are the dielectric constant and magnetic permeability of the medium.

An immediate consequence of Maxwell equations is *conservation of charge*: divergence of the  $\nabla \times H$  equation gives

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0, \qquad (5.13)$$

which expresses conservation of charge for a volume element.

We now define energy density and energy flux of the electromagnetic field. Consider the work done per unit volume on a particle distribution, (cf. the second Eq. (5.9)):

$$\boldsymbol{j} \cdot \boldsymbol{E} = (\boldsymbol{\nabla} \times \boldsymbol{H}) \cdot \boldsymbol{E} - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}, \qquad (5.14)$$

Using the vector identity

$$\boldsymbol{E} \cdot (\boldsymbol{\nabla} \times \boldsymbol{H}) = \boldsymbol{H} \cdot (\boldsymbol{\nabla} \times \boldsymbol{E}) - \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}), \qquad (5.15)$$

we rewrite Eq. (5.14) into the form

$$\boldsymbol{j} \cdot \boldsymbol{E} = -\boldsymbol{H} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \boldsymbol{\nabla} \cdot (\boldsymbol{E} \times \boldsymbol{H}) - \boldsymbol{E} \cdot \frac{\partial \boldsymbol{D}}{\partial t}.$$
(5.16)

If  $\epsilon$  and  $\mu$  are not time-dependent, the above relation may be written as (using Eqs. (5.12))

$$\boldsymbol{j} \cdot \boldsymbol{E} + \frac{1}{2} \frac{\partial}{\partial t} \left( \epsilon E^2 + \frac{B^2}{\mu} \right) = -\boldsymbol{\nabla} \cdot \left( \boldsymbol{E} \times \boldsymbol{H} \right).$$
(5.17)

Equation (5.17) represents the Poynting theorem in differential form: The rate of change of mechanical energy per unit volume plus the rate of change of field energy per unit volume is equal to negative divergence of the electromagnetic flux. We thus set the electromagnetic energy density equal to

$$U_{\text{field}} = \frac{1}{2} \frac{\partial}{\partial t} \left( \epsilon E^2 + \frac{B^2}{\mu} \right) = U_E + U_B, \qquad (5.18)$$

and the vector of electromagnetic flux, or Poynting vector, equal to

$$\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{H}.\tag{5.19}$$

Integrating the above over a volume element and using the divergence theorem:

$$\int_{V} \boldsymbol{j} \cdot \boldsymbol{E} \, \mathrm{d}V + \frac{1}{2} \, \frac{\partial}{\partial t} \int_{V} \left( \epsilon E^{2} + \frac{B^{2}}{\mu} \right) \, \mathrm{d}V = -\int_{\Sigma} \boldsymbol{S} \cdot \, \mathrm{d}\boldsymbol{A}, \tag{5.20}$$

or shortly

$$\frac{\partial}{\partial t} \left( U_{\text{mech}} + U_{\text{field}} \right) = -\int_{\Sigma} \boldsymbol{S} \cdot \, \mathrm{d}\boldsymbol{A}.$$
(5.21)

That is, the rate of change of total (mechanical plus field) energy within the volume V is equal to the net inward flow of energy through the bounding surface  $\Sigma$ .

Although  $U_{\text{field}}$  is called a field energy, there are contributions from the matter, because  $\epsilon$  and  $\mu$  represent material properties. If we treat all charges (free and bound) as part of the mechanical system, then we would use only the *microscopic* fields  $\boldsymbol{E}$  and  $\boldsymbol{B}$ . Then  $\boldsymbol{j}$  would be replaced by the sum of the conduction and induced molecular currents and  $\boldsymbol{S} \to \boldsymbol{E} \times \boldsymbol{B}/\mu$ . When matter and field is present, the allocation of energy into matter and field is somewhat arbitrary, while the total energy is always conserved.

If we now consider only the microscopic energy flux or the flux in vacuum with use of Eq. (4.10) where p = E/c for photons, we can write the electromagnetic momentum density as

$$\boldsymbol{g} = \frac{\boldsymbol{S}}{c^2} = \frac{1}{\mu c^2} \boldsymbol{E} \times \boldsymbol{B}.$$
 (5.22)

The angular momentum carried by the field is given by the angular momentum density  $\mathcal{L}$ ,

$$\mathcal{L} = \mathbf{r} \times \mathbf{g},\tag{5.23}$$

where  $\boldsymbol{r}$  is the position vector from the point about which the angular momentum is computed.

In electrostatics and magnetostatics both  $\boldsymbol{E}$  and  $\boldsymbol{B}$  decrease like  $r^{-2}$  as  $r \to \infty$ . This implies that  $\boldsymbol{S}$  decreases like  $r^{-4}$  in static problems. Thus the right-hand side integral in Eqs. (5.20) and (5.21) goes to zero, since the surface area increases only as  $r^2$ . Because for time-varying fields  $\boldsymbol{E}$  and  $\boldsymbol{B}$  may decrease only as  $r^{-1}$ , the integral then contributes a finite amount to the rate of change of energy of the system. This finite energy flowing outward (or inward) at large distances is called radiation. Those parts of  $\boldsymbol{E}$  and  $\boldsymbol{B}$  that decrease as  $r^{-1}$  at large distances constitute the radiation field.

## 5.2 Electromagnetic Potentials

Following the Maxwell's equations, we now express  $\boldsymbol{E}$  and  $\boldsymbol{B}$  fields in terms of a scalar potential  $\phi(\boldsymbol{r},t)$  and a vector potential  $\boldsymbol{A}(\boldsymbol{r},t)$ : Defining the vector potential as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A} \tag{5.24}$$

(so that we hold the  $\nabla \cdot B$  equation) and inserting it into  $\nabla \times E$  equation, we obtain

$$\nabla \times \left( \boldsymbol{E} + \frac{\partial \boldsymbol{A}}{\partial t} \right) = 0, \text{ which yields } \boldsymbol{E} = -\nabla \phi - \frac{\partial \boldsymbol{A}}{\partial t}$$
 (5.25)

for any arbitrary scalar function (scalar potential)  $\phi$ . Substituting this into Maxwell equations, we obtain

$$\nabla^2 \phi + \frac{\partial}{\partial t} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) = -\frac{\rho}{\epsilon}, \qquad (5.26)$$

$$\nabla^2 \mathbf{A} - \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} - \boldsymbol{\nabla} \left( \boldsymbol{\nabla} \cdot \mathbf{A} + \epsilon \mu \frac{\partial \phi}{\partial t} \right) = -\mu \mathbf{j}.$$
(5.27)

The potentials are not uniquely determined, for example, the addition of the gradient of an arbitrary scalar function  $\psi$  to **A** will leave **B** unchanged. Following Eqs. (5.24) and (5.25), we may set

$$\mathbf{A}' \to \mathbf{A} + \nabla \psi, \quad \phi' \to \phi - \frac{\partial \psi}{\partial t}.$$
 (5.28)

We can calibrate the potentials to satisfy the *Lorentz gauge* condition,

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} + \epsilon \mu \frac{\partial \phi}{\partial t} = 0, \qquad (5.29)$$

hence the term in bracket in Eq. (5.27) is zero. With this gauge, Eqs. (5.26) and (5.27) become inhomogeneous wave equations:

$$\nabla^2 \phi - \epsilon \mu \frac{\partial^2 \phi}{\partial t^2} = -\frac{\rho}{\epsilon},\tag{5.30}$$

$$\nabla^2 \mathbf{A} - \epsilon \mu \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{j}.$$
(5.31)

The particular form of the Lorentz gauge (Eq. (5.29)) is the Coulomb gauge condition,

$$\boldsymbol{\nabla} \cdot \boldsymbol{A} = \boldsymbol{\nabla} \phi = 0. \tag{5.32}$$

The solutions of Eqs. (5.30) and (5.31) may be written as integrals over the sources:

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \int \frac{[\rho] \,\mathrm{d}^3 \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|},\tag{5.33}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi} \int \frac{[\boldsymbol{j}] \,\mathrm{d}^3 \boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}.$$
(5.34)

Equations (5.33) and (5.34) are the retarded potentials, since the quantity [Q] means that Q is evaluated at the retarded time,  $[Q] \equiv Q(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)$ , which refers to conditions at the point  $\mathbf{r}'$  that existed at a time earlier than t by the time  $|\mathbf{r} - \mathbf{r}'|/c$  required for light to travel between  $\mathbf{r}$  and  $\mathbf{r}'$ , so that the potentials at point  $\mathbf{r}$  can only be affected by conditions at point  $\mathbf{r}'$  at such a retarded time.

### 5.3 Plane Electromagnetic Waves

Maxwell equations in vacuum become (cf. Eqs. (5.10))

$$\nabla \cdot \boldsymbol{E} = 0 \qquad \nabla \cdot \boldsymbol{B} = 0$$
$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \nabla \times \boldsymbol{B} = \epsilon_0 \mu_0 \frac{\partial \boldsymbol{E}}{\partial t}. \tag{5.35}$$

Solution of these equations proves the existence of traveling waves that carry energy. Taking the curl of the third equation and combining it with the fourth and the first equation (using the vector identity  $\nabla \times (\nabla \times \mathbf{E}) = \nabla (\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$ ), we obtain the wave equation for  $\mathbf{E}$ :

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{E}) = -\epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2}, \quad \text{that is} \quad \boldsymbol{\nabla}^2 \boldsymbol{E} - \epsilon_0 \mu_0 \frac{\partial^2 \boldsymbol{E}}{\partial t^2} = 0.$$
 (5.36)

An identical equation holds for  $\boldsymbol{B}$ , since Eq. (5.35) is invariant under  $\boldsymbol{E} \to \boldsymbol{B}, \, \boldsymbol{B} \to -\boldsymbol{E}$ .

Let us now suppose the solution of the form

$$\boldsymbol{E} = \hat{\boldsymbol{a}}_1 E_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \quad \boldsymbol{B} = \hat{\boldsymbol{a}}_2 B_0 e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}, \tag{5.37}$$

where  $\hat{\boldsymbol{a}}_1$  and  $\hat{\boldsymbol{a}}_2$  are unit vectors,  $E_0$  and  $B_0$  are complex constants, and  $\boldsymbol{k} = k\boldsymbol{n}$  and w are the "wave vector" and angular frequency, respectively. Such solutions represent waves traveling in the direction  $\boldsymbol{n}$ . By superposing such waves propagating in all directions with all frequencies, we construct the most general solution to the source-free Maxwell equations. Substitution into vacuum Maxwell's equations (5.35) yields:

$$i\mathbf{k} \cdot \hat{\mathbf{a}}_{1} E_{0} = 0 \qquad i\mathbf{k} \cdot \hat{\mathbf{a}}_{2} B_{0} = 0$$
  
$$i\mathbf{k} \times \hat{\mathbf{a}}_{1} E_{0} = i\omega \, \hat{\mathbf{a}}_{2} B_{0} \qquad i\mathbf{k} \times \hat{\mathbf{a}}_{2} B_{0} = -i\omega \, \hat{\mathbf{a}}_{1} \epsilon_{0} \mu_{0} E_{0}. \qquad (5.38)$$

From the top two equations we see that  $\hat{a}_1$  and  $\hat{a}_2$  are perpendicular to k. From the bottom two equations we see that  $\hat{a}_1$  and  $\hat{a}_2$  are perpendicular to each other. The vectors  $\hat{a}_1$ ,  $\hat{a}_2$ , and k form a right-hand triad of perpendicular vectors,  $E_0$  and  $B_0$  are thus related by

$$E_0 = \frac{\omega}{k} B_0, \quad B_0 = \frac{\omega}{k} \epsilon_0 \mu_0 E_0, \quad \text{so that} \quad E_0 = \left(\frac{\omega}{k}\right)^2 \epsilon_0 \mu_0 E_0 \quad \text{and} \quad \omega^2 = \frac{k^2}{\epsilon_0 \mu_0}. \tag{5.39}$$

Taking k and  $\omega$  positive, we have

$$\omega = \frac{k}{\sqrt{\epsilon_0 \mu_0}} \quad \text{and} \quad E_0 = \frac{1}{\sqrt{\epsilon_0 \mu_0}} B_0. \tag{5.40}$$

The waves propagate with a phase velocity  $v_{\rm ph} = \omega/k$ , and, as expected, in a vacuum with the group velocity,  $v_{\rm g} = \partial \omega/\partial k$ , so that

$$v_{\rm ph} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c, \quad v_{\rm g} = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = c.$$
 (5.41)

We can now compute the energy flux and energy density of the waves. Since E and B both vary sinusoidally in time, the Poynting vector and the energy density actually fluctuate; however, we take a *time average*, since this is in most cases what is measured. If A(t) and B(t) are two complex quantities with the same sinusoidal time dependence

$$A(t) = \mathcal{A} e^{i\omega t}, \quad B(t) = \mathcal{B} e^{i\omega t},$$
 (5.42)

then the time average of the product of their real parts is

$$\langle \operatorname{Re} A(t) \cdot \operatorname{Re} B(t) \rangle = \frac{1}{2} \operatorname{Re} \left( \mathcal{A} \mathcal{B}^* \right) = \frac{1}{2} \operatorname{Re} \left( \mathcal{A}^* \mathcal{B} \right),$$
 (5.43)

where the asterisk denotes complex conjugation. The time-averaged Poynting vector satisfies (cf. Eq. (5.19) with  $E_0 = cB_0$ )

$$\langle \mathbf{S} \rangle = \frac{1}{T} \int_0^T \mathbf{S}(t) \, \mathrm{d}t = \frac{1}{2\mu_0} \operatorname{Re}\left(E_0 B_0^*\right), \quad \text{so that} \quad \langle \mathbf{S} \rangle = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} \left|E_0\right|^2 = \frac{1}{2\mu_0} \frac{\left|B_0\right|^2}{\sqrt{\epsilon_0\mu_0}}.$$
 (5.44)

Similarly, the time-averaged energy density is (cf. Eq. (5.18))

$$\langle U \rangle = \frac{1}{4} \operatorname{Re} \left( \epsilon_0 E_0 E_0^* + \frac{1}{\mu_0} B_0 B_0^* \right), \quad \text{so that} \quad \langle U \rangle = \frac{\epsilon_0}{2} |E_0|^2 = \frac{1}{2\mu_0} |B_0|^2.$$
(5.45)

Therefore, the velocity of energy flow is also  $\langle S \rangle / \langle U \rangle = 1 / \sqrt{\epsilon_0 \mu_0} = c$ .

This was vacuum solution. Similar results hold if we use a constant permitivity and permeability. However, in practice these quantities depend on frequency, so a more careful approach is required.

## 5.4 The Radiation Spectrum

The radiation spectrum depends on temporal variations of electric field (we can ignore the magnetic field, since it mimics the electric field). One cannot give a meaning to the spectrum at a precisely instant time, knowing only the electric field at one point. Instead, one must take into account the spectrum of many waves, or of the radiation at one point during a sufficiently long time  $\Delta t$ . However, having such a record of the radiation field in time  $\Delta t$ , we still can only define the spectrum within a frequency resolution  $\Delta \omega$ , where

$$\Delta\omega\Delta t > 1. \tag{5.46}$$

This uncertainty relation is a property of any wave theory of light.

Let us assume, for simplicity, that the radiation is in the form of a finite pulse (in practice, we only require that  $\boldsymbol{E}(t)$  vanishes sufficiently rapidly for  $t \to \pm \infty$ ). Let us treat only one of the two independent components of the transverse electric field, say  $E(t) \equiv \hat{\boldsymbol{a}} \cdot \boldsymbol{E}(t)$ . We may thus express E(t) in terms of a Fourier integral (Fourier transform):

$$\hat{E}(\omega) = \int_{-\infty}^{\infty} E(t) e^{-i\omega t} dt, \qquad E(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{E}(\omega) e^{i\omega t} d\omega.$$
(5.47)

The function  $\widehat{E}(\omega)$  is complex, however, E(t) is real, so we can write

$$\hat{E}(-\omega) = \int_{-\infty}^{\infty} E(t) e^{i\omega t} dt = \hat{E}^*(\omega), \qquad (5.48)$$

(the asterisk denotes a complex conjugate function) so that the negative frequencies can be eliminated.

 $\vec{E}(\omega)$  contains the whole information about the frequency behavior of E(t). To convert this into frequency information about the energy, we express the work per unit time per unit area in terms of the Poynting vector as a representative of directional energy flux (energy or work transfer per unit area A per unit time t):

$$\frac{\mathrm{d}W}{\mathrm{d}t\,\mathrm{d}A} = \sqrt{\frac{\epsilon_0}{\mu_0}} E^2(t),\tag{5.49}$$

while the total energy (or work) per unit area within the pulse is

$$\frac{\mathrm{d}W}{\mathrm{d}A} = \sqrt{\frac{\epsilon_0}{\mu_0}} \int_{-\infty}^{\infty} E^2(t) \,\mathrm{d}t.$$
(5.50)

Equation (5.48) gives  $|\hat{E}(\omega)|^2 = |\hat{E}(-\omega)|^2$  and from the Parseval's theorem

$$\int_{-\infty}^{\infty} |x(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 \, \mathrm{d}\omega, \qquad (5.51)$$

(where  $X(\omega) = \mathcal{F}_{\omega}\{x(t)\}$  represents the continuous Fourier transform in normalized, unitary form) follows

$$\int_{-\infty}^{\infty} E^2(t) \,\mathrm{d}t = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 \,\mathrm{d}\omega = \frac{1}{\pi} \int_0^{\infty} |\hat{E}(\omega)|^2 \,\mathrm{d}\omega.$$
(5.52)

Inserting Eq. (5.52) into Eq. (5.50) gives the energy per unit area per unit frequency,

$$\frac{\mathrm{d}W}{\mathrm{d}A} = \frac{1}{\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} \int_0^\infty |\hat{E}(\omega)|^2 \,\mathrm{d}\omega, \quad \text{so that} \quad \frac{\mathrm{d}W}{\mathrm{d}A \,\mathrm{d}\omega} = \frac{1}{\pi} \sqrt{\frac{\epsilon_0}{\mu_0}} |\hat{E}(\omega)|^2. \tag{5.53}$$

We note that this is the total energy per area per frequency range in the entire pulse; we do not write "per unit time", because to write both dt and  $d\omega$  would lead to violation of the uncertainty relation between  $\omega$  and t. However, if the pulse is repeated on an average time scale T, then we may formally write

$$\frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega\,\mathrm{d}t} = \frac{1}{\pi}\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{|\hat{E}(\omega)|^2}{T}.$$
(5.54)

This formula can be used to define the portion of spectrum of a length T of much longer signal. If a very long signal has more or less the same properties over its entire length (*time stationarity*) then the result will be independent of t for large T and we may write

$$\frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega\,\mathrm{d}t} = \frac{1}{\pi}\sqrt{\frac{\epsilon_0}{\mu_0}} \lim_{T \to \infty} \frac{1}{T} |\hat{E}_T(\omega)|^2,\tag{5.55}$$

where the subscript T on  $\hat{E}_T(\omega)$  emphasizes the transform of a portion of the function E(t) of length T. We can include infinitely long waves (such as sine waves) using the formulas based on finite pulses.

The efficiency of the concept of local spectrum depends on whether the changes of character of E(t) occur on a time scale long enough that one can still define time interval T in which a suitable frequency resolution  $\Delta \omega \sim 1/T$  can be obtained. Otherwise one must consider the spectrum of the entire pulse as the basic entity.

## 5.5 Polarization and Stokes Parameters

#### 5.5.1 Monochromatic Waves

The monochromatic plane waves in Eq. (5.37) are linearly polarized; the electric vector oscillates in the direction  $\hat{a}_1$ , which, together with k, defines the *plane of polarization*. By superposing two such waves in perpendicular directions, we can construct the general state of polarization for a given k and  $\omega$ . We consider only the electric vector  $\boldsymbol{E}$ ; the magnetic vector stays perpendicular to it with the same magnitude as  $\boldsymbol{E}$ . Let us examine the electric vector at an arbitrary point (say,  $\boldsymbol{r} = 0$ ) and choose axes x and y with corresponding unit vectors  $\hat{\boldsymbol{x}}$  and  $\hat{\boldsymbol{y}}$ . The direction of the wave is right-handed, along the axis z with corresponding unit vector  $\hat{\boldsymbol{z}}$ . Then the electric vector is the real part of

$$\boldsymbol{E} = (\hat{\boldsymbol{x}} E_1 + \hat{\boldsymbol{y}} E_2) e^{-i\omega t} = \boldsymbol{E}_0 e^{-i\omega t}.$$
(5.56)

This generalization of Eq. (5.37) replaces  $\hat{a}_1 E_1$  by the complex vector  $\boldsymbol{E}_0$ . The complex amplitudes  $\mathcal{E}_1$  and  $\mathcal{E}_2$  are

$$E_1 = \mathcal{E}_1 e^{i\phi_1}, \quad E_2 = \mathcal{E}_2 e^{i\phi_2}.$$
 (5.57)

From the real part of **E** we find the physical components of the electric field along  $\hat{x}$  and  $\hat{y}$ ,

$$E_x = \mathcal{E}_1 \cos(\omega t - \phi_1), \quad E_y = \mathcal{E}_2 \cos(\omega t - \phi_2).$$
(5.58)

These equations describe the tip of the electric field vector in the x-y plane.

The figure traced out is an ellipse, the general wave is said to be elliptically polarized. The equations for a general ellipse with its principal axes x' and y', which are tilted at an angle  $\varphi$  to the axes x and y, are

$$E'_{x} = \mathcal{E}_{0} \cos\beta\cos\omega t, \quad E'_{y} = -\mathcal{E}_{0} \sin\beta\sin\omega t, \tag{5.59}$$

where  $-\pi/2 \leq \beta \leq \pi/2$ . The magnitudes of the principal axes are  $\mathcal{E}_0|\cos\beta|$  and  $\mathcal{E}_0|\sin\beta|$ , since  $(E'_x/\mathcal{E}_0\cos\beta)^2 + (E'_y/\mathcal{E}_0\sin\beta)^2 = 1$ . The ellipse will be traced out in a clockwise sense for  $0 < \beta < \pi/2$  and counter-clockwise sense for  $-\pi/2 < \beta < 0$ , as viewed by an observer toward whom the wave is propagating. This is called *right-* and *left-handed* elliptical polarization (or *negative* and *positive helicity*), respectively.

There are two degenerate cases of elliptical polarization: When  $\beta = \pm \pi/4$  the ellipse becomes a circle, the wave is circularly polarized. When  $\beta = 0$  or  $\beta = \pm \pi/2$ , the ellipse becomes a straight line, the wave is linearly polarized. In the latter case the wave is neither right-handed nor left-handed.

Let us now connect the quantities in Eq. (5.58) and those defining the principal axes of the ellipse. We transform  $\boldsymbol{E}$  components in Eq. (5.59) to the x- and y-axes by rotating through the angle  $\chi$ . This yields

$$E_x = \mathcal{E}_0 \left( \cos\beta\cos\chi\cos\omega t + \sin\beta\sin\chi\sin\omega t \right), \qquad (5.60)$$

$$E_y = \mathcal{E}_0 \left( \cos\beta \sin\chi \cos\omega t - \sin\beta \cos\chi \sin\omega t \right). \tag{5.61}$$

This becomes identical with Eq. (5.58) if we take

$$\mathcal{E}_{1} \cos \phi_{1} = \mathcal{E}_{0} \cos \beta \cos \chi, \qquad \qquad \mathcal{E}_{1} \sin \phi_{1} = \mathcal{E}_{0} \sin \beta \sin \chi, \\ \mathcal{E}_{2} \cos \phi_{2} = \mathcal{E}_{0} \cos \beta \sin \chi, \qquad \qquad \mathcal{E}_{2} \sin \phi_{2} = -\mathcal{E}_{0} \sin \beta \cos \chi. \tag{5.62}$$

Given  $\mathcal{E}_1$ ,  $\phi_1$ ,  $\mathcal{E}_2$ ,  $\phi_2$ , these equations can be solved for  $\mathcal{E}_0$ ,  $\beta$ , and  $\chi$  by means of the *Stokes* parameters for monochromatic waves:

$$I \equiv \mathcal{E}_1^2 + \mathcal{E}_2^2 = \mathcal{E}_0^2,$$
  

$$Q \equiv \mathcal{E}_1^2 - \mathcal{E}_2^2 = \mathcal{E}_0^2 \cos 2\beta \cos 2\chi,$$
  

$$U \equiv 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) = \mathcal{E}_0^2 \cos 2\beta \sin 2\chi,$$
  

$$V \equiv 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) = \mathcal{E}_0^2 \sin 2\beta.$$
(5.63)

From Eqs. (5.62) we have

$$\mathcal{E}_0 = \sqrt{I}, \quad \sin 2\beta = \frac{V}{I}, \quad \tan 2\chi = \frac{U}{Q}.$$
 (5.64)

Since an elliptical polarization is determined by the three parameters  $\mathcal{E}_0$ ,  $\beta$ , and  $\chi$ , we relate

$$I^2 = Q^2 + U^2 + V^2 \tag{5.65}$$

for a monochromatic wave.

The meanings of the Stokes parameters are: I is the flux or intensity, V is the circularity parameter that measures the ratio of principal axes of the ellipse (the wave has right- or lefthanded polarization when V is positive or negative, V = 0 for linear polarization). There is only one remaining independent parameter, Q or U, which measures the orientation of the ellipse relative to the x-axis; in case of a circular polarization Q = U = 0.

#### 5.5.2 Quasi-monochromatic Waves

The monochromatic waves are 100% polarized if the electric vector displays a simple, nonrandom directional behavior in time. However, in practice we never see a single monochromatic component but rather a superposition of many components, each with its own polarization. An important case of interest occurs when the amplitudes and phases of the wave relatively slowly vary in time, so that instead of Eq. (5.57) we have

$$E_1(t) = \mathcal{E}_1(t) e^{i\phi_1(t)}, \quad E_2(t) = \mathcal{E}_2(t) e^{i\phi_2(t)}.$$
 (5.66)

We assume that over short times, of order  $1/\omega$ , the wave looks completely elliptically polarized but over much longer times,  $\Delta t \gg 1/\omega$ , over which  $\mathcal{E}_1$ ,  $\mathcal{E}_2$ ,  $\phi_1$ , and  $\phi_2$  change substantially, this state of polarization can change completely. Such a wave is no longer monochromatic; by the uncertainty relation its frequency range  $\Delta \omega$  can be estimated as  $\Delta \omega > 1/\Delta t$  so that  $\Delta \omega \ll \omega$ . The wave is called *quasi-monochromatic*; the frequency range  $\Delta \omega$  is called the *bandwidth* of the wave, and the time  $\Delta t$  is called the *coherence time*.

The quantitative characterization of quasi-monochromatic waves depends on a kind of measurements that can be made. In principle, for strong waves the precise time variations of the quantities  $\mathcal{E}_1$ ,  $\mathcal{E}_2$ ,  $\phi_1$ , and  $\phi_2$  could be measured; this would be the most detailed characterization possible. On the other hand, most measurements usually involve some apparatus in which the characteristics of radiation are delayed. If we suppose that any time delays involved are short compared to the coherence time of the wave, then we can show that the outcome of a measurement with such a device depends on simple extensions of the Stokes parameters.

The most general linear transformation of field components due to measuring devices can be written

$$E_1' = \lambda_{11}E_1 + \lambda_{12}E_2, \quad E_2' = \lambda_{21}E_1 + \lambda_{22}E_2 \tag{5.67}$$

where the complex constants  $\lambda_{i,j}$  characterize the measuring apparatus. We measure the average sum of the squares of the 1' and 2' components of electric field, where 1' is and 2' are

$$\left\langle \left(\operatorname{Re} E_{1}^{\prime} \operatorname{e}^{-\mathrm{i}\omega t}\right)^{2} \right\rangle = |\lambda_{11}|^{2} \left\langle E_{1} E_{1}^{*} \right\rangle + \lambda_{11} \lambda_{12}^{*} \left\langle E_{1} E_{2}^{*} \right\rangle + \lambda_{12} \lambda_{11}^{*} \left\langle E_{2} E_{1}^{*} \right\rangle + |\lambda_{12}|^{2} \left\langle E_{2} E_{2}^{*} \right\rangle, \quad (5.68)$$

$$\left\langle \left(\operatorname{Re} E_{2}^{\prime} e^{-\mathrm{i}\omega t}\right)^{2} \right\rangle = |\lambda_{22}|^{2} \left\langle E_{2} E_{2}^{*} \right\rangle + \lambda_{22} \lambda_{21}^{*} \left\langle E_{2} E_{1}^{*} \right\rangle + \lambda_{21} \lambda_{22}^{*} \left\langle E_{1} E_{2}^{*} \right\rangle + |\lambda_{21}|^{2} \left\langle E_{1} E_{1}^{*} \right\rangle.$$
(5.69)

We modified Eq. (5.43) to average over the "fast" variations in the field described by the  $e^{-i\omega t}$  term. The brackets  $\langle \rangle$  on the right-hand side refer to only time averaging of the slowly varying combinations of  $E_1(t)$  and  $E_2(t)$ , where, for example,

$$\langle E_1 E_2^* \rangle = \frac{1}{T} \int_0^T E_1(t) E_2^*(t) \,\mathrm{d}t$$
 (5.70)

is the integral over the time interval of the measurement.

A set of four real quantities used to express  $\langle E_i E_j^* \rangle$  are the Stokes parameters for quasimonochromatic waves,

$$I \equiv \langle E_1 E_1^* \rangle + \langle E_2 E_2^* \rangle = \langle \mathcal{E}_1^2 + \mathcal{E}_2^2 \rangle, \qquad (5.71)$$

$$Q \equiv \langle E_1 E_1^* \rangle - \langle E_2 E_2^* \rangle = \left\langle \mathcal{E}_1^2 - \mathcal{E}_2^2 \right\rangle, \qquad (5.72)$$

$$U \equiv \langle E_1 E_2^* \rangle + \langle E_2 E_1^* \rangle = \langle 2\mathcal{E}_1 \mathcal{E}_2 \cos(\phi_1 - \phi_2) \rangle, \qquad (5.73)$$

$$V \equiv \frac{1}{i} \left( \langle E_1 E_2^* \rangle - \langle E_2 E_1^* \rangle \right) = \left\langle 2\mathcal{E}_1 \mathcal{E}_2 \sin(\phi_1 - \phi_2) \right\rangle, \tag{5.74}$$

where we used Eq. (5.57). Equations (5.71) - (5.74) are the time-dependent generalizations of Eqs. (5.63). The Stokes parameters most completely describe the radiation field, providing that two waves with the same parameters cannot be distinguished by measurements that use an apparatus of the above type.

Equation (5.65) does not apply for arbitrary quasi-monochromatic waves. The Schwartz inequality

$$\langle E_1 E_1^* \rangle \langle E_2 E_2^* \rangle \ge \langle E_1 E_2^* \rangle \langle E_2 E_1^* \rangle \tag{5.75}$$

implies the equality sign in (5.75) only if  $E_1(t)/E_2(t)$  is a time-independent complex constant. In this case the electric vector traces out an ellipse of fixed shape and fixed orientation and only its overall size changes slowly with time. Such a wave is equivalent to a pure elliptically polarized monochromatic wave because their Stokes parameters are the same. In other words, Eqs. (5.71) and (5.75) give

$$I^2 \ge Q^2 + U^2 + V^2, \tag{5.76}$$

where the equality holds for a completely elliptically polarized wave.

The other extreme is the completely *unpolarized* wave with unrelated phases between  $E_1$ and  $E_2$  and with no preferred orientation in the x-y plane, so that  $\langle \mathcal{E}_1^2 \rangle = \langle \mathcal{E}_2^2 \rangle$  and

$$Q = U = V = 0$$
, or  $Q^2 + U^2 + V^2 = 0.$  (5.77)

An important property of the Stokes parameters is that they are additive for a superposition of independent waves, that is, the waves with no permanent relations between phases of the various waves, and which are randomly and uniformly distributed from 0 to  $2\pi$  over the relevant time scales. A superposition

$$E_1 = \sum_k E_1^{(k)}, \quad E_2 = \sum_l E_2^{(l)}$$
 (5.78)

of different waves, each having its own  $E_1^{(k)}$  and  $E_1^{(k)}$ , gives

$$\langle E_i E_j^* \rangle = \sum_k \sum_l \langle E_1^{(k)} E_2^{(l)*} \rangle = \sum_k \langle E_1^{(k)} E_2^{(k)*} \rangle, \qquad (5.79)$$

where, due to phase randomness, only terms with k = l survive the averaging, as indicated. It follows that

$$I = \sum I^{(k)}, \quad Q = \sum Q^{(k)}, \quad U = \sum U^{(k)}, \quad V = \sum V^{(k)}, \quad (5.80)$$

proving the additivity.

By the superposition principle, an arbitrary set of Stokes parameters can be represented as

$$\begin{pmatrix} I\\Q\\U\\V\\V \end{pmatrix} = \begin{pmatrix} I - \sqrt{Q^2 + U^2 + V^2}\\0\\0\end{pmatrix} + \begin{pmatrix} \sqrt{Q^2 + U^2 + V^2}\\Q\\U\\V \end{pmatrix},$$
(5.81)

where the first term on the right-hand side represents a completely unpolarized wave of intensity  $I - \sqrt{Q^2 + U^2 + V^2}$  and the second term represents a completely (elliptically) polarized wave of intensity  $\sqrt{Q^2 + U^2 + V^2}$  (see Eq. (5.65)). The Stokes parameters for a quasi-monochromatic wave can be decomposed into the previously given forms of the completely polarized plus the unpolarized part. Such a wave is therefore said to be *partially polarized*. The degree of polarization is defined as the ratio (percentage) of the intensity of the polarized part to the total intensity,

$$\Pi = \frac{I_{\text{pol}}}{I} = \frac{\sqrt{Q^2 + U^2 + V^2}}{I}.$$
(5.82)

A special case is partial linear polarization, V = 0, with the maximum and minimum values of intensity

$$I_{\max} = \frac{1}{2}I_{\text{unpol}} + I_{\text{pol}}, \quad I_{\min} = \frac{1}{2}I_{\text{unpol}},$$
 (5.83)

where  $I_{\text{unpol}} = I - \sqrt{Q^2 + U^2}$  and  $I_{\text{pol}} = \sqrt{Q^2 + U^2}$ . Equation (5.82) finally gives

$$\Pi = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}}.$$
(5.84)

This formula applies only if the polarization is of plane type, while in case of present circular or elliptical polarization it underestimates its true degree.

#### 5.5.3 Limits of Phenomenological Transfer Theory

The specific intensity and associated concept of rays was used as a fundamental variable. However, there are certain limitations imposed on transfer theory by the wave or quantum nature of light. For example, we defined specific intensity by  $dE = I_{\nu} dA dt d\Omega d\nu$  (Eq. (4.2)), where dA,  $d\Omega$ ,  $d\nu$ , and dt were presumed to be infinitesimal. However, dA and  $d\Omega$  cannot be both arbitrarily small because the uncertainty principle for photons constrains

$$dx dp_x dy dp_y = p^2 dA d\Omega \ge h^2, \quad \text{so that} \quad dA d\Omega \ge \lambda^2.$$
(5.85)

As soon as the linear size of dA is of order of the wavelength, the direction becomes uncertain and the concept of rays breaks down. Another limitation results from the energy uncertainty principle,

$$dE dt \ge h^2$$
, so that  $d\nu dt \ge 1$ . (5.86)

Equations (5.85) and (5.86) imply that when the wavelength of light is larger than atomic dimensions (like in the optics), we cannot describe the interaction of light on the atomic scale in terms of specific intensity. However, we may regard transfer theory as a valid *macroscopic theory*, provided the absorption and emission properties are correctly calculated from quantum electrodynamics.

A more precise, classical treatment of the validity of rays is known as the *eikonal approxima*tion. This approach treats a scalar field rather than the vector electromagnetic fields. Rays are curves whose tangents at each point lie along the direction of the wave propagation. The rays are well defined only if their direction and amplitude is practically constant over a distance of a wavelength  $\lambda$ . This limit is called the *geometrical optics limit*. Suppose the wave is represented by a function

$$g(\mathbf{r},t) = a(\mathbf{r},t) e^{i\psi(\mathbf{r},t)}, \qquad (5.87)$$

where  $a(\mathbf{r},t)$  is the slowly varying *amplitude* and  $\psi(\mathbf{r},t)$  is the rapidly varying *phase*. The behavior of a and  $\psi$  is constrained by the wave equation

$$\boldsymbol{\nabla}^2 g(\boldsymbol{r}, t) - \frac{1}{c^2} \frac{\partial^2 g}{\partial t^2} = 0.$$
(5.88)

Substituting Eq. (5.87) in (5.88) gives

$$\nabla^2 a - \frac{1}{c^2} \frac{\partial^2 a}{\partial t^2} + ia \left( \nabla^2 \psi - \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2} \right) + 2i \left( \nabla a \cdot \nabla \psi - \frac{1}{c^2} \frac{\partial a}{\partial t} \frac{\partial \psi}{\partial t} \right) -a \left( \nabla \psi \right)^2 + \frac{a}{c^2} \left( \frac{\partial \psi}{\partial t} \right)^2 = 0.$$
(5.89)

The slow and fast variability of amplitude and phase, respectively, implies the constraints

$$\frac{1}{a} |\nabla a| \ll |\nabla \psi|, \quad |\nabla^2 \psi| \ll |\nabla \psi|^2, \quad \frac{1}{a} |\nabla^2 a| \ll |\nabla \psi|^2, \\
\frac{1}{a} \left| \frac{\partial a}{\partial t} \right| \ll \left| \frac{\partial \psi}{\partial t} \right|, \quad \left| \frac{\partial^2 \psi}{\partial t^2} \right| \ll \left| \frac{\partial \psi}{\partial t} \right|^2,$$
(5.90)

that reduce Eq. (5.89) to the *eikonal equation*:

$$(\nabla\psi)^2 - \frac{1}{c^2} \left(\frac{\partial\psi}{\partial t}\right)^2 = 0.$$
(5.91)

If a in Eq. (5.87) is a constant, then the local wavevector  $\boldsymbol{k}$  (normal to the surfaces of constant phase  $\psi$ ) and the local frequency  $\omega$  are

$$\boldsymbol{k} = \boldsymbol{\nabla}\psi, \quad \boldsymbol{\omega} = -\frac{\partial\psi}{\partial t}, \quad (5.92)$$

and, substituting Eq. (5.92) into Eq. (5.91), we obtain Eq. (5.39) as the relationship between wavenumber and frequency of a plane wave.

## Chapter 6

# **Radiation from Moving Charges**

## 6.1 Liénard-Wiechert Potentials

(Rybicki & Lightman 1979; Mihalas & Mihalas 1984): Consider a particle of charge q that moves along a trajectory  $\mathbf{r} = \mathbf{r}_0(t)$  whose velocity is  $\mathbf{u}(t) = \dot{\mathbf{r}}_0(t)$ . We express its charge and current as

$$q = \int q\delta(\mathbf{r} - \mathbf{r}_0(t)) \,\mathrm{d}^3\mathbf{r}, \quad q\mathbf{u} = \int q\mathbf{u}\delta(\mathbf{r} - \mathbf{r}_0(t)) \,\mathrm{d}^3\mathbf{r}, \tag{6.1}$$

where the charge and current densities are  $\rho = q\delta(\mathbf{r} - \mathbf{r}_0(t))$  and  $\mathbf{j} = q\mathbf{u}\delta(\mathbf{r} - \mathbf{r}_0(t))$ , respectively, and where the general property of the Dirac  $\delta$ -function is a localization of an integral given by  $\int f(x)\delta(x-x_0) \, dx = f(x_0)$ .

We evaluate the retarded potentials (5.33) and (5.34) via these charge and current densities. Since the scalar potential is

$$\phi(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0} \int d^3\mathbf{r}' \int dt' \frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|} \delta\left(t'-t+\frac{|\mathbf{r}-\mathbf{r}'|}{c}\right),\tag{6.2}$$

the substitution of the charge and current densities yields integrals over the single variable t',

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \int R^{-1}(t') \,\delta\left[t' - t + \frac{R(t')}{c}\right] \,\mathrm{d}t',\tag{6.3}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0 q}{4\pi} \int \boldsymbol{u}(t') R^{-1}(t') \,\delta\left[t' - t + \frac{R(t')}{c}\right] \,\mathrm{d}t',\tag{6.4}$$

where  $R(t') = r - r_0(t')$  and R(t') = |R(t')|.

The argument of the  $\delta$ -function vanishes for a value of  $t' = t_{ret}$  given by

$$c\left(t - t_{\rm ret}\right) = R(t_{\rm ret}).\tag{6.5}$$

We substitute a new variable t'' = t' - t + R(t')/c whose differential

$$dt'' = \left[1 + \frac{1}{c}\dot{R}(t')\right] dt' = \left[1 - \frac{1}{c}\boldsymbol{n}(t') \cdot \boldsymbol{u}(t')\right] dt', \tag{6.6}$$

where the latter we obtain by differentiating the identity  $R^2(t') = \mathbf{R}^2(t')$  to  $2R(t')\dot{R}(t') = -2\mathbf{R}(t') \cdot \mathbf{u}(t')$ , where  $\dot{\mathbf{R}}(t') = -\mathbf{u}(t')$  and the unit vector  $\mathbf{n} = \mathbf{R}/R$ . Equations (6.3) and (6.4)

take the form

$$\phi(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0} \int R^{-1}(t') \left[ 1 - \frac{1}{c} \mathbf{n}(t') \cdot \mathbf{u}(t') \right]^{-1} \delta(t'') \,\mathrm{d}t'', \tag{6.7}$$

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0 q}{4\pi} \int \boldsymbol{u}(t') R^{-1}(t') \left[ 1 - \frac{1}{c} \boldsymbol{n}(t') \cdot \boldsymbol{u}(t') \right]^{-1} \delta(t'') \, \mathrm{d}t'', \tag{6.8}$$

Setting t'' = 0 or equivalently  $t' = t_{\text{ret}}$  yields

$$\phi = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{\kappa R} \right], \quad \mathbf{A} = \frac{\mu_0}{4\pi} \left[ \frac{q\mathbf{u}}{\kappa R} \right], \tag{6.9}$$

where we keep the bracket notation for retarded potentials and where

$$\kappa(t_{\rm ret}) = 1 - \frac{1}{c} \boldsymbol{n}(t_{\rm ret}) \cdot \boldsymbol{u}(t_{\rm ret}).$$
(6.10)

Equations (6.9) are the *Liénard-Wiechert Potentials*. They differ from static electromagnetic theory in the factor  $\kappa(t_{\text{ret}})$  that becomes very important at velocities close to c, where it concentrates the potentials into a narrow cone about the particle velocity (it is related to the beaming effect - see Sect. 7.2.4).

The second difference is that the quantities are evaluated at the retarded time  $t_{\rm ret}$  which enables a particle to radiate. The potentials fall off as  $\propto 1/r$  so that the fields would decrease  $\propto 1/r^2$  if the differentiation of potentials acted solely on the  $\propto 1/r$  factor. Retardation involves an implicit dependence on position via the definition of retarded time, and differentiation with respect to this dependence carries the 1/r behavior of the potentials into the fields themselves. This allows radiation energy to flow to infinite distances.

## 6.2 The Velocity and Radiation Fields

The differentiation of the potentials to obtain the fields (using Eqs. (5.24) and (5.25)) is straightforward. We first determine the retarded position and time of the particle  $r_{\rm ret}$  and  $t_{\rm ret}$  when particle's velocity  $\boldsymbol{u} = \dot{\boldsymbol{r}}_0(t_{\rm ret})$  and acceleration  $\dot{\boldsymbol{u}} = \ddot{\boldsymbol{r}}_0(t_{\rm ret})$ . Using the notation

$$\boldsymbol{\beta} = \frac{\boldsymbol{u}}{c}, \quad \kappa = 1 - \boldsymbol{n} \cdot \boldsymbol{\beta}, \tag{6.11}$$

the fields are

$$\boldsymbol{E}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0} \left\{ \frac{q\left(\boldsymbol{n}-\boldsymbol{\beta}\right)\left(1-\boldsymbol{\beta}^2\right)}{\kappa^3 R^2} + \frac{1}{c} \frac{q\boldsymbol{n}}{\kappa^3 R} \times \left[\left(\boldsymbol{n}-\boldsymbol{\beta}\right) \times \dot{\boldsymbol{\beta}}\right] \right\},\tag{6.12}$$

$$\boldsymbol{B}(\boldsymbol{r},t) = \frac{1}{c}\boldsymbol{n} \times \boldsymbol{E}(\boldsymbol{r},t).$$
(6.13)

The electric field in Eq. (6.12) is composed of two terms. The first is the velocity field that falls off as  $1/r^2$  and is the generalization of the Coulomb law to moving particles, for  $u \ll c$ this becomes precisely Coulomb law. In case of a constant velocity, only this term contributes to the fields. The electric field in this case always points along the line toward the current position of the particle. This follows from the fact that the displacement to the field point from the retarded point is  $\mathbf{n}c\bar{t}$ , where  $\bar{t} = t - t_{\text{ret}}$  is the light travel time. In the same time the particle undergoes a displacement  $\boldsymbol{\beta}c\bar{t}$ . The displacement between the field point and the particle position is  $(\mathbf{n} - \boldsymbol{\beta})c\bar{t}$ , which is the direction of the velocity field in Eq. (6.12). The second term, the *acceleration field*, falls off as 1/R, is proportional to the particle's acceleration and is perpendicular to  $\boldsymbol{n}$ . This electric field, along with the corresponding magnetic field, constitutes the *radiation field*,

$$\boldsymbol{E}_{\mathrm{rad}}(\boldsymbol{r},t) = \frac{1}{4\pi\epsilon_0} \frac{q\boldsymbol{n}}{\kappa^3 Rc} \times \left[ (\boldsymbol{n} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right], \tag{6.14}$$

$$\boldsymbol{B}_{\mathrm{rad}}(\boldsymbol{r},t) = \frac{1}{c}\boldsymbol{n} \times \boldsymbol{E}_{\mathrm{rad}}.$$
(6.15)

 $\boldsymbol{E}$ ,  $\boldsymbol{B}$  and  $\boldsymbol{n}$  form a right-hand triad of mutually perpendicular vectors and  $|\boldsymbol{E}_{rad}| = |\boldsymbol{B}_{rad}|$ , this is consistent with the radiation solution of the source-free Maxwell equations.

## 6.3 Radiation from Non-relativistic System of Particles

We could describe radiation processes involving particles moving relativistically. However, this would be made easier after the section on special relativity. Therefore, we shall now focus to nonrelativistic particles.

#### 6.3.1 Larmor Formula

When  $|\beta| \ll 1$ , we can simplify Eqs. (6.14) and (6.15) to

$$\boldsymbol{E}_{\rm rad} = \frac{1}{4\pi\epsilon_0} \frac{q\boldsymbol{n}}{Rc^2} \times (\boldsymbol{n} \times \dot{\boldsymbol{u}}), \qquad (6.16)$$

$$\boldsymbol{B}_{\mathrm{rad}} = \frac{1}{c} \boldsymbol{n} \times \boldsymbol{E}_{\mathrm{rad}}.$$
 (6.17)

The magnitudes of  $\boldsymbol{E}_{rad}$  and  $\boldsymbol{B}_{rad}$  are

$$|\boldsymbol{E}_{\rm rad}| = \frac{1}{4\pi\epsilon_0} \frac{q\dot{u}}{Rc^2} \sin\Theta, \quad |\boldsymbol{B}_{\rm rad}| = \frac{1}{4\pi} \sqrt{\frac{\mu_0}{\epsilon_0}} \frac{q\dot{u}}{Rc^2} \sin\Theta, \tag{6.18}$$

where  $\Theta$  is the angle between the Poynting vector (5.19) direction **n** and the particle's acceleration  $\dot{\boldsymbol{u}}$ . The magnitude of the Poynting vector is

$$S = \sqrt{\frac{\epsilon_0}{\mu_0}} E_{\rm rad}^2 = \frac{1}{(4\pi)^2 \epsilon_0} \frac{q^2 \dot{u}^2}{R^2 c^3} \sin^2 \Theta.$$
(6.19)

We evaluate the energy dW emitted per unit time into solid angle  $d\Omega$  about **n** can be evaluated by multiplying the Poynting vector  $(J s^{-1} m^{-2})$  by the area  $dA = R^2 d\Omega$ ,

$$\frac{\mathrm{d}W}{\mathrm{d}t\,\mathrm{d}\Omega} = \frac{1}{(4\pi)^2\epsilon_0} \frac{q^2\dot{u}^2}{c^3}\sin^2\Theta.$$
(6.20)

We obtain the total power emitted into all angles by integrating (6.20),

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} = \frac{1}{(4\pi)^2 \epsilon_0} \frac{q^2 \dot{u}^2}{c^3} \int_{\Omega} \sin^2 \Theta \,\mathrm{d}\Omega = \frac{1}{8\pi\epsilon_0} \frac{q^2 \dot{u}^2}{c^3} \int_{-1}^{1} (1-\mu^2) \,\mathrm{d}\mu = \frac{1}{6\pi\epsilon_0} \frac{q^2 \dot{u}^2}{c^3}.$$
 (6.21)

This is the Larmor formula for emission from a single accelerated charge q.

There are two points to notice about Eqs. (6.20) and (6.21):

- 1. The characteristic dipole pattern  $\propto \sin^2 \Theta$ : no radiation is emitted along  $\dot{\boldsymbol{u}}$ , the maximum is emitted perpendicular to  $\dot{\boldsymbol{u}}$ .
- 2. The instantaneous direction of  $\boldsymbol{E}_{rad}$  is determined by  $\boldsymbol{\dot{u}}$  and  $\boldsymbol{n}$ . If the particle accelerates along a line, the radiation will be 100% linearly polarized in the plane of  $\boldsymbol{\dot{u}}$  and  $\boldsymbol{n}$ .

#### 6.3.2 The dipole approximation

In a system of N particles with positions  $\mathbf{r}_i$ , velocities  $\mathbf{u}_i$ , and charges  $q_i$ , i = 1, 2, ..., N, we can find the radiation field at large distances by the vector sum of the  $\mathbf{E}_{i, \text{rad}}$ . However, the radiation field refers to conditions at retarded times which differ for each particle. We must also keep track of the phase relations between the various pieces of the radiating system introduced by retardation.

In some situations, however, we may ignore this difficulty. Let the typical size of the system be L, and let the typical time scale for changes within the system be  $\tau$ . If  $\tau$  is much longer than the time it takes light to travel a distance L,  $\tau \gg L/c$ , then the differences in retarded time across the source are negligible. We may also characterize  $\tau$  as the time scale over which significant changes in the radiation field  $\mathbf{E}_{rad}$  occur, and this in turn determines the typical characteristic frequency of the emitted radiation. Calling this frequency  $\nu$ , we write

$$\nu \approx \frac{1}{\tau}, \quad \text{so that} \quad \frac{c}{\nu} \gg L \quad \text{or} \quad \lambda \gg L.$$
(6.22)

The differences in retarded times can be ignored when the size of the system is small compared to a wavelength.

We may also characterize  $\tau$  as the time a particle takes to change its motion substantially. If  $\ell < L$  is a characteristic scale of the particle's orbit and u its typical velocity, then  $\tau \sim \ell/u$  and the condition  $\tau \gg L/c$  implies  $u/c \ll \ell/L$ , which is equivalent to the nonrelativistic condition  $u \ll c$ . We may therefore consistently use the nonrelativistic form of the radiation fields for these problems. This implies

$$\boldsymbol{E}_{\text{rad}} = \frac{1}{4\pi\epsilon_0} \sum_{i} \frac{q_i}{c^2} \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \dot{\boldsymbol{u}}_i)}{R_i}.$$
(6.23)

Let  $R_0$  be the distance from a system point to the field point and  $\boldsymbol{d} = \sum_i q_i \boldsymbol{r}_i$  is the *dipole* moment. Because for  $R_0 \to \infty$  are the differences in the actual  $R_i$  negligible,

$$\boldsymbol{E}_{\rm rad} = \frac{1}{4\pi\epsilon_0} \frac{\boldsymbol{n} \times (\boldsymbol{n} \times \ddot{\boldsymbol{d}})}{c^2 R_0}.$$
 (6.24)

The right-hand side of Eq. (6.24) must still be evaluated at  $t_{\text{ret}}$ , using any point within the field, say, the point used to define  $R_0$ .

The generalization of Eqs. (6.20) and (6.21) become

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{1}{(4\pi)^2 \epsilon_0} \frac{\ddot{\boldsymbol{\sigma}}^2}{c^3} \sin^2 \Theta, \quad P = \frac{1}{6\pi\epsilon_0} \frac{\ddot{\boldsymbol{\sigma}}^2}{c^3}.$$
(6.25)

This is the *dipole approximation*, where the instantaneous polarization of  $\boldsymbol{E}$  lies in the plane of  $\ddot{\boldsymbol{d}}$  and  $\boldsymbol{n}$ .

Let us consider the spectrum of radiation in the dipole approximation. For simplicity we assume that d always lies in a single direction. Equation (6.24) gives

$$E(t) = \frac{1}{4\pi\epsilon_0} \frac{\ddot{d}(t)}{c^2 R_0} \sin\Theta, \qquad (6.26)$$

where E(t) and d(t) are the magnitudes of  $\boldsymbol{E}(t)$  and  $\boldsymbol{d}(t)$ , respectively. The Fourier transform

$$d(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{d}(\omega) e^{i\omega t} d\omega, \qquad (6.27)$$

implies

$$\ddot{d}(t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 \hat{d}(\omega) e^{i\omega t} d\omega = -\omega^2 d(t)$$
(6.28)

and Eq. (6.26) gives

$$E(t) = -\frac{1}{4\pi\epsilon_0} \frac{\omega^2}{c^2 R_0} d(t) \sin\Theta, \quad \text{so that} \quad \hat{E}(\omega) = -\frac{1}{4\pi\epsilon_0} \frac{\omega^2}{c^2 R_0} \hat{d}(\omega) \sin\Theta.$$
(6.29)

The energy per unit solid angle per frequency range and the total energy per frequency range, using Eqs. (5.53), (6.28) and  $dA = R_0^2 d\Omega$ , is

$$\frac{\mathrm{d}W}{\mathrm{d}\omega\,\mathrm{d}\Omega} = \frac{1}{16\pi^3\epsilon_0} \frac{\omega^4}{c^3} |\hat{d}(\omega)|^2 \sin^2\Theta, \quad \frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{1}{6\pi^2\epsilon_0} \frac{\omega^4}{c^3} |\hat{d}(\omega)|^2. \tag{6.30}$$

According to these formulas is the spectrum of the emitted radiation directly related to the frequencies of oscillation of the dipole moment. However, this is not true for particles with relativistic velocities.

#### 6.3.3 The general multiple expansion

We now indicate the features of the general case. Since  $\boldsymbol{E}$  and  $\boldsymbol{B}$  are related well outside the source, we may consider the vector potential  $\boldsymbol{A}$  contains all the necessary information. A Fourier analysis of the sources and fields is

$$\boldsymbol{j}_{\omega}(\boldsymbol{r}) = \int \boldsymbol{j}(\boldsymbol{r},t) \,\mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}t, \quad \boldsymbol{A}_{\omega}(\boldsymbol{r}) = \int \boldsymbol{A}(\boldsymbol{r},t) \,\mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}t.$$
(6.31)

Using Eq. (6.4) for the vector potential in the form

$$\boldsymbol{A}(\boldsymbol{r},t) = \frac{\mu_0}{4\pi} \int d^3 \boldsymbol{r}' \int dt' \frac{\boldsymbol{j}(\boldsymbol{r}',t')}{|\boldsymbol{r}-\boldsymbol{r}'|} \delta\left(t'-t+\frac{|\boldsymbol{r}-\boldsymbol{r}'|}{c}\right), \tag{6.32}$$

we take the Fourier transform of Eq. (6.32),

$$\boldsymbol{A}_{\omega}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \int \frac{\boldsymbol{j}_{\omega}(\boldsymbol{r}')}{|\boldsymbol{r} - \boldsymbol{r}'|} e^{-ik|\boldsymbol{r} - \boldsymbol{r}'|} d^3 \boldsymbol{r}', \qquad (6.33)$$

where  $k|\mathbf{r} - \mathbf{r}'| = (\omega/c)|\mathbf{r} - \mathbf{r}'| = \omega t$ . Equation (6.33) relates single Fourier components of  $\mathbf{j}$  and  $\mathbf{A}$ .

Suppose an origin of coordinates inside the source of size L. At field points such that  $r \gg L$ , we approximate

$$|\mathbf{r} - \mathbf{r}'| \approx r - \mathbf{n} \cdot \mathbf{r}',\tag{6.34}$$

where **n** points toward the field point **r** and  $r = |\mathbf{r}|$ . Substituting Eq. (6.34) in (6.33), we obtain

$$\boldsymbol{A}_{\omega}(\boldsymbol{r}) \approx \frac{\mu_0}{4\pi} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}}{r} \int \boldsymbol{j}_{\omega}(\boldsymbol{r}') \,\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{n}\cdot\boldsymbol{r}'} \,\mathrm{d}^3\boldsymbol{r}', \tag{6.35}$$

The factor  $\exp(ikr)$  outside the integral expresses the effect of retardation from the source as a whole. The factor  $\exp(ik\mathbf{n} \cdot \mathbf{r'})$  inside the integral expresses the relative retardation of each

element of the source. In our slow-motion approximation,  $kL \ll 1$ , we expand the exponential in the integral (6.35),

$$\boldsymbol{A}_{\omega}(\boldsymbol{r}) \approx \frac{\mu_0}{4\pi} \frac{\mathrm{e}^{\mathrm{i}k\boldsymbol{r}}}{r} \sum_{n=0}^{\infty} \frac{1}{n!} \int \boldsymbol{j}_{\omega}(\boldsymbol{r}') (\mathrm{i}k\boldsymbol{n} \cdot \boldsymbol{r}')^n \,\mathrm{d}^3 \boldsymbol{r}'.$$
(6.36)

Equation (6.36) is an expansion in the small dimensionless parameter  $kL = 2\pi L/\lambda$ . The dipole approximation results from taking just n = 0,

$$\boldsymbol{A}_{\omega}(\boldsymbol{r})|_{\text{dipole}} \approx \frac{\mu_0}{4\pi} \frac{\mathrm{e}^{\mathrm{i}\boldsymbol{k}\boldsymbol{r}}}{r} \int \boldsymbol{j}_{\omega}(\boldsymbol{r}') \,\mathrm{d}^3 \boldsymbol{r}'$$
(6.37)

and the quadrupole term is the term n = 1,

$$\boldsymbol{A}_{\omega}(\boldsymbol{r})|_{\text{quad}} \approx \frac{\mu_0}{4\pi} \frac{\mathrm{i}k \mathrm{e}^{\mathrm{i}kr}}{r} \int \boldsymbol{j}_{\omega}(\boldsymbol{r}')(\boldsymbol{n} \cdot \boldsymbol{r}') \,\mathrm{d}^3 \boldsymbol{r}'.$$
(6.38)

Although the frequencies of the vector potential (and hence in the radiation) are identical to those in the current density, these frequencies may differ from the frequencies of particle orbits. For example, if a particle orbits in a circle with angular frequency  $\omega_0$ , the function  $j_{\omega}(\mathbf{r})$  contains frequencies not only at  $\omega_0$  but at all harmonics  $2\omega_0, 3\omega_0...$  In the dipole approximation contributes only  $\omega_0$ , in the quadrupole approximation only  $2\omega_0$ , and so on.

## 6.4 Thomson Scattering

We apply the dipole formula to the process when a free charge radiates in response to an incident electromagnetic wave. If the charge oscillates at  $v \ll c$ , we neglect magnetic forces, B = E/c. The force due to a linearly polarized wave is

$$\boldsymbol{F} \equiv m\ddot{\boldsymbol{r}} = q\boldsymbol{e}E_0\sin\omega_0 t,\tag{6.39}$$

where q is the electric charge and e is the electric field direction. Equation (6.39) in terms of the dipole moment, d = qr, describes an oscillating dipole,

$$\ddot{\boldsymbol{d}} = \frac{q^2 E_0}{m} \boldsymbol{e} \sin \omega_0 t, \quad \text{so that} \quad \boldsymbol{d} = -\left(\frac{q^2 E_0}{m\omega_0^2}\right) \boldsymbol{e} \sin \omega_0 t, \tag{6.40}$$

with an amplitude  $\boldsymbol{d}_0$  given by  $\boldsymbol{e}$  times the bracket in Eq. (6.40).

Following Eqs. (6.25), the time-averaged power is

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^4 E_0^2}{32\pi^2 \epsilon_0 m^2 c^3} \sin^2 \Theta, \quad P = \frac{q^4 E_0^2}{12\pi \epsilon_0 m^2 c^3}, \tag{6.41}$$

where the time average of  $\sin^2 \omega_0 t$  gives a factor 1/2. The incident flux  $\langle S \rangle = \frac{1}{2} \sqrt{\epsilon_0/\mu_0} E_0^2$  (see Eq. (5.44)). We define the *differential cross section* d $\sigma$  for scattering into d $\Omega$ ,

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \langle S \rangle \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{2} \sqrt{\frac{\epsilon_0}{\mu_0}} E_0^2 \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega},\tag{6.42}$$

implying (using Eq. (6.41))

$$\left. \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \right|_{\text{polarized}} = \frac{q^4}{(4\pi\epsilon_0)^2 m^2 c^4} \sin^2\Theta = r_0^2 \sin^2\Theta. \tag{6.43}$$

The quantity  $r_0$  gives a "size" of the point charge, assuming its rest energy  $m_0c^2$  is purely electromagnetic. For an electron  $r_0 \equiv r_e \approx 2.82 \times 10^{-15}$  m is the *classical electron radius*. The total cross section we obtain by integrating over solid angle, using  $\mu \equiv \cos \Theta$ ,

$$\sigma = \int_{\Omega} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \,\mathrm{d}\Omega = \frac{q^4}{8\pi\epsilon_0^2 m^2 c^4} \int_{-1}^1 (1-\mu^2) \,\mathrm{d}\mu = \frac{q^4}{6\pi\epsilon_0^2 m^2 c^4} = \frac{8\pi}{3} r_{\mathrm{e}}^2. \tag{6.44}$$

Electron or Thomson cross section  $\sigma_T \approx 6.65 \times 10^{-29} \,\mathrm{m}^2$ .

The total as well as the differential cross sections are frequency independent, so that the scattering is equally effective at all frequencies. However, this is valid only for sufficiently low frequencies, at high frequencies, where  $h\nu \sim m_e c^2$  (X-rays with  $h\nu \geq 0.511$  MeV), the quantum mechanical effects must be involved (see Appendix 9). For intense radiation fields the electron also moves relativistically and the dipole approximation ceases to be valid.

The scattered radiation is linearly polarized in the plane of the incident polarization vector  $\boldsymbol{e}$  and the direction of scattering  $\boldsymbol{n}$ . We get the differential cross section for scattering of unpolarized radiation by recognizing that an unpolarized beam can be regarded as the superposition of two linearly polarized beams with perpendicular axes. We choose one such beam along  $\boldsymbol{e}_1$ , which is in the plane of the incident and scattered directions, and the second along  $\boldsymbol{e}_2$ , perpendicular to this plane. We denote  $\Theta$  the angle between  $e_1$  and  $\boldsymbol{n}$ , while the angle between  $\boldsymbol{e}_2$  and  $\boldsymbol{n}$  is  $\pi/2$ . We also classify the angle  $\theta$  between the scattered wave and incident wave,  $\theta = \pi/2 - \Theta$ . The differential cross section for unpolarized radiation is then the average of the cross sections for scattering of linearly polarized radiation through angles  $\Theta$  and  $\pi/2$ ,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\Big|_{\mathrm{unpol}} = \frac{1}{2} \left[ \frac{\mathrm{d}\sigma(\Theta)}{\mathrm{d}\Omega} \Big|_{\mathrm{pol}} + \frac{\mathrm{d}\sigma(\pi/2)}{\mathrm{d}\Omega} \Big|_{\mathrm{pol}} \right] = \frac{r_0^2}{2} \left( 1 + \sin^2 \Theta \right) = \frac{r_0^2}{2} \left( 1 + \cos^2 \theta \right), \qquad (6.45)$$

which depends only on the angle between the incident and scattered directions.

There are several important features of electron scattering of unpolarized radiation:

- Forward-backward symmetry: The scattering cross section, Eq. (6.45), is symmetric under the reflection  $\theta \to -\theta$ .
- Total cross section: The total scattering cross sections of unpolarized and polarized incident radiation are identical,  $\sigma_{\rm unpol} = \sigma_{\rm pol} = (8\pi/3)r_{\rm e}^2$ . This is because the electron at rest has no net direction intrinsically defined.
- Polarization of scattered radiation: The last two terms in Eq. (6.45) refer to intensities in two perpendicular directions in the plane normal to  $\boldsymbol{n}$ , since they arise from the two perpendicular components of the incident wave. Because the ratio of polarized intensities in the plane and perpendicular to the plane of scattering is  $\cos^2 \theta$ , the degree of polarization of the scattered wave is (cf. Eq. (5.84))

$$\Pi = \frac{1 - \cos^2 \theta}{1 + \cos^2 \theta}.\tag{6.46}$$

Since  $\Pi \geq 0$ , the electron scattering of a completely unpolarized incident wave produces a scattered wave with some degree of polarization that depends on the viewing angle with respect to the incident direction. Looking along the incident direction ( $\theta = 0$ ), we see no net polarization, since, by symmetry, all directions in the plane are equivalent. If we look perpendicularly to the incident wave ( $\theta = \pi/2$ ) we see 100% polarization, since the electron's motion is confined to a plane normal to the incident direction.

## 6.5 Lorentz-Abraham Force (Radiation Reaction Force)

is a force exerted on a particle by the radiation it produces. An accelerating charge emits radiation (according to the Larmor formula), which carries momentum away from the charge. Since momentum is conserved, the charge is pushed in the direction opposite the direction of the emitted radiation.

Let T be the time interval over which the particle's kinetic energy is changed substantially by the radiative emission. From the Larmor formula (6.21) with  $a = \dot{u}$ ,

$$T \sim \frac{v^2}{P_{\rm rad}} \sim \frac{6\pi\epsilon_0 mc^3}{q^2} \left(\frac{v}{a}\right)^2,\tag{6.47}$$

where m is the mass of the particle, and v its velocity. Let us estimate  $v/a \sim t_p$  as the typical orbital time scale for the particle. The condition  $T/t_p \gg 1$  requires  $t_p \gg \tau$ , where (in case of an electron) the time for radiation to cross the classical electron radius (Eq. (6.41)) is

$$\tau = \frac{1}{6\pi\epsilon_0} \frac{e^2}{m_{\rm e}c^3} \sim 10^{-23} \,\mathrm{s.} \tag{6.48}$$

If we are consider processes that occur on a time scale much longer than  $\tau$ , we can treat radiation reaction as a perturbation.

We derive the formula for the radiation reaction force from considerations of energy balance. When the radiation reaction force is relatively small, we may define the force as a term added to the existing external force, such that the radiated energy must be compensated by the work done against the radiation reaction force. Following again the Larmor formula (6.21) for electrons, we set

$$-\boldsymbol{F}_{\rm rad} \cdot \boldsymbol{u} = \frac{1}{6\pi\epsilon_0} \frac{e^2 \dot{u}^2}{c^3},\tag{6.49}$$

However, Eq. (6.49) brings a contradiction;  $\mathbf{F}_{rad}$  cannot depend on  $\mathbf{u}$ , because this would imply a preferred frame relative to which  $\mathbf{u}$  is measured. We satisfy this equation in an average sense, by its integration (by parts) over a time interval  $(t_2 - t_1) \gg \tau$ :

$$-\int_{t_1}^{t_2} \boldsymbol{F}_{\mathrm{rad}} \cdot \boldsymbol{u} \,\mathrm{d}t = \frac{1}{6\pi\epsilon_0} \frac{e^2}{c^3} \int_{t_1}^{t_2} \dot{\boldsymbol{u}} \cdot \dot{\boldsymbol{u}} \,\mathrm{d}t = \frac{1}{6\pi\epsilon_0} \frac{e^2}{c^3} \left( \dot{\boldsymbol{u}} \cdot \boldsymbol{u} |_{t_1}^{t_2} - \int_{t_1}^{t_2} \ddot{\boldsymbol{u}} \cdot \boldsymbol{u} \,\mathrm{d}t \right).$$
(6.50)

We may assume that the initial and final states are the same (within the long term average) or that  $\dot{\boldsymbol{u}} \cdot \boldsymbol{u}(t_1) = \dot{\boldsymbol{u}} \cdot \boldsymbol{u}(t_2)$ , the first term on the right-hand side of Eq. (6.50) vanishes, leaving

$$-\int_{t_1}^{t_2} \left( \boldsymbol{F}_{\mathrm{rad}} - \frac{1}{6\pi\epsilon_0} \frac{e^2 \ddot{\boldsymbol{u}}}{c^3} \right) \cdot \boldsymbol{u} \,\mathrm{d}t = 0, \quad \text{so that} \quad \boldsymbol{F}_{\mathrm{rad}} = \frac{1}{6\pi\epsilon_0} \frac{e^2 \ddot{\boldsymbol{u}}}{c^3} = m_{\mathrm{e}} \tau \ddot{\boldsymbol{u}}, \tag{6.51}$$

where  $\ddot{\boldsymbol{u}}$  is the jerk (the derivative of acceleration, or the third derivative of displacement). The radiation reaction force in Eq. (6.51) depends on the jerk, this increases the degree of the particle's equation of motion and implies a nonphysical behavior if not used consistently.

The equation of motion for a particle with zero total force,

$$m\left(\dot{\boldsymbol{u}} - \tau \ddot{\boldsymbol{u}}\right) = \boldsymbol{F} = 0 \tag{6.52}$$

gives the trivial solution  $\boldsymbol{u} = \text{const.}$ , which is physically correct. However, there is also a non-trivial solution

$$\boldsymbol{u} = \boldsymbol{u}_0 \,\mathrm{e}^{t/\tau},\tag{6.53}$$

("runaway" solution) which we must exclude, because  $\dot{\boldsymbol{u}} \cdot \boldsymbol{u}(t_1) \neq \dot{\boldsymbol{u}} \cdot \boldsymbol{u}(t_2)$  or, because of the rapid velocity increase, it violates the restriction that the motion will not change on a time scale short compared to  $\tau$  (we say that such solutions are spurious).
## 6.6 Radiation from Harmonically Bound Particles

#### 6.6.1 Undriven Harmonically Bound Particles

A harmonically bound particle to a center of force  $(\mathbf{F} = -k\mathbf{r} = -m\omega_0^2\mathbf{r})$  will oscillate sinusoidally with frequency  $\omega_0$ . Such a system, although rarely found in nature, gives the only possible classical model of a spectral line. Many of the quantum results are consistent with this model ("oscillator strengths", "classical damping widths"). Since there is always a small damping by the radiation reaction force, the oscillation is not purely harmonic. We assume  $\omega_0 \tau \ll 1$ , so that the radiation reaction formula is valid. If the particle oscillates along the x axis (cf. Eq. (6.51)),

$$\ddot{x} + \omega_0^2 x - \tau \, \ddot{x} = 0, \tag{6.54}$$

which is a third-order differential equation with constant coefficients. Since the third derivative term is small, we may approximate the motion as harmonic to first order,

$$x(t) \propto \cos(\omega_0 t + \phi_0), \quad \text{so that} \quad \ddot{x} \propto -\omega_0^2 \dot{x}.$$
 (6.55)

This approximation preserves an important feature of damping: it is expressed as an odd number of time derivatives and is therefore not time reversible. Equation (6.54) becomes

$$\ddot{x} + \omega_0^2 (\tau \dot{x} + x) = 0$$
, so that  $\alpha^2 + \omega_0^2 \tau \alpha + \omega_0^2 = 0$ , (6.56)

by assuming x(t) is of the form  $e^{\alpha t}$ . Expanding in powers of  $\omega_0 \tau$ , we obtain

$$\alpha = -\frac{\omega_0^2 \tau}{2} \pm \mathrm{i}\omega_0 + \mathcal{O}(\omega_0^3 \tau^2), \qquad (6.57)$$

where the discriminant of the solution of Eq. (6.56),  $\omega_0^2 \tau^2 - 4 \approx -4$ . Taking the initial conditions  $x(0) = x_0, \dot{x}(0) \approx 0$ , involving Eqs. (6.55), (6.57), and (6.48) gives

$$x(t) = x_0 e^{-\Gamma t/2} \cos \omega_0 t = \frac{1}{2} x_0 \left( e^{-\Gamma t/2 + i\omega_0 t} + e^{-\Gamma t/2 - i\omega_0 t} \right), \quad \Gamma = \omega_0^2 \tau = \frac{1}{6\pi\epsilon_0} \frac{e^2 \omega_0^2}{m_e c^3}.$$
 (6.58)

The Fourier transform of x(t) is (cf. Eq. (5.47)),

$$\hat{x}(\omega) = \int_0^\infty x(t) \,\mathrm{e}^{-\mathrm{i}\omega t} \,\mathrm{d}t = \frac{x_0}{2} \left[ \frac{1}{\Gamma/2 + \mathrm{i}(\omega - \omega_0)} + \frac{1}{\Gamma/2 + \mathrm{i}(\omega + \omega_0)} \right]. \tag{6.59}$$

This is large near  $\omega = \pm \omega_0$ . Since we interested only in positive frequencies, and only in regions where values are large, we approximate

$$\hat{x}(\omega) \approx \frac{x_0}{2} \frac{1}{\Gamma/2 + i(\omega - \omega_0)}, \quad |\hat{x}(\omega)|^2 = \left(\frac{x_0}{2}\right)^2 \frac{1}{(\Gamma/2)^2 + (\omega - \omega_0)^2}.$$
 (6.60)

The energy radiated per unit frequency is (cf. Eq. (6.30))

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{1}{6\pi^2\epsilon_0} \frac{e^2\omega^4}{c^3} |\hat{x}(\omega)|^2 = \frac{\omega^4}{24\pi^2\epsilon_0 c^3} \frac{e^2x_0^2}{(\Gamma/2)^2 + (\omega - \omega_0)^2}.$$
(6.61)

Equation (6.61) gives the frequency spectrum typical of a "decaying oscillator", which has a sharp maximum near  $\omega = \omega_0$ , since  $\Gamma/\omega_0 \ll 1$  (cf. Eq. (6.58), for example, for the blue edge of the visible light,  $\Gamma/\omega_0 \approx 10^{-9}$ ), where  $\Gamma$  is the full width at half maximum (FWHM).

Using the definition of  $\Gamma$  and  $k = m_e \omega_0^2 =$  force constant of the spring, Eq. (6.61) is

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{1}{6\pi^2\epsilon_0} \frac{e^2\omega^4}{c^3} |\hat{x}(\omega)|^2 = \left(\frac{kx_0^2}{2}\right) \frac{\Gamma/(2\pi)}{(\Gamma/2)^2 + (\omega - \omega_0)^2}.$$
(6.62)

The first factor gives the initial potential energy of the particle (of the spring) while the second factor gives the distribution of the radiated energy over frequency. The integral over  $\omega$  is

$$W = \int_{-\infty}^{\infty} \frac{\mathrm{d}W}{\mathrm{d}\omega} \,\mathrm{d}\omega = \int_{-\infty}^{\infty} \frac{\Gamma k x_0^2 / (4\pi)}{\left(\Gamma/2\right)^2 + \left(\omega - \omega_0\right)^2} \,\mathrm{d}\omega = \frac{k x_0^2}{2\pi} \operatorname{atan} \left[\frac{2\left(\omega - \omega_0\right)}{\Gamma}\right]_{-\infty}^{\infty} = \frac{k x_0^2}{2}.$$
 (6.63)

The profile of the emitted spectrum,

$$\frac{\Gamma/(2\pi)}{(\Gamma/2)^2 + (\omega - \omega_0)^2},$$
(6.64)

is known as a Lorentz profile. The line width  $\Delta \omega = \Gamma$  is a universal constant for electron oscillators. In terms of wavelength (cf. Eq. (6.58)) it is

$$\Delta \lambda = \frac{c}{\nu^2} \Delta \nu = \frac{2\pi c}{\omega^2} \Delta \omega = 2\pi c \tau \approx 1.2 \times 10^{-4} \,\text{\AA}.$$
(6.65)

#### 6.6.2 Driven Harmonically Bound Particles

We now consider forced oscillations due to an incident radiation. We write (cf. Eq. (6.54))

$$m_{\rm e}\ddot{x} + m_{\rm e}\omega_0^2 x - m_{\rm e}\tau \ddot{x} = eE_0\cos\omega t, \qquad (6.66)$$

where the right-hand side represents the force due to a sinusoidally varying field. Following Eq. (6.56) and representing x by a complex variable, we have

$$\ddot{x} + \omega_0^2 \left(\tau \dot{x} + x\right) = \frac{eE_0}{m_e} \operatorname{Re}\left(e^{\mathrm{i}\omega t}\right),\tag{6.67}$$

where we take the real part of x. The steady-state solution of Eq. (6.67) (cf. Eq. (6.57) and Sect. 3.2.1 in Kurfürst (2017)) is

$$x = |x_0| e^{i(\omega t + \delta)}, \quad \text{where} \quad x_0 = -\frac{eE_0}{m_e} \left(\omega^2 - \omega_0^2 - i\omega_0^3 \tau\right)^{-1}, \quad \delta = \operatorname{atan}\left(\frac{\omega^3 \tau}{\omega^2 - \omega_0^2}\right). \quad (6.68)$$

There is a *phase shift* caused by the odd time derivative damping term. For  $\omega > \omega_0$  the particle "leads" the driving force and for  $\omega < \omega_0$  it "lags behind". Taking the real part of x we have an oscillating dipole of charge e and amplitude  $|x_0|$  with frequency  $\omega$ . The time-averaged total radiated power (cf. Eqs. (6.21), (6.41) and (6.68)) is

$$P = \frac{1}{12\pi\epsilon_0} \frac{e^2 |x_0|^2 \omega^4}{c^3} = \frac{e^4 E_0^2}{12\pi\epsilon_0 m_{\rm e}^2 c^3} \frac{\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + \left(\omega_0^3 \tau\right)^2}.$$
(6.69)

Dividing Eq. (6.69) by the time-average Poynting vector  $\langle S \rangle = \frac{1}{2} \sqrt{\epsilon_0/\mu_0} E_0^2$  (see Eq. (5.44)), we obtain the scattering cross section as a function of frequency,

$$\sigma(\omega) = \sigma_T \frac{\omega^4}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega_0^3 \tau)^2},$$
(6.70)

where  $\sigma_T$  is the Thomson cross section (6.44).

We identify from Eq. (6.70) three characteristic regimes for  $\omega$ :

- $\omega \gg \omega_0$ : In this case  $\sigma(\omega) \to \sigma_T$ , the value for free electrons, since at high incident energies the binding becomes negligible.
- $\omega \ll \omega_0$ : This gives

$$\sigma(\omega) \to \sigma_T \left(\frac{\omega}{\omega_0}\right)^4,$$
(6.71)

which corresponds to the electron fully responding to the incident field with no inertial effects, so that  $kx \approx eE$  (since  $\omega \ll \omega_0$ , the electric field appears nearly static and produces a nearly static force). The dipole moment is directly proportional to the incident field, the radiation is scattered as  $\omega^4$ , and the scattering is called *Rayleigh scattering*. It is responsible for the blue color of the sky and the red color of the sun at sunset, because it favors the scattering of higher frequency (bluer) light.

•  $\omega \approx \omega_0$ : Since  $\omega^2 - \omega_0^2 \to 0$ , that is  $\omega^2 - \omega_0^2 = (\omega - \omega_0)(\omega + \omega_0) \approx \omega - \omega_0$ , so that

$$\sigma(\omega) \approx \frac{\sigma_T}{4\tau} \frac{(\omega_0^2 \tau)}{(\omega - \omega_0)^2 + \left(\frac{\omega_0^2 \tau}{2}\right)^2} \equiv \frac{\pi \sigma_T}{2\tau} \frac{\Gamma/(2\pi)}{(\omega - \omega_0)^2 + (\Gamma/2)^2},\tag{6.72}$$

using  $\Gamma = \omega_0^2 \tau$ . With the definitions of  $\sigma_T$  and  $\tau$  from (6.44) and (6.48), Eq. (6.72) becomes

$$\sigma(\omega) \approx \frac{\pi}{2\epsilon_0} \frac{e^2}{m_{\rm e}c} \frac{\Gamma/(2\pi)}{(\omega - \omega_0)^2 + (\Gamma/2)^2},\tag{6.73}$$

Near the resonance,  $\omega \approx \omega_0$ , the shape of the scattering cross section is the same as the emission from the free oscillator (cf. Eq. (6.64)). This can be explained, since the free oscillations are excited by a pulse of radiation,  $E(t) \propto \delta(t)$ . The spectrum of this pulse is independent of  $\omega$  (white spectrum), so that we may regard the free oscillations as the scattering of a white spectrum, yielding emission proportional to the scattering cross section.

We obtain an important result by integrating  $\sigma(\omega)$  over  $\omega$  (in SI units),

$$\int_0^\infty \sigma(\omega) \,\mathrm{d}\omega = \frac{\pi e^2}{2\epsilon_0 m_\mathrm{e}c}, \quad \text{or} \quad \int_0^\infty \sigma(\nu) \,\mathrm{d}\nu = \frac{e^2}{4\epsilon_0 m_\mathrm{e}c}, \tag{6.74}$$

while in cgs units Eq. (6.74) becomes

$$\int_0^\infty \sigma(\omega) \,\mathrm{d}\omega = \frac{2\pi^2 e^2}{m_{\rm e}c}, \quad \text{or} \quad \int_0^\infty \sigma(\nu) \,\mathrm{d}\nu = \frac{\pi e^2}{m_{\rm e}c}.$$
(6.75)

We have neglected a divergence, since the cross section actually approaches  $\sigma_T$  for large  $\omega$ . This may be justified as follows: the radiation reaction formula is only valid for  $\omega \tau \ll 1$ , so that we must cut off the integral at  $\omega_{\max}$  such that  $\omega_{\max} \ll 1/\tau$ . The contribution to the integral from the Thomson limit is less than

$$\int_{0}^{\omega_{\max}} \sigma_T \,\mathrm{d}\omega = \sigma_T \omega_{\max},\tag{6.76}$$

which is negligible comparing to the integral (6.74), since  $\sigma_T \omega_{\text{max}} \ll \sigma_T / \tau \equiv e^2 / (\epsilon_0 m_e c)$ . In the quantum theory one obtains similar formulas, which correspond to the above results,

$$\int_0^\infty \sigma(\nu) \,\mathrm{d}\nu = \frac{e^2}{4\epsilon_0 m_e c} f_{nn'} \quad \left( = \frac{\pi e^2}{m_e c} f_{nn'} \quad \text{in cgs} \right), \tag{6.77}$$

where  $f_{nn'}$  is called the *oscillator strength* or *f*-value for the transition between states *n* and *n'*.

## Chapter 7

# **Relativistic Effects**

#### 7.1 Tensor analysis

(Rybicki & Lightman 1979; Mihalas & Mihalas 1984): For basics of tensor algebra and analysis, including the terminology and explanation of quantities and operations introduce in this Section - see Kurfürst (2017), Sect. 2.3 (in Czech). We now extend it including the consequences of *special relativity* which is based on two fundamental postulates:

- The laws of nature are identical in every inertial frame of reference.
- The speed of light is c in all such frames.

Considering two *inertial frames*  $\mathcal{K}$  and  $\mathcal{K}'$ , with a relative (constant) velocity v along the x axis, whose origins coincide at t = 0. We emit a pulse of light at the origin at t = 0; each observer then detects a spherical wavefront centered on his own origin. This is a consequence of the second postulate and is inconsistent with "classical" physics. This result implies the fact that time and space are specific in each frame and not universal. The expanding "sphere of light" is thus in each frame described as

$$c^{2}t^{2} - x^{2} - y^{2} - z^{2} = 0, \quad c^{2}t'^{2} - x'^{2} - y'^{2} - z'^{2} = 0,$$
(7.1)

where, unlike the Newtonian physics,  $t' \neq t$  (we note that another formalisms express Eqs. (7.1) with opposite signs, yielding however the same physics). The relations between x, y, z, t and x', y', z', t' are called the Lorentz transformation, which is represented (regarding a boost along the x axis) by:

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma\left(t - \frac{v}{c^2}x\right) \quad \text{where} \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}, \quad (7.2)$$

while the inverse transformation is:

$$x = \gamma(x' + vt'), \quad y = y', \quad z = z', \quad t = \gamma\left(t' + \frac{v}{c^2}x'\right).$$
 (7.3)

Transformation relation in Eq. (7.2) can be written as the Lorentz matrix (where  $\beta \equiv v/c$ )

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
 (7.4)

Equation (7.2) can be thus written in the compact form

$$x^{\prime \mu} = \Lambda^{\mu}{}_{\nu}x^{\nu}. \tag{7.5}$$

Lorentz transformation of a general second-rank tensor T is defined by

$$T^{\mu\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}T^{\sigma\tau}, \qquad (7.6)$$

which explicitly means  $T'^{00} = \Lambda^0_0 \Lambda^0_0 T^{00} + \Lambda^0_1 \Lambda^0_0 T^{10} + \Lambda^0_0 \Lambda^0_1 T^{01} + \Lambda^0_1 \Lambda^0_1 T^{11}$ ,  $T'^{01} = \Lambda^0_0 \Lambda^1_0 T^{00} + \Lambda^0_1 \Lambda^1_0 T^{10} + \Lambda^0_0 \Lambda^1_1 T^{01} + \Lambda^0_1 \Lambda^1_1 T^{11}$ ,  $T'^{02} = \Lambda^0_0 \Lambda^2_2 T^{02} + \Lambda^0_1 \Lambda^2_2 T^{12}$ , etc., using all non-zero terms of  $\Lambda^{\mu}_{\nu}$ . The resulting tensor  $T'^{\mu\nu}$ , expressed in terms of  $T^{\mu\nu}$ , using all non-zero terms of  $\Lambda^{\mu}_{\nu}$ .

$$T^{\mu\nu} = \gamma^{2} \begin{bmatrix} \left(T^{00} + \beta^{2}T^{11} - \beta T^{ij}\right) & \left(T^{01} + \beta^{2}T^{10} - \beta T^{ii}\right) & \frac{T^{02} - \beta T^{12}}{\gamma} & \frac{T^{03} - \beta T^{13}}{\gamma} \\ \left(T^{10} + \beta^{2}T^{01} - \beta T^{ii}\right) & \left(T^{11} + \beta^{2}T^{00} - \beta T^{ij}\right) & \frac{T^{12} - \beta T^{02}}{\gamma} & \frac{T^{13} - \beta T^{03}}{\gamma} \\ \frac{T^{20} - \beta T^{21}}{\gamma} & \frac{T^{21} - \beta T^{20}}{\gamma} & T^{22} / \gamma^{2} & T^{23} / \gamma^{2} \\ \frac{T^{30} - \beta T^{31}}{\gamma} & \frac{T^{31} - \beta T^{30}}{\gamma} & T^{32} / \gamma^{2} & T^{33} / \gamma^{2} \end{bmatrix}, \quad (7.7)$$

where in this case  $T^{ii} = T^{00} + T^{11}$  and  $T^{ij} = T^{01} + T^{10}$ .

We now define the  $4 \times 4$  Minkowski metric

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$
(7.8)

which represents the space-time "length"

$$ds^{2} = \eta_{\mu\nu} dx^{\mu} dx^{\nu} = \eta^{\mu\nu} dx_{\mu} dx_{\nu}.$$
(7.9)

We obtain the covariant components  $T_{\mu\nu}$  by lowering indexes as

$$T_{\mu\nu} = \eta_{\mu\sigma}\eta_{\nu\tau}T^{\sigma\tau}.$$
 (7.10)

(cf. Eqs. 2.59 - 2.61 in Kurfürst (2017)). Lorentz transformaton of the covariant components now becomes

$$T'_{\mu\nu} = \tilde{\Lambda}^{\ \sigma}_{\mu} \tilde{\Lambda}^{\ \tau}_{\nu} T_{\sigma\tau}, \qquad (7.11)$$

where the coefficients  $\tilde{\Lambda}^{\nu}_{\mu} = \eta_{\mu\tau} \Lambda^{\tau}{}_{\sigma} \eta^{\sigma\nu}$ . The explicit form of  $\tilde{\Lambda}^{\nu}_{\mu}$  is thus

$$\tilde{\Lambda}_{\mu}^{\ \nu} = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0\\ \beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
(7.12)

while the explicit form of  $T'_{\mu\nu}$  is

$$T'_{\mu\nu} = \gamma^2 \begin{bmatrix} \left(T_{00} + \beta^2 T_{11} + \beta T_{ij}\right) & \left(T_{01} + \beta^2 T_{10} + \beta T_{ii}\right) & \frac{T_{02} + \beta T_{12}}{\gamma} & \frac{T_{03} + \beta T_{13}}{\gamma} \\ \left(T_{10} + \beta^2 T_{01} + \beta T_{ii}\right) & \left(T_{11} + \beta^2 T_{00} + \beta T_{ij}\right) & \frac{T_{12} + \beta T_{02}}{\gamma} & \frac{T_{13} + \beta T_{03}}{\gamma} \\ \frac{T_{20} + \beta T_{21}}{\gamma} & \frac{T_{21} + \beta T_{20}}{\gamma} & T_{22} / \gamma^2 & T_{23} / \gamma^2 \\ \frac{T_{30} + \beta T_{31}}{\gamma} & \frac{T_{31} + \beta T_{30}}{\gamma} & T_{32} / \gamma^2 & T_{33} / \gamma^2 \end{bmatrix},$$
(7.13)

where in this case  $T_{ii} = T_{00} + T_{11}$  and  $T_{ij} = T_{01} + T_{10}$ .

In a similar way we can also define *mixed components* of a tensor T,

$$T^{\mu}{}_{\nu} = \eta_{\nu\tau}T^{\mu\tau}, \quad T^{\mu}{}_{\mu} = \eta_{\mu\sigma}T^{\sigma\nu}.$$
 (7.14)

whose explicit forms, expressed in terms of contravariant tensors, are

$$T^{\mu}{}_{\nu} = \begin{pmatrix} T^{00} & -T^{01} & -T^{02} & -T^{03} \\ T^{10} & -T^{11} & -T^{12} & -T^{13} \\ T^{20} & -T^{21} & -T^{22} & -T^{23} \\ T^{30} & -T^{31} & -T^{32} & -T^{33} \end{pmatrix}, \quad T^{\nu}_{\mu} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ -T^{10} & -T^{11} & -T^{12} & -T^{13} \\ -T^{20} & -T^{21} & -T^{22} & -T^{23} \\ -T^{30} & -T^{31} & -T^{32} & -T^{33} \end{pmatrix}.$$
(7.15)

The mixed tensors are Lorentz-transformed:

$$T^{\prime \mu}{}_{\nu} = \Lambda^{\mu}{}_{\sigma}\tilde{\Lambda}^{\ \tau}_{\nu}T^{\sigma}{}_{\tau}, \quad T^{\prime}{}_{\mu}{}^{\nu} = \tilde{\Lambda}^{\ \sigma}_{\mu}\Lambda^{\nu}{}_{\tau}T^{\ \tau}_{\sigma}.$$
(7.16)

The explicit forms in this case are

$$T^{\prime \mu}{}_{\nu} = \gamma^{2} \begin{bmatrix} \left(T^{0}{}_{0} - \beta^{2}T^{1}{}_{1} + \beta T^{i}{}_{j}\right) & \left(T^{0}{}_{1} - \beta^{2}T^{1}{}_{0} + \beta T^{i}{}_{i}\right) & \frac{T^{0}{}_{2} - \beta T^{1}{}_{2}}{\gamma} & \frac{T^{0}{}_{3} - \beta T^{1}{}_{3}}{\gamma} \\ \left(T^{1}{}_{0} - \beta^{2}T^{0}{}_{1} - \beta T^{i}{}_{i}\right) & \left(T^{1}{}_{1} - \beta^{2}T^{0}{}_{0} - \beta T^{i}{}_{j}\right) & \frac{T^{1}{}_{2} - \beta T^{0}{}_{2}}{\gamma} & \frac{T^{1}{}_{3} - \beta T^{0}{}_{3}}{\gamma} \\ \frac{T^{2}{}_{0} + \beta T^{2}{}_{1}}{\gamma} & \frac{T^{2}{}_{1} + \beta T^{2}{}_{0}}{\gamma} & T^{2}{}_{2}/\gamma^{2} & T^{2}{}_{3}/\gamma^{2} \\ \frac{T^{3}{}_{0} + \beta T^{3}{}_{1}}{\gamma} & \frac{T^{3}{}_{1} + \beta T^{3}{}_{0}}{\gamma} & T^{3}{}_{2}/\gamma^{2} & T^{3}{}_{3}/\gamma^{2} \end{bmatrix}, \quad (7.17)$$

where  $T^{i}{}_{i} = T^{0}{}_{0} - T^{1}{}_{1}$  and  $T^{i}{}_{j} = T^{0}{}_{1} - T^{1}{}_{0}$ , and

$$T'_{\mu}{}^{\nu} = \gamma^{2} \begin{bmatrix} \left(T_{0}{}^{0} - \beta^{2}T_{1}{}^{1} - \beta T_{i}{}^{j}\right) & \left(T_{0}{}^{1} - \beta^{2}T_{1}{}^{0} - \beta T_{i}{}^{i}\right) & \frac{T_{0}{}^{2} + \beta T_{1}{}^{2}}{\gamma} & \frac{T_{0}{}^{3} + \beta T_{1}{}^{3}}{\gamma} \\ \left(T_{1}{}^{0} - \beta^{2}T_{0}{}^{1} + \beta T_{i}{}^{i}\right) & \left(T_{1}{}^{1} - \beta^{2}T_{0}{}^{0} + \beta T_{i}{}^{j}\right) & \frac{T_{1}{}^{2} + \beta T_{0}{}^{2}}{\gamma} & \frac{T_{1}{}^{3} + \beta T_{0}{}^{3}}{\gamma} \\ \frac{T_{2}{}^{0} - \beta T_{2}{}^{1}}{\gamma} & \frac{T_{2}{}^{1} - \beta T_{2}{}^{0}}{\gamma} & T_{2}{}^{2} / \gamma^{2} & T_{2}{}^{3} / \gamma^{2} \\ \frac{T_{3}{}^{0} - \beta T_{3}{}^{1}}{\gamma} & \frac{T_{3}{}^{1} - \beta T_{3}{}^{0}}{\gamma} & T_{3}{}^{2} / \gamma^{2} & T_{3}{}^{3} / \gamma^{2} \end{bmatrix},$$
(7.18)

where  $T_i^{\ i} = T_0^{\ 0} - T_1^{\ 1}$  and  $T_i^{\ j} = T_0^{\ 1} - T_1^{\ 0}$ .

Here are some other examples of Lorentz transformation of vector or tensor fields:

• Transformation of two vectors (which form a tensor product),

$$A^{\prime\mu}B^{\prime\nu} = \Lambda^{\mu}{}_{\sigma}\Lambda^{\nu}{}_{\tau}A^{\sigma}B^{\tau}, \qquad (7.19)$$

which is another expression for (7.2).

• Transformation of the components of the second rank Minkowski metric tensor  $\eta^{\mu\nu}$ :

$$\eta^{\prime\alpha\beta} = \Lambda^{\alpha}{}_{\mu}\Lambda^{\beta}{}_{\nu}\eta^{\mu\nu}, \qquad (7.20)$$

where  $\eta'^{\alpha\beta} = \eta^{\alpha\beta}$  (thus  $\eta^{\alpha\beta}$  has the same components in all frames). The proof of this identity is directly given by Eq. (7.20). Another proof follows from symmetricity of  $\eta^{\alpha\beta}$  and from its decomposition into the sum of four tensor products  $\eta^{\alpha}\eta^{\beta}$  where each of them can be transformed according to Eq. (7.19) as  $\eta'^{\alpha}\eta'^{\beta} = \Lambda^{\alpha}{}_{\beta}\eta^{\beta}\Lambda^{\beta}{}_{\alpha}\eta^{\alpha} = \Lambda^{\alpha}{}_{\beta}\Lambda^{\beta}{}_{\alpha}\eta^{\beta}\eta^{\alpha}$ .

• We can construct also the second rank Kronecker-delta tensor  $\delta^{\mu}{}_{\nu}$  (represented by unit matrix) following the identity

$$\Lambda^{\sigma}{}_{\nu}\tilde{\Lambda}^{\mu}_{\sigma} = \delta^{\mu}{}_{\nu}. \tag{7.21}$$

The proof is obtained from the following identities:  $s^2 = \eta_{\sigma\tau} x^{\sigma} x^{\tau} = x^{\sigma} x_{\sigma} = x'^{\sigma} x'_{\sigma}$  (where  $s^2$  is the Lorentz invariant "space-time" square of line length in the observer's frame of reference, cf. Eq. (7.9), so that  $x'^{\sigma} x'_{\sigma} = \Lambda^{\sigma}{}_{\nu} \tilde{\Lambda}^{\mu}_{\sigma} x^{\nu} x_{\mu} = s^2 \equiv x^{\nu} x_{\mu} \delta^{\mu}{}_{\nu}$ . Multiplying Eq. (7.5) by  $\tilde{\Lambda}^{\alpha}_{\mu}$  and using Eq. (7.21) thus yields the inverse transformation of Eq. (7.5):

$$x^{\alpha} = \tilde{\Lambda}^{\ \alpha}_{\mu} x^{\prime \mu}. \tag{7.22}$$

Higher-rank tensors can be defined in a similar way. The Lorentz transformation involves a factor  $\Lambda$  for each contravariant index and a factor  $\tilde{\Lambda}$  for each covariant index.

All tensor operations follow the basic principles described in Sect. 2.3 of the lecture notes Kurfürst (2017). We introduce here only the specific rules that are connected with covariant transformations:

• *Raising and Lowering Indexes.* We use Minkowski metric in special relativity to change contravariant indexes into covariant ones, and vice versa. The proof of this is given by the mutually commuting pair of transformation equations:

$$\eta_{\mu\nu}\Lambda^{\mu}{}_{\sigma} = \tilde{\Lambda}^{\ \tau}_{\nu}\eta_{\tau\sigma}, \quad \eta^{\mu\nu}\tilde{\Lambda}^{\ \sigma}_{\mu} = \Lambda^{\nu}{}_{\tau}\eta^{\tau\sigma}. \tag{7.23}$$

The lowering operator  $\eta_{\mu\nu}$  thus changes the Lorentz transformation coefficients  $\Lambda$  to  $\tilde{\Lambda}$  while the latter changes a contravariant index into a covariant one.

• Contraction of Tensor. The scalar product of two vectors  $A^{\mu}B_{\mu}$  can be regarded as the contraction of the second-rank tensor  $A^{\mu}B_{\nu}$ . Let  $T^{\mu\nu}{}_{\sigma}$  being a third-order tensor, then  $T^{\mu\nu}{}_{\nu}$  is a vector. If  $F^{\mu\nu}G_{\sigma\tau}$  is the fourth-rank tensor, we can form the invariant  $F^{\mu\nu}G_{\mu\nu}$ . We prove this property of contraction for the above example of a third-order tensor  $T^{\mu\nu}{}_{\nu}$ . From the transformation law for  $T^{\mu\nu}{}_{\sigma}$  we obtain

$$T^{\mu\nu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}\tilde{\Lambda}^{\ \tau}{}_{\nu}T^{\alpha\beta}{}_{\tau}. \tag{7.24}$$

But  $\Lambda^{\nu}{}_{\beta}\tilde{\Lambda}^{\ \tau}_{\nu} = \delta^{\tau}{}_{\beta}$  (cf. Eq. (7.21)), so that

$$T^{\prime\mu\nu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha}T^{\alpha\beta}{}_{\beta}, \qquad (7.25)$$

which shows, according to one  $\Lambda$  operator needed, that  $T^{\mu\nu}{}_{\nu}$  is indeed a vector. In a similar manner we can make the general proof of this property.

• Gradients of Tensor. A covariant tensor field is defined as a tensor that is a function of the space-time coordinates  $x^0, x^1, x^2, x^3$ . Then the gradient operation  $\partial/\partial x^{\mu}$  acting on such a field produces a tensor field of one higher rank with  $\mu$  as a new covariant index. A convenient notation for the gradient operation is a comma followed by the index  $\mu$ . Thus, for example, if  $\lambda$  is a scalar, then  $\lambda_{,\mu} \equiv \partial \lambda/\partial x^{\mu}$  is a covariant vector. Similarly  $T^{\mu\nu}{}_{,\sigma} \equiv \partial T^{\mu\nu}/\partial x^{\sigma}$  is a third-rank tensor. We shall prove this rule for the special case of the vector field  $A^{\mu}$ . Differentiating the transformation

$$A^{\prime\mu} = \Lambda^{\mu}{}_{\sigma}A^{\sigma}, \quad (\text{or reversely}A^{\mu} = \tilde{\Lambda}^{\mu}{}_{\sigma}A^{\sigma\prime},)$$
(7.26)

gives

$$\frac{\partial A'^{\mu}}{\partial x'^{\nu}} = \Lambda^{\mu}{}_{\sigma} \frac{\partial A^{\sigma}}{\partial x'^{\nu}} = \Lambda^{\mu}{}_{\sigma} \frac{\partial x^{\alpha}}{\partial x'^{\nu}} \frac{\partial A^{\sigma}}{\partial x^{\alpha}} = \Lambda^{\mu}{}_{\sigma} \tilde{\Lambda}^{\alpha}{}_{\nu}{}^{\alpha} \frac{\partial A^{\sigma}}{\partial x^{\alpha}}, \tag{7.27}$$

where we have used Eq. (7.22) to evaluate  $\partial x^{\alpha}/\partial x'^{\nu}$ . This is the transformation for a second-rank tensor with contravariant index  $\mu$  and covariant index  $\nu$ . However, introducing here only the partial derivatives, we have assumed that the velocity components in  $\Lambda$  are constant, which applies only in Cartesian coordinate systems. In general (e.g., spherical) coordinate systems we need *covariant derivatives*  $\nabla_{\mu}$  (whose complete formalism for various either orthogonal or general non-orthogonal coordinate systems is introduced in Sects. A.1.1, A.2.1, A.3.1, and A.7.1, in Kurfürst (2017).

We note that although the summation convention allows summation over any two indexes, in the covariant formalism only a subscript-superscript pair forms a tensor. Thus we have to carefully define superscripts and subscripts usage, to satisfy this principle, for example  $\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^{\mu}{}_{\nu}$  forms the Kronecker tensor, while  $\eta^{\mu\sigma}\eta^{\sigma\nu} = \text{Tr}(\delta^{\mu}{}_{\nu})$  which is its four-space trace.

Let us define some further rules: The divergence of a tensor is a gradient followed by a contraction of the gradient index with one of the other contravariant indexes. For example,  $\partial A^{\mu}/\partial x^{\mu} \equiv A^{\mu}{}_{,\mu}$ , while  $\nabla_{\mu}A^{\mu} \equiv A^{\mu}{}_{;\mu} \equiv$  covariant divergence of the vector  $A^{\mu}$ , and  $\nabla_{\nu}T^{\mu\nu} \equiv T^{\mu\nu}{}_{;\nu} \equiv$  covariant divergence of the tensor  $T^{\mu\nu}$  (noting the various types of notation).

A statement that two tensors of the same rank and type are equal is called *tensor equation*. If a tensor equation holds in one Lorentz frame, then it holds in all Lorentz frames. This property is called Lorentz covariance or simply covariance (this meaning of the word "covariance" has nothing to do with covariant components of tensors).

### 7.2 Basic Relativistic Transformations

Since space and time are Lorentz-transformed simultaneously and mutually, the basic characteristic is now called an *event*, specified by a location in space and by the time it happened. We now describe some consequences of Lorentz transformations (whose elementary principles we have already introduced in Sect. 7.1).

#### 7.2.1 Dilation of Time

Suppose a clock device is at rest at the origin of the frame  $\mathcal{K}'$  and measures an interval of time  $T' = t'_2 - t'_1$ . What is the time interval in  $\mathcal{K}$ ? Note that in  $\mathcal{K}'$ , the spatial part of an event is: x' = y' = z' = 0. We obtain

$$T = t_2 - t_1 = \gamma(t'_2 - t'_1) = \gamma T', \tag{7.28}$$

where the factor  $\gamma = (1 - v^2/c^2)^{-1/2}$ , so that the moving clock appears to have slowed down. The effect is quite symmetrical between the two frames: observer in  $\mathcal{K}'$  measures clocks in  $\mathcal{K}$  also have slowed down. This apparent contradiction is a result of measuring a time interval between two events separated in space:  $\mathcal{K}$  measures  $t_1$  as a clock passes  $x_1$ , and  $t_2$  as it passes  $x_2$ ; then he subtracts  $t_2 - t_1$ , assuming that the clocks at  $x_1$  and  $x_2$  are synchronized. Observations of  $\mathcal{K}'$  prove that the two clocks in  $\mathcal{K}$  are not synchronized at all.

#### 7.2.2 Contraction of Length

A rigid rod of length  $L' = x'_2 - x'_1$  is at rest in the frame  $\mathcal{K}'$ . Its length measured in  $\mathcal{K}$  is  $L = x_2 - x_1$ , where  $x_2$  and  $x_1$  are the positions of the ends of the rod *at the same time t* in the frame  $\mathcal{K}$ . We have the result

$$L' = x'_2 - x'_1 = \gamma(x_2 - x_1) = \gamma L, \quad L = L'/\gamma.$$
(7.29)

The rod in  $\mathcal{K}$  appears shorter by a factor  $\gamma^{-1} = (1 - v^2/c^2)^{1/2}$ . The effect is again quite symmetric between the two observers. If the rod were at rest in  $\mathcal{K}$ , then  $\mathcal{K}'$  would measure its length contracted. How it is possible? The point is that the measurements did not happen at the same time by both observers. Since the Lorentz transformation of time depends on position and vice versa, simultaneity of events is not Lorentz invariant.

In both the time-dilation and length-contraction effects plays a crucial role the synchronization of clocks and the principle of simultaneity. Many of the apparent curiosities of special relativity are simply a result of the *relativity of simultaneity* between two events separated in space.

#### 7.2.3 Proper Time

Although intervals of space and time differ in various frames of reference, there are some quantities that are identical in all Lorentz frames. An important *Lorentz invariant* quantity is called the *proper time*  $d\tau$  defined as

$$c^{2} d\tau^{2} = c^{2} dt^{2} - dx^{2} - dy^{2} - dz^{2}.$$
(7.30)

It is easily shown from Eqs. (7.2) that  $d\tau = d\tau'$ , and it gives the time intervals between events that occur at the same spatial location (dx = dy = dz = 0), as measured by an observer in his own time.

If we relate our measurements to another reference frame with relative velocity v, then we have

$$\mathrm{d}\tau = \gamma^{-1}\,\mathrm{d}t,\tag{7.31}$$

where Eq. (7.31) is the time dilation formula (7.28) in which  $d\tau$  is measured by the observer "in motion".

#### 7.2.4 Transformation of Velocities

How is related a velocity  $\boldsymbol{u}'$  of a point in frame  $\mathcal{K}'$  to a velocity  $\boldsymbol{u}$  of the same point in frame  $\mathcal{K}$ ? From the differential form of Lorentz transformations (7.2)

$$dx = \gamma (dx' + v dt'), \quad dy = dy', \quad dz = dz', \quad dt = \gamma \left( dt' + \frac{v}{c^2} dx' \right), \tag{7.32}$$

we obtain the relations

$$u_x \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\gamma(\mathrm{d}x' + v\,\mathrm{d}t')}{\gamma\left(\mathrm{d}t' + \frac{v}{c^2}\,\mathrm{d}x'\right)} = \frac{u'_x + v}{1 + \frac{v}{c^2}u'_x}, \ u_y = \frac{u'_y}{\gamma\left(1 + \frac{v}{c^2}u'_x\right)}, \ u_z = \frac{u'_z}{\gamma\left(1 + \frac{v}{c^2}u'_x\right)}.$$
 (7.33)

The generalization of these equations to an arbitrary velocity  $\boldsymbol{v}$ , not necessarily parallel to the x axis, can be expressed in terms of the components of  $\boldsymbol{u}$  perpendicular to and parallel to v:

$$u_{||} = \frac{u'_{||} + v}{1 + \frac{v}{c^2}u'_{||}}, \quad u_{\perp} = \frac{u'_{\perp}}{\gamma\left(1 + \frac{v}{c^2}u'_{||}\right)}.$$
(7.34)

Direction of velocities in  $\mathcal{K}'$  and  $\mathcal{K}$  are then related by *aberration formula*:

$$\operatorname{tg} \theta = \frac{u_{\perp}}{u_{||}} = \frac{u'_{\perp}}{\gamma\left(u'_{||} + v\right)} = \frac{u'\sin\theta'}{\gamma\left(u'\cos\theta' + v\right)},\tag{7.35}$$

where  $u' = |\mathbf{u}'|$ , while the azimuthal angle  $\phi$  remains unchanged. Identifying u' = u = c, then  $\cos \theta = u_{\parallel}/c$ ,  $\sin \theta = u_{\perp}/c$ , and the aberration formula becomes the *aberration of light*,

$$tg\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}, \quad \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}, \quad \sin\theta = \frac{\sin\theta'}{\gamma(1 + \beta\cos\theta')}.$$
 (7.36)

In case  $\theta' = \pi/2$ , that is, a photon is emitted perpendicularly to v in  $\mathcal{K}'$ , we have

$$\operatorname{tg} \theta = \frac{c}{\gamma v}, \quad \sin \theta = \frac{1}{\gamma}, \quad \text{and for } \gamma \gg 1 \to \theta \sim \frac{1}{\gamma}.$$
 (7.37)

If photons are emitted in  $\mathcal{K}'$  isotropically, then for half of them  $\theta' < \pi/2$  and for the second half  $\theta' > \pi/2$ . Equation (7.37) thus shows that in  $\mathcal{K}$  are the photons concentrated in the forward direction, with half of them lying within a cone of half-angle  $1/\gamma$ , while a minority will be emitted having  $\theta \gg 1/\gamma$ . This is called the *beaming effect*.

#### 7.2.5 Doppler Effect

Any periodic phenomenon in the "moving" frame  $\mathcal{K}'$  will appear to have a longer period by a factor  $\gamma$  when measured by observers in  $\mathcal{K}$ . If, on the other hand, we measure the arrival times of periodic phenomena that propagate with the velocity of light, then there will be an additional effect on the observed period due to the delay times for light propagation. The joint effect is called the *Doppler effect*.

In the rest frame  $\mathcal{K}$  of the observer, the moving source emits one period of radiation as it moves from point 1 to point 2 with velocity v (Fig. 7.1). If the angular frequency of the radiation in the rest frame  $\mathcal{K}'$  of the source is  $\omega'$ , then the time taken to move from point 1 to point 2 in  $\mathcal{K}$  is given by (time-dilation effect):

$$\Delta t = \gamma \Delta t' = \frac{2\pi\gamma}{\omega'}.\tag{7.38}$$

We note  $l = v\Delta t$  and  $d = v\Delta t\cos\theta$  in Fig. 7.1. The difference in arrival times  $\Delta t_A$  of the radiation emitted at 1 and at 2 is equal to

$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t \left( 1 - \frac{v}{c} \cos \theta \right)$$
(7.39)

and the observed angular frequency  $\omega$  will be

$$\omega = \frac{2\pi}{\Delta t_A} = \frac{\omega'}{\gamma \left(1 - \frac{v}{c} \cos \theta\right)} = \omega' \gamma \left(1 + \frac{v}{c} \cos \theta'\right), \quad \omega' = \omega \gamma \left(1 - \frac{v}{c} \cos \theta\right). \tag{7.40}$$



Figure 7.1: Illustration of the relativistic Doppler effect and aberration.

which is the relativistic Doppler formula. The classical Doppler effect requires to take into account not only the relative velocity between source and observer but also their velocities relative to the medium carrying the waves (say, air in case of sound waves). The relativistic formula does not refer to any medium for the propagation of light, it involves only the relative velocity of source and observer.

#### 7.3 Four-Vectors

A four-vectors are transformed in the same manner as coordinates of events (7.2). Most physical quantities can be related to four-vectors or to their generalizations - the tensors. We should point out that the invariance of  $(x^{\mu})^2$ , is a general property of four-vectors. However, in Minkowski space, it is possible for the "square" of a four-vector to be positive, zero, or negative; these possibilities are called, respectively, a space-like, null (or light-like), or time-like four-vector.

Let us introduce some physically important four-vectors other than the general type  $x^{\mu}$ . The difference between two infinitesimally neighboring events,  $dx^{\mu}$ , is also a four-vector. Dividing now  $dx^{\mu}$  by the element of proper time  $d\tau$  defines the *four-velocity* (where i = 1, 2, 3),

$$U^{\mu} = \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \begin{pmatrix} c \,\mathrm{d}t/\mathrm{d}\tau\\ \mathrm{d}x^{i}/\mathrm{d}\tau \end{pmatrix} = \gamma_{u} \begin{pmatrix} c\\ u^{i} \end{pmatrix} = \gamma_{u} \begin{pmatrix} c\\ u \end{pmatrix}, \tag{7.41}$$

where  $\gamma_u = (1 - u^2/c^2)^{-1/2}$ , and u is the magnitude of the ordinary (or "three-velocity"), u = dx/dt. The transformation of  $U^{\mu}$  under the boost along the x-axis is

$$U'^{0} = \gamma \left( U^{0} - \beta U^{1} \right), \quad U'^{1} = \gamma \left( -\beta U^{0} + U^{1} \right), \quad U'^{2} = U^{2}, \quad U'^{3} = U^{3}.$$
(7.42)

With the above definitions we have

$$\gamma_{u'}c = \gamma\gamma_u \left(c - \beta u^1\right), \quad \gamma_{u'}{u'}^1 = \gamma\gamma_u \left(-\beta c + u^1\right), \quad \gamma_{u'}{u'}^2 = \gamma_u u^2, \quad \gamma_{u'}{u'}^3 = \gamma_u u^3, \quad (7.43)$$

where the first two of them we can rewrite as

$$\gamma_{u'} = \gamma \gamma_u \left( 1 - \frac{v}{c^2} u^1 \right), \quad \gamma_{u'} {u'}^1 = \gamma \gamma_u \left( u^1 - v \right).$$
(7.44)

Since  $u^1 = u \cos \theta$ , we obtain the transformation for speed in terms of the factors  $\gamma$ :

$$\gamma_{u'} = \gamma \gamma_u \left( 1 - \frac{vu}{c^2} \cos \theta \right). \tag{7.45}$$

Dividing the second Eq. (7.44) by the first Eq. (7.44), we obtain the already derived formula (7.33):

$$u'^{1} = \frac{u^{1} - v}{1 - \frac{vu^{1}}{c^{2}}}.$$
(7.46)

The square of  $U^{\mu}$  is clearly Lorentz invariant,

$$U^{\mu}U_{\mu} = (\gamma_{u}c)^{2} - (\gamma_{u}u)^{2} = c^{2}.$$
(7.47)

The four-velocity takes an especially simple form in the rest frame (where the ordinary velocity  $\boldsymbol{u}$  vanishes). We have

$$U'^{\mu} = c \left(1, 0, 0, 0\right)^{T}, \tag{7.48}$$

where only the time component is nonzero. This property makes four-velocity a useful tool in picking out the time component of an arbitrary vector (since  $U'_{\mu}A'^{\mu} = U_{\mu}A^{\mu}$  is an invariant) as measured by an observer with four-velocity  $U^{\mu}$ :

$$A'^{0} = \frac{1}{c} U'_{\mu} A'^{\mu} = \frac{1}{c} U_{\mu} A^{\mu}, \qquad (7.49)$$

where  $U_{\mu}A^{\mu}$  can be evaluated in any convenient frame, not necessarily in the rest frame. Two examples of this formula can be checked immediately: First, setting  $A^{\mu} = U^{\mu}$ , we obtain the trivial result  $U'^0 = c$ . Setting  $A^{\mu} = x^{\mu}$ , we find

$$x'^{0} = \frac{1}{c} x_{\mu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}\tau} = \frac{1}{2c} \frac{\mathrm{d}}{\mathrm{d}\tau} (x_{\mu} x^{\mu}) = \frac{1}{2c} \frac{\mathrm{d}}{\mathrm{d}\tau} (c^{2} \tau^{2}) = c\tau, \qquad (7.50)$$

which is obviously correct, since the proper time is physically equal to the time of a clock in the rest frame.

In a quite similar manner we can construct the *four-acceleration* (do not confuse its notation  $A^{\mu}$  with notation of an arbitrary four-vector)

$$A^{\mu} = \frac{\mathrm{d}U^{\mu}}{\mathrm{d}\tau} = \frac{\mathrm{d}^2 x^{\mu}}{\mathrm{d}\tau^2} = \gamma_u^2 \begin{pmatrix} \gamma_u^2 \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{c} \\ \boldsymbol{a} + \gamma_u^2 \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{c^2} \boldsymbol{u} \end{pmatrix},$$
(7.51)

which is defined as the rate of change of four-velocity with respect to the particle's proper time along its worldline, where  $\mathbf{a} = d\mathbf{u}/dt$  is the three-acceleration and  $\mathbf{u}$  is the three-velocity. The four-acceleration geometrically represents a curvature vector of a worldline. Therefore, the magnitude of the four-acceleration (which is a Lorentz scalar) is equal to the proper acceleration that a moving particle "feels" when moving along a worldline. A worldline having constant fouracceleration is a *Minkowski-circle*, i.e., hyperbola. The scalar product of a particle's four-velocity and its four-acceleration is always zero,

$$u^{\mu}a_{\mu} = 0. \tag{7.52}$$

Another four-vector, the "wavenumber vector"  $\mathbf{k}$ : A planar electromagnetic wave has space and time dependence proportional to  $\exp(i\mathbf{k} \cdot \mathbf{x} - i\omega t)$ . The phase of this wave must be an invariant to all observers, since the vanishing of the electric and magnetic fields in one frame implies their vanishing in all frames (a charged particle moving on an unaccelerated straight-line trajectory in one frame must have such a trajectory in all frames, by the relativity principle). Notice that we may write

$$\mathbf{k} \cdot \mathbf{x} - \omega t = -k_{\mu} x^{\mu}, \quad \text{where} \quad k^{\mu} = \begin{pmatrix} \omega/c \\ \mathbf{k} \end{pmatrix}.$$
 (7.53)

The product  $k_{\mu}x^{\mu}$  is an invariant and since  $x^{\mu}$  is an arbitrary four-vector, then  $k^{\mu}$  must be a four-vector also. Therefore, we can write the transformation for  $k^{\mu}$ ,

$$k'^{0} = \gamma \left(k^{0} - \beta k^{1}\right), \quad k'^{1} = \gamma \left(-\beta k^{0} + k^{1}\right), \quad k'^{2} = k^{2}, \quad k'^{3} = k^{3}.$$
(7.54)

Since  $|\mathbf{k}| = \omega/c$  for electromagnetic waves, we have  $k^1 = (\omega/c) \cos \theta$ , and the zeroth component  $k'^0$  of the transformation reduces to the Doppler formula

$$\omega' = \omega \gamma \left( 1 - \frac{v}{c} \cos \theta \right). \tag{7.55}$$

Another way that leads to (7.55) is to use (7.49) with  $A^{\mu} = k^{\mu}$ , which is obviously a null vector, since Eq. (5.39) gives

$$k_{\mu}k^{\mu} = \frac{\omega^2}{c^2} - |\mathbf{k}|^2 = 0.$$
(7.56)

The construction of four-vectors is by no means an automatic procedure; in two cases  $(x^{\mu})$  and  $k^{\mu}$  we have simply used a known three-vector for the spatial part and added an appropriate time component. In one case  $(U^{\mu})$  we multiplied the speed of light and a three-velocity by a factor  $\gamma_u$  to make the four-vector. In some cases (electric and magnetic fields) there is no four-vector that corresponds to a given three-vector. However, the best way to systematic construction of four-vectors is using the means of tensor analysis, introduced in Sect. 7.1.

## 7.4 Transformation of Electromagnetic Fields

It is empirically confirmed that Maxwell equations are Lorentz invariant in form; then the speed of light c and the elementary charge e are the Lorentz invariants (*Lorentz scalars*). Since the four-volume element  $dx_0 dx_1 dx_2 dx_3$  is Lorentz invariant and setting  $\rho$  the charge density, we have  $de = d\rho dx_1 dx_2 dx_3$ , thus  $\rho$  must transform as the zeroth component of a four-vector.

We write the equation of charge conservation (5.13) using the notation introduced,

$$j^{\mu}_{,\mu} = 0, \quad \text{where} \quad j^{\mu} = \begin{pmatrix} \rho c \\ \boldsymbol{j}, \end{pmatrix}$$
 (7.57)

obtaining the four-vector of current density or simply the *four-current*. Following Eqs. (5.30) and (5.31), we define another four-vector, the *four-potential* 

$$A^{\mu} = \begin{pmatrix} \phi/c \\ \mathbf{A} \end{pmatrix}, \quad A'^{\mu} = \Lambda^{\mu}{}_{\nu}A^{\nu}, \quad A^{\mu} = \tilde{\Lambda}^{\mu}{}_{\nu}A'^{\nu}, \tag{7.58}$$

thus we can rewrite the Lorentz gauge condition (5.29) as a covariant scalar product

$$A^{\alpha}{}_{,\alpha} = 0 \tag{7.59}$$

and Eqs. (5.30) and (5.31) as tensor equations

$$A^{\alpha}_{,\beta}{}^{,\beta} = \mu j^{\alpha}. \tag{7.60}$$

Using the four-potential, we can construct the covariant *tensor of electromagnetic field* that combines the electric and magnetic field as a whole (see Kurfürst (2017), Sect. 2.3):

$$F_{\mu\nu} = \frac{\partial A_{\nu}}{\partial x^{\mu}} - \frac{\partial A_{\mu}}{\partial x^{\nu}} = A_{\nu}{}^{,\mu} - A_{\mu}{}^{,\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{pmatrix},$$
(7.61)

while the contravariant tensor  $F^{\mu\nu}$  takes the form

$$F^{\mu\nu} = \frac{\partial A^{\nu}}{\partial x_{\mu}} - \frac{\partial A^{\mu}}{\partial x_{\nu}} = A^{\nu}_{,\mu} - A^{\mu}_{,\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}.$$
 (7.62)

Using Eqs. (7.61) and (7.62) we can construct Maxwell's equations in tensor form: The two Maxwell equations containing sources (for  $\nabla \cdot \boldsymbol{E}$  and  $\nabla \times \boldsymbol{B}$ ) are

$$F^{\mu\nu\,,\nu} = -\mu_0 \, j^\mu,\tag{7.63}$$

while the other two Maxwell's equations (for  $\nabla \cdot \boldsymbol{B}$  and  $\nabla \times \boldsymbol{E}$ ) we find as

$$F_{\mu\nu}{}^{,\sigma} + F_{\nu\sigma}{}^{,\mu} + F_{\sigma\mu}{}^{,\nu} = 0.$$
(7.64)

Since  $F_{\mu\nu}$  is a second-rank tensor, its components are transformed in the usual way, that is,

$$F^{\prime\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}, \quad F^{\prime}_{\mu\nu} = \tilde{\Lambda}^{\alpha}_{\mu}\tilde{\Lambda}^{\beta}_{\nu}F_{\alpha\beta}$$
(7.65)

Following Eq. (7.65) we obtain the transformation law for the fields  $\boldsymbol{E}$  and  $\boldsymbol{B}$ . For a boost with velocity  $\boldsymbol{v}$ , these equations can be written in the form:

$$\boldsymbol{E}_{||}^{\prime} = \boldsymbol{E}_{||} \qquad \qquad \boldsymbol{B}_{||}^{\prime} = \boldsymbol{B}_{||} \qquad (7.66)$$

$$\boldsymbol{E}_{\perp}^{\prime} = \gamma \left( \boldsymbol{E}_{\perp} + \boldsymbol{v} \times \boldsymbol{B} \right) \qquad \qquad \boldsymbol{B}_{\perp}^{\prime} = \gamma \left( \boldsymbol{B}_{\perp} - \frac{\boldsymbol{v}}{c^{2}} \times \boldsymbol{E} \right). \tag{7.67}$$

Let us apply Eqs. (7.66) to express the case of a constant velocity  $\mathbf{v} \equiv v_x \equiv v$  parallel with x axis. Then we have

$$E'_{x} = E_{x}, \qquad E'_{y} = \gamma \left( E_{y} - vB_{z} \right), \qquad E'_{z} = \gamma \left( E_{z} + vB_{y} \right),$$
(7.68)

$$B'_{x} = B_{x}, \qquad \qquad B'_{y} = \gamma \left( B_{y} + \frac{v}{c^{2}} E_{z} \right), \qquad \qquad B'_{z} = \gamma \left( B_{z} - \frac{v}{c^{2}} E_{y} \right), \tag{7.69}$$

while the inverse transformations of the perpendicular components are

$$E_y = \gamma \left( E'_y + vB'_z \right), \qquad \qquad E_z = \gamma \left( E'_z - vB'_y \right), \qquad (7.70)$$

$$B_y = \gamma \left( B'_y - \frac{v}{c^2} E'_z \right), \qquad \qquad B_z = \gamma \left( B'_z + \frac{v}{c^2} E'_y \right). \tag{7.71}$$

One consequence of these equations is that a pure electric or pure magnetic field is not Lorentz invariant. If the field is purely electric ( $\mathbf{B} = 0$ ) in one frame, in another frame it will be, in general, a mixed electric and magnetic field.

Any scalar formed from  $F_{\mu\nu}$  represents a function of **E** and **B** which is a Lorentz invariant. One such scalar is just the dot product of F with itself,

$$F_{\mu\nu}F^{\mu\nu} = 2\left(\boldsymbol{B}^2 - \frac{\boldsymbol{\mathcal{E}}^2}{c^2}\right) = F'_{\mu\nu}F'^{\mu\nu} = 2\left(\boldsymbol{B'}^2 - \frac{\boldsymbol{\mathcal{E}'}^2}{c^2}\right).$$
 (7.72)

Another Lorentz scalar is the determinant of F:

$$\det F = \left(\frac{\boldsymbol{E} \cdot \boldsymbol{B}}{c}\right)^2 \tag{7.73}$$

Thus  $\boldsymbol{E} \cdot \boldsymbol{B} = \boldsymbol{E}' \cdot \boldsymbol{B}'$  is also an invariant. We can prove that the determinant of any second-rank tensor is a Lorentz scalar, since

$$\det A^{\mu\nu} = \det \left(\Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}A^{\alpha\beta}\right) = (\det \Lambda)^2 \det A^{\alpha\beta} = \det A^{\alpha\beta}.$$
(7.74)

## 7.5 Electromagnetic Field of a Uniformly Moving Charge

Let us apply Eqs. (7.66) to find the fields of a charge q moving with constant velocity  $\mathbf{v}$  along the x axis. In the rest frame of the charged particle ( $\mathbf{v'} = \mathbf{0}$ ) the fields are

$$E'_{x} = \frac{1}{4\pi\epsilon_{0}} \frac{qx'}{r'^{3}} \qquad \qquad B'_{x} = 0 \qquad (7.75)$$

$$E'_{y} = \frac{1}{4\pi\epsilon_{0}} \frac{qy'}{r'^{3}} \qquad \qquad B'_{y} = 0 \tag{7.76}$$

$$E'_{z} = \frac{1}{4\pi\epsilon_{0}} \frac{qz'}{r'^{3}} \qquad \qquad B'_{z} = 0 \qquad (7.77)$$

where  $r'^3 = (x'^2 + y'^2 + z'^2)^{3/2}$ . We can Lorentz transform (Eqs. (7.2) and (7.58)) the primed coordinates to give

$$E_x = \frac{1}{4\pi\epsilon_0} \frac{q\gamma(x-vt)}{r^3} \qquad B_x = 0 \qquad (7.78)$$

$$E_y = \frac{1}{4\pi\epsilon_0} \frac{q\gamma y}{r^3} \qquad \qquad B_y = -\frac{1}{4\pi\epsilon_0} \frac{q\beta\gamma z}{r^3}$$
(7.79)

$$E_z = \frac{1}{4\pi\epsilon_0} \frac{q\gamma z}{r^3} \qquad \qquad B_z = \frac{1}{4\pi\epsilon_0} \frac{q\beta\gamma y}{r^3} \tag{7.80}$$

where  $r^3 = \left[\gamma^2 (x - vt)^2 + y^2 + z^2\right]^{3/2}$ .

An important case is that of a highly relativistic charge,  $\gamma \gg 1$ . For simplicity, we choose the field point at a distance b from the origin along the y axis, so that its coordinates are (0, b, 0)while the coordinates of a particle are (vt, 0, 0). Then we have

$$E_x = \frac{-q\gamma vt}{4\pi\epsilon_0 \left(\gamma^2 v^2 t^2 + b^2\right)^{3/2}} \qquad B_x = 0$$
(7.81)

$$E_y = \frac{q\gamma b}{4\pi\epsilon_0 \left(\gamma^2 v^2 t^2 + b^2\right)^{3/2}} \qquad B_y = 0$$
(7.82)

$$E_z = 0 \qquad \qquad B_z = \frac{q\beta\gamma b}{4\pi\epsilon_0 \left(\gamma^2 v^2 t^2 + b^2\right)^{3/2}} \equiv \beta E_y \qquad (7.83)$$

For large  $\gamma$  we have  $\beta \approx 1$  and  $E_y \approx B_z$ . The fields are mostly transverse, since  $\max(E_x) = q/(6\sqrt{3}\pi\epsilon_0 b^2) \approx 10^{-2}q/(\pi\epsilon_0 b^2)$  while  $\max(E_y) \approx \max(B_z) = q\gamma/(4\pi\epsilon_0 b^2) \approx 25\gamma \max(E_x)$ . Therefore, the field of a highly relativistic charge appears to be a *pulse of radiation* traveling in the same direction as the charge (Eq. (5.19)) and is confined to the transverse plane roughly to a time interval  $\Delta t \sim b/(\gamma v)$ . This connection between the fields of a highly relativistic charge and an associated radiation field is an important one and is used in the method of virtual quanta (will be discussed in Appendix 8).

We can now Fourier transform this pulse of virtual radiation (cf. Sect. 5.4). We find

$$\hat{E}(\omega) = \int_{-\infty}^{\infty} E(t) e^{-i\omega t} dt = \frac{q\gamma b}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{e^{-i\omega t} dt}{\left(\gamma^2 v^2 t^2 + b^2\right)^{3/2}},$$
(7.84)

which we integrate using the solution

$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}\alpha x} \,\mathrm{d}x}{(1+x^2)^{3/2}} = 2\alpha \,K_1(\alpha), \quad \text{where} \quad K_1(\alpha) = \frac{\mathrm{i}\pi}{2} \frac{J_1(\mathrm{i}\alpha) + J_{-1}(\mathrm{i}\alpha)}{2\sin\pi} \tag{7.85}$$

is the modified Bessel function of order one that must be solved as limit. The functions in Eq. (7.85)

$$J_1(i\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+1)!} \left(\frac{i\alpha}{2}\right)^{2k+1} \quad \text{and} \quad J_{-1}(i\alpha) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k-1)!} \left(\frac{i\alpha}{2}\right)^{2k-1}$$
(7.86)

are the Bessel functions of the first kind of order one (see corresponding examples in Sect. B.2.4 in Kurfürst (2017)).

Solving Eq. (5.4) we obtain

$$\hat{E}(\omega) = \frac{q}{2\pi\epsilon_0 bv} \frac{\omega b}{\gamma v} K_1\left(\frac{\omega b}{\gamma v}\right),\tag{7.87}$$

thus the spectrum is (cf. Eq. (5.53))

$$\frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega} = \frac{c}{4\pi^2} |\hat{E}(\omega)|^2 = \frac{q^2c}{16\pi^4\epsilon_0^2 b^2 v^2} \left(\frac{\omega b}{\gamma v}\right)^2 K_1^2\left(\frac{\omega b}{\gamma v}\right). \tag{7.88}$$

The spectrum starts to cut off for  $\omega > \gamma v/b$ , which we could have predicted on the basis of the uncertainty principle, since the pulse is confined roughly to a time interval of order  $b/\gamma v$ . In fact, the complete behavior of  $\hat{E}(\omega)$  can be estimated to within a factor  $\sim 2$  just by analysis of the picture of E(t): E(t) has a maximum  $q\gamma/b^2$  for a time interval  $\sim b/\gamma v$ . Thus we approximate

$$\hat{E}_{\max}(\omega) \sim E_{\max}(t)\Delta t \sim \left(\frac{q\gamma}{b^2}\right) \left(\frac{b}{\gamma v}\right), \quad \Delta \omega \sim \frac{1}{\Delta t} \sim \frac{\gamma v}{b}.$$
 (7.89)

We have thus found the spectrum per unit area at a distance b from the line of the charge's motion. To find the total energy per unit frequency range, we must integrate this over  $dA = 2\pi b db$ :

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = 2\pi \int_{b_{\min}}^{b_{\max}} \frac{\mathrm{d}W}{\mathrm{d}A\,\mathrm{d}\omega} \, b\,\mathrm{d}b. \tag{7.90}$$

The lower limit has been chosen not as zero but as some minimum distance  $b_{\min}$ , such that the approximation of the field by means of classical electrodynamics and a point charge is valid.

Two possible choices are: either  $b_{\min}$  = radius of ion, if field is that of an ion, or  $b_{\min} \sim \hbar/(mc) =$ Compton wavelength of particle. The integral is now

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{q^2c}{8\pi^3\epsilon_0^2 v^2} \int_x^\infty y K_1^2(y) \,\mathrm{d}y, \quad \text{where} \quad y \equiv \frac{\omega b}{\gamma v}, \quad x \equiv \frac{\omega b_{\min}}{\gamma v}.$$
(7.91)

This integral can be solved again via the Bessel functions:

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{q^2c}{8\pi^3\epsilon_0^2 v^2} \left[ xK_0(x)K_1(x) - \frac{x^2}{2} \left( K_1^2(x) - K_0^2(x) \right) \right].$$
(7.92)

Two limiting forms occur:  $\omega$  is small,  $\omega \ll \gamma v/b_{\min}$  and  $\omega$  is large,  $\omega \gg \gamma v/b_{\min}$ , which give the two forms, respectively:

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{q^2c}{8\pi^3\epsilon_0^2 v^2} \ln\left(\frac{0.68\gamma v}{\omega b_{\min}}\right), \quad \frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{q^2c}{16\pi^2\epsilon_0^2 v^2} \exp\left(-\frac{2\omega b_{\min}}{\gamma v}\right). \tag{7.93}$$

These forms can be derived approximately by direct integration of  $xK_1^2(x)$ , using the asymptotic results  $K_1(x) \sim 1/x$ ,  $x \ll 1$ , and  $K_1(x) \sim (\pi/2x)^{1/2} e^{-x}$ ,  $x \gg 1$ .

### 7.6 Relativistic Dynamics and Four-Force

For a particle of constant invariant rest mass is the four-momentum of a particle  $P^{\mu}$  defined by

$$P^{\mu} = m_0 U^{\mu} \tag{7.94}$$

In the nonrelativistic limit are the spatial components of the four-momentum just the components of the ordinary three-momentum,  $m_0 \mathbf{v}$ . For the relativistic expression of all the components we consider the expansion of the time component  $P^0c$  (using Eq. (7.41)) for  $\mathbf{v} \ll c$ :

$$P^{0}c = m_{0}c U^{0} = \gamma m_{0}c^{2} \approx m_{0}c^{2} + \frac{1}{2}m_{0}\mathbf{v}^{2} + \dots$$
(7.95)

The second term in the latter expression in (7.95) is the classical nonrelativistic expression for the kinetic energy of the particle while the first term  $m_0c^2$  that is independent of  $\boldsymbol{u}$  is the rest energy of the particle. Therefore, we interpret  $E = P^0c$  as the total energy of the particle. If the spatial part of the relativistic momentum is  $\boldsymbol{p} = \gamma m_0 \boldsymbol{v}$  then

$$P^{\mu} = \begin{pmatrix} E/c \\ \boldsymbol{\rho} \end{pmatrix}. \tag{7.96}$$

Involving Eq. (7.41) and the Minkowski metric tensor Eq. (7.8), we thus obtain

$$(P^{\mu})^{2} = P^{\mu}P_{\mu} = m_{0}^{2}c^{2} = \frac{E^{2}}{c^{2}} - |\mathbf{p}|^{2}, \text{ so that } E^{2} = |\mathbf{p}|^{2}c^{2} + m_{0}^{2}c^{4}.$$
 (7.97)

The four-momentum of massless photons cannot be defined by Eq. (7.95). We can however define Eq. (7.96), where we use the quantum relations

$$E = \hbar \omega \quad \text{and} \quad \boldsymbol{\rho} = \hbar \boldsymbol{k}.$$
 (7.98)

From Eq. (7.53) we then have

$$P^{\mu} = \hbar k^{\mu} = \begin{pmatrix} \hbar \omega / c \\ \hbar \mathbf{k} \end{pmatrix}.$$
(7.99)

The photon four-momentum is a null (light-like) four-vector,  $P^{\mu}P_{\mu} = 0$ , since  $E = |\mathbf{p}|c = |\hbar \mathbf{k}|c$ .

Having defined the four-acceleration  $a^{\mu}$  in Eq. (7.51), we introduce the *four-force*  $F^{\mu}$  as a relativistic form of second Newton's law:

$$F^{\mu} \equiv m_0 a^{\mu} = \frac{\mathrm{d}P^{\mu}}{\mathrm{d}\tau}.$$
(7.100)

Substituting Eq. (7.100) into Eq. (7.51), we have

$$F^{\mu} = \gamma_u^2 m_0 \begin{pmatrix} \gamma_u^2 \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{c} \\ \boldsymbol{a} + \gamma_u^2 \frac{\boldsymbol{u} \cdot \boldsymbol{a}}{c^2} \boldsymbol{u} \end{pmatrix}, \qquad (7.101)$$

Equation (7.101) can be also expressed in terms of ordinary three-force  $\mathbf{f} = d\mathbf{p}/dt$  as

$$F^{\mu} = \gamma_u \begin{pmatrix} \boldsymbol{u} \cdot \boldsymbol{f}/c \\ \boldsymbol{f} \end{pmatrix}, \qquad (7.102)$$

where the scalar product

$$\boldsymbol{u} \cdot \boldsymbol{f} = \mathrm{d/dt} \left( \gamma_u \, m_0 c^2 \right) = \gamma_u^3 \, m_0 \, \boldsymbol{u} \cdot \boldsymbol{a} = \mathrm{d}E/\mathrm{d}t.$$
(7.103)

We evaluate in electrodynamics  $F^{\mu}$ , having known the electromagnetic Lorentz three-force,  $f^{\text{LF}} = q (\mathbf{E} + \mathbf{u} \times \mathbf{B})$ , in such a way that the Lorentz four-force involves the electromagnetic field embodied in the tensor  $F^{\mu\nu}$  and the particle three-velocity embodied in the four-velocity  $U^{\mu}$  and should also be a four-vector proportional to the (scalar) charge q of the body. The simplest way is

$$F^{\mu} = q F^{\mu\nu} U_{\nu}, \quad \text{so that} \quad a^{\mu} = \frac{q}{m_0} F^{\mu\nu} U_{\nu}.$$
 (7.104)

We check Eq. (7.104) by substituting Eqs. (7.41), (7.62), and (7.103). Since  $F^{00} \equiv 0$  and noting that, due to the formalism of raising and lowering indexes, the spatial part of  $U_{\mu} = -$  (the spatial part of)  $U^{\mu}$ , we have:

$$a^{0} = \frac{q}{m_{0}} \left( F^{01} U_{1} + F^{02} U_{2} + F^{03} U_{3} \right) = \frac{\gamma_{u} \, q \, \boldsymbol{u} \cdot \boldsymbol{E}}{m_{0} c} = \frac{\gamma_{u} \, \boldsymbol{u} \cdot \boldsymbol{f}}{m_{0} c} = \frac{\gamma_{u}^{4} \, \boldsymbol{u} \cdot \boldsymbol{a}}{c}, \tag{7.105}$$

which confirms the solution for the time componenta in Eqs. (7.51) and (7.102). In a similar way we check also the spatial part,

$$a^{1} = \frac{q}{m_{0}} \left( F^{10}U_{0} + F^{12}U_{2} + F^{13}U_{3} \right) = \frac{\gamma_{u} q}{m_{0}} \left( \mathbf{E} + \mathbf{u} \times \mathbf{B} \right)_{x} = \gamma_{u} \frac{\mathbf{f}_{x}^{\text{LF}}}{m_{0}},$$
(7.106)

while the solution of  $a^2$  and  $a^3$  is obviously corresponding, confirming thus the spatial part of Eqs. (7.51) and (7.102).

In accordance with Eq. (7.52) is the four-force, *regardless of its origin*, always orthogonal to the four-velocity:

$$F^{\mu}U_{\mu} \equiv m_0 A^{\mu}U_{\mu} = 0. \tag{7.107}$$

where Eq. (7.107) implies that every four-force is velocity dependent, which might be negligible in the nonrelativistic limit. For example, for the Lorentz four-force we find

$$F_{\rm LF}^{\mu}U_{\mu} = qF^{\mu\nu}U_{\mu}U_{\nu} = \gamma_u^2 \,\boldsymbol{u} \cdot \boldsymbol{f} - \gamma_u^2 \,\boldsymbol{u} \cdot \boldsymbol{f} = 0, \qquad (7.108)$$

which is however obvious from Eq. (7.108) remembering that  $F^{\mu\nu}$  (Eq. (7.62)) is antisymmetric and the four-tensor

$$U_{\mu}U_{\nu} = \gamma_u^2 \begin{pmatrix} c^2 & -c \, \boldsymbol{u} \\ -c \, \boldsymbol{u} & |\boldsymbol{u}|^2 \end{pmatrix}$$
(7.109)

is symmetric.

## 7.7 Radiation from relativistically moving charges

#### 7.7.1 Total emission

Let us consider an instantaneous rest frame  $\mathcal{K}'$ , where a charged particle has zero velocity in a certain, infinitesimally small time interval dt' (which is just the proper time of the particle), so we can calculate the emitted radiation using the dipole (Larmor) formula. Suppose a total amount of energy dW' is emitted within the time interval dt'. Since the emission is isotropic, the three-momentum of this radiation in this frame is zero,  $d\mathbf{p}' = 0$ .

The energy in a frame  $\mathcal{K}$  moving with relative velocity -v with respect to the particle is therefore  $dW = \gamma dW'$ , while the time interval  $dt = \gamma dt'$ . We can evaluate the total power Pand P' emitted in frames  $\mathcal{K}$  and  $\mathcal{K}'$ , respectively, as

$$P = \frac{\mathrm{d}W}{\mathrm{d}t}, \quad P' = \frac{\mathrm{d}W'}{\mathrm{d}t'} = \frac{\mathrm{d}W}{\mathrm{d}t}, \quad \text{so that} \quad P = P'. \tag{7.110}$$

Thus the total emitted power is a Lorentz invariant for any isotropic (or at least for front-back symmetrical) emission in its instantaneous rest frame. We use this fact to express the power in covariant form. From the Larmor formula (6.21), we have

$$P' = \frac{1}{6\pi\epsilon_0} \frac{q^2}{c^3} |\mathbf{a}'|^2, \tag{7.111}$$

however, since  $U'^{\mu} = (c, \mathbf{0})$  in particle's rest frame and since  $A'^{\mu}U'_{\mu} = 0$ , we have

$$A'^{0} = 0$$
, thus  $|\mathbf{a}'|^{2} = A'^{\mu}A'_{\mu}$ , so that  $P' = P = \frac{1}{6\pi\epsilon_{0}}\frac{q^{2}}{c^{3}}A'^{\mu}A'_{\mu}$ . (7.112)

The emitted power in any frame is thus the square of  $A^{\mu}$  in that frame.

We now express the radiation power P in terms of the three-acceleration **a** rather than in terms of the four-acceleration  $A^{\mu}$ . If  $\mathcal{K}'$  is an instantaneous rest frame of a particle, then  $u'_{||} = u'_{\perp} = 0$  and  $u_{||} \equiv v$ ,  $u_{\perp} = 0$ , so that  $\gamma_u = \gamma$ . From  $A'_{\mu} = \tilde{\Lambda}^{\nu}_{\mu} A_{\nu}$ , with use of Eq. (7.51), we have

$$a'_{||} = \beta \gamma a_0 + \gamma a_{||} = -\beta \gamma^5 \frac{v a_{||}}{c} + \gamma^3 a_{||} + \gamma^5 \frac{v^2 a_{||}}{c^2} = \gamma^3 a_{||},$$
(7.113)

$$a'_{\perp} = \gamma^2 a_{\perp} \left( 1 + \gamma^2 \frac{u_{\perp}^2}{c^2} \right) = \gamma^2 a_{\perp}, \qquad (7.114)$$

We now write Eq. (7.112),

$$P = \frac{1}{6\pi\epsilon_0} \frac{q^2}{c^3} \left( a'_{||}^2 + a'_{\perp}^2 \right) = \frac{1}{6\pi\epsilon_0} \frac{q^2}{c^3} \gamma^4 \left( a_{\perp}^2 + \gamma^2 a_{||}^2 \right).$$
(7.115)

#### 7.7.2 Angular Distribution of Radiative Power

Let us now consider an amount of energy dW' is emitted in  $\mathcal{K}'$  into the solid angle  $d\Omega' = \sin \theta' d\theta' d\phi'$  about the direction at angle  $\theta'$  to the (say) x' axis. We denote

$$\mu = \cos \theta, \quad \mu' = \cos \theta' \quad \text{so that} \quad d\Omega = d\mu \, d\phi, \quad d\Omega' = d\mu' \, d\phi'.$$
 (7.116)

Since energy and momentum are components of a four-momentum  $P^{\mu}$  (Eqs. (7.96) and (7.99)), transformation of the radiation energy, according to P' = W'/c and  $P'^{x} = P' \cos \theta' \equiv P' \mu'$ , is

$$dP^{0} = \gamma dP'^{0} + \beta \gamma dP'^{1} \text{ so that } dW = \gamma \left( dW' + v dP'^{x} \right) = \gamma \left( 1 + \beta \mu' \right) dW'.$$
(7.117)

Differentiating Eq. (7.36), and, since  $d\phi = d\phi'$ , Eq. (7.116) yields

$$\frac{\mathrm{d}\mu}{\mathrm{d}\mu'} = \frac{\mathrm{d}}{\mathrm{d}\mu'} \left(\frac{\mu' + \beta}{1 + \beta\mu'}\right) = \frac{1}{\gamma^2 \left(1 + \beta\mu'\right)^2}, \quad \text{and so} \quad \mathrm{d}\Omega = \frac{\mathrm{d}\Omega'}{\gamma^2 \left(1 + \beta\mu'\right)^2} \tag{7.118}$$

and Eq. (7.117) gives

$$\frac{\mathrm{d}W}{\mathrm{d}\Omega} = \gamma^3 \left(1 + \beta \mu'\right)^3 \frac{\mathrm{d}W'}{\mathrm{d}\Omega'}.\tag{7.119}$$

The power P' emitted in the rest frame is simply given by dW'/dt'. However, in frame K there are two possible choices We further outline two time intervals to divide dW in  $\mathcal{K}$ :

- $dt = \gamma dt'$  defines the time interval  $P_e$  during which the emission occurs in  $\mathcal{K}$ .
- $dt_A = \gamma(1 \beta \mu) dt'$  defines the time interval  $P_r$  of reception of a radiative power by a stationary observer in  $\mathcal{K}$  (see Eqs. (7.39) and (7.40)).

we thus obtain the two different results:

$$\frac{\mathrm{d}P_{\mathrm{e}}}{\mathrm{d}\Omega} = \frac{\mathrm{d}W}{\mathrm{d}\Omega\,\mathrm{d}t} = \gamma^2 \left(1 + \beta\mu'\right)^3 \frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \frac{1}{\gamma^4 \left(1 - \beta\mu\right)^3} \frac{\mathrm{d}P'}{\mathrm{d}\Omega'},\tag{7.120}$$

$$\frac{\mathrm{d}P_{\mathrm{r}}}{\mathrm{d}\Omega} = \frac{\mathrm{d}W}{\mathrm{d}\Omega\,\mathrm{d}t_A} = \gamma^4 \left(1 + \beta\mu'\right)^4 \frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \frac{1}{\gamma^4 \left(1 - \beta\mu\right)^4} \frac{\mathrm{d}P'}{\mathrm{d}\Omega'},\tag{7.121}$$

where we converted  $\mu$  to  $\mu'$  and vice versa using Eq. (7.36).

Which of these two equations should we use? Since  $P_r$  is the power actually measured by an observer, it would seem to be the natural one. Also in favor of  $P_r$  is that Eq. (7.121) has the expected symmetry property of yielding the inverse transformation by interchanging primed and unprimed variables, along with a change of sign of  $\beta$ . For these reasons we hereafter in this section deal with  $P_r$ , calling it shortly P.

In practice, the distinction between emitted and received power is often not important, since  $P_{\rm e}$  and  $P_{\rm r}$  are equal in an average sense for stationary distributions of particles. We will discuss this further in the context of synchrotron emission (still in progress).

Let us now return to Eq. (7.121). If the radiation is isotropic in the particle's frame (or nearly isotropic), then the angular distribution in the observer's frame will be more or less peaked in the forward direction for relativistic velocities ( $\beta \rightarrow 1$ ). Regarding small  $\theta$  angle and  $\beta \sim 1$ , we may expand

$$\mu \approx 1 - \frac{\theta^2}{2}, \quad \beta = \sqrt{1 - \frac{1}{\gamma^2}} \approx 1 - \frac{1}{2\gamma^2}, \quad \text{thus} \quad \frac{1}{\gamma^4 \left(1 - \beta \mu\right)^4} \approx \left(\frac{2\gamma}{1 + \gamma^2 \theta^2}\right)^4. \tag{7.122}$$

The latter factor is sharply peaked near  $\theta \simeq 0$  with an angular scale of order  $1/\gamma$ .

We apply the previous considerations to an emitting particle. In the instantaneous rest frame of the particle the angular distribution is given by (cf. Eq. (6.20))

$$\frac{\mathrm{d}P'}{\mathrm{d}\Omega'} = \frac{1}{(4\pi)^2 \epsilon_0} \frac{q^2 {a'}^2}{c^3} \sin^2 \Theta',$$
(7.123)

where  $\Theta'$  is the angle between the acceleration and the direction of emission. Spitting  $\mathbf{a}' = \mathbf{a}'_{||} + \mathbf{a}'_{\perp}$  and using Eqs. (7.113), (7.114), and (7.121), we obtain

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{q^2}{(4\pi)^2 \epsilon_0 c^3} \frac{\left(\gamma^2 a_{||}^2 + a_{\perp}^2\right)}{\left(1 - \beta\mu\right)^4} \sin^2 \Theta',\tag{7.124}$$

To use this formula we relate  $\Theta'$  to the angles in the frame  $\mathcal{K}$ . This is difficult in general case, so we work out the angular distribution of the received power for special cases:

• Acceleration is parallel to velocity: In this case  $\Theta' = \theta'$  and according to Eq. (7.36),

$$\sin^2 \Theta' = \frac{\sin^2 \theta}{\gamma^2 \left(1 - \beta \mu\right)^2}.$$
(7.125)

Substituting Eq. (7.125) into Eq. (7.124) with  $a_{\perp} = 0$ , we obtain

$$\frac{\mathrm{d}P_{||}}{\mathrm{d}\Omega} = \frac{q^2}{(4\pi)^2 \epsilon_0 c^3} a_{||}^2 \frac{\sin^2 \theta}{(1-\beta\mu)^6}.$$
(7.126)

• Acceleration is perpendicular to velocity: Choosing  $\mathbf{a}'$  such that the angle  $\phi = 0$  (see Eq. (7.116)) gives  $\cos \Theta' = \sin \theta' \cos \phi'$ , so that

$$\sin^2 \Theta' = 1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 \left(1 - \beta \mu\right)^2}$$
(7.127)

and substituting Eq. (7.127) into Eq. (7.124) with  $a_{\parallel} = 0$ , we obtain

$$\frac{\mathrm{d}P_{\perp}}{\mathrm{d}\Omega} = \frac{q^2 a_{\perp}^2}{(4\pi)^2 \epsilon_0 c^3} \frac{1}{(1-\beta\mu)^4} \left[ 1 - \frac{\sin^2 \theta \cos^2 \phi}{\gamma^2 \left(1-\beta\mu\right)^2} \right].$$
(7.128)

• Extreme Relativistic Limit: When  $\gamma \gg 1$ , the quantity  $(1 - \beta \mu)$  in the denominators becomes small and the radiation becomes strongly peaked in the forward direction. Using the same arguments as in Eq. (7.122), we obtain

$$1 - \beta \mu \approx \frac{1 + \gamma^2 \theta^2}{2\gamma^2},\tag{7.129}$$

which gives radiation power for the parallel acceleration,

$$\frac{\mathrm{d}P_{||}}{\mathrm{d}\Omega} = \frac{4q^2}{\pi^2\epsilon_0 c^3} \,\gamma^{10}a_{||}^2 \,\frac{\gamma^2\theta^2}{\left(1+\gamma^2\theta^2\right)^6}.\tag{7.130}$$

while for the perpendicular acceleration,

$$\frac{\mathrm{d}P_{\perp}}{\mathrm{d}\Omega} = \frac{q^2}{\pi^2 \epsilon_0 c^3} \,\gamma^8 a_{\perp}^2 \,\frac{1 - 2\gamma^2 \theta^2 \cos 2\phi + \gamma^4 \theta^4}{\left(1 + \gamma^2 \theta^2\right)^6}.\tag{7.131}$$

Since both the expressions in Eqs. (7.130) and (7.131) are angle dependent merely through the combination  $\gamma\theta$ , the peaking is for angles  $\theta \sim 1/\gamma$ .

## 7.8 Relativistic Invariants and Specific Intensity

Consider a group of particles that occupy a spatial volume element  $d^3\mathbf{x}' = dx' dy' dz'$  and a momentum volume element  $d^3\mathbf{p}' = dP'_x dP'_y dP'_z$  in a frame  $\mathcal{K}'$  comoving with the particles, but no spread in energy,  $dW' = dP'_0 = 0$ , because the contribution to the energy from the space momentum in the rest frame is quadratic and thus vanishes to the first order. The group occupies a phase space element  $d\mathfrak{V}' = d^3\mathbf{p}' d^3\mathbf{x}'$ . For any observer in any frame  $\mathcal{K}$  not comoving with the particles they occupy the same amount of phase space in his frame  $d\mathfrak{V}' = d\mathfrak{V} = d^3\mathbf{p} d^3\mathbf{x}$ , a phase space element is Lorentz invariant.

We prove it in the following way: Let the observer moves along the x axis with velocity parameter  $\beta$  with respect to  $\mathcal{K}'$ . Consider the spatial volume element  $d^3\mathbf{x}$  in  $\mathcal{K}$ , occupied by the particles. Perpendicular distances are unaffected, dy = dy' and dz = dz', but there is a length contraction in the x direction (cf. Eq. (7.29)). This yields

$$\mathrm{d}^3 \mathbf{x} = \gamma^{-1} \, \mathrm{d}^3 \mathbf{x}'. \tag{7.132}$$

The momentum volume element measured in  $\mathcal{K}$  is  $\mathrm{d}^{3}\boldsymbol{p}$ . The momentum transforms as a fourvector (Eq. (7.26)),  $\mathrm{d}P_{x} = \beta\gamma \,\mathrm{d}P'_{x} + \gamma \,\mathrm{d}P'_{0}$ ,  $\mathrm{d}P_{y} \,\mathrm{d}P_{z} = \mathrm{d}P'_{y} \,\mathrm{d}P'_{z}$ , but since the particles have the same energy in  $\mathcal{K}$  as in  $\mathcal{K}'$ ,  $\mathrm{d}P_{x} = \gamma \,\mathrm{d}P'_{x}$ , we obtain

$$\mathrm{d}^{3}\boldsymbol{\rho} = \gamma \,\mathrm{d}^{3}\boldsymbol{\rho}' \tag{7.133}$$

and, combining Eqs. (7.132) and (7.133),

$$d\mathfrak{V} = d\mathfrak{V}', \text{ so that } d\mathfrak{V} = \text{Lorentz invariant.}$$
 (7.134)

Equation (7.134) was derived for particles of finite mass, where  $\mathcal{K}'$  is a rest frame, however, since there occurs no reference to particle mass, it is applicable also to the limiting case of photons. From Eq. (7.134), it follows the phase space density

$$f = \frac{\mathrm{d}N}{\mathrm{d}\mathfrak{V}} = \text{Lorentz invariant},$$
 (7.135)

because the number of particles dN within the phase volume element is a countable quantity and therefore itself invariant.

We relate the phase space density of photons to the specific intensity  $I_{\nu}$  by evaluating the energy density  $u_{\nu}(\Omega)$  per unit solid angle per frequency range (Eqs. (4.9) and (4.10)), using f:  $dE_{\nu}(\Omega) = h\nu dN d\Omega = u_{\nu} d\Omega d\nu$ , so that  $h\nu f d\mathfrak{V} d\Omega = u_{\nu} d\Omega d\nu d^{3}\mathbf{x}$ , and, since  $U_{n}u(\Omega) = I_{\nu}/c$ ,  $p = h\nu/c$ , this yields

$$\frac{I_{\nu}}{\nu^3} = \frac{h^4}{c^2} f = \text{Lorentz invariant.}$$
(7.136)

Because the source function occurs in RTE as  $I_{\nu} - S_{\nu}$ , it must have the same transformation properties as  $I_{\nu}$ ,

$$\frac{S_{\nu}}{\nu^3} = \text{Lorentz invariant.}$$
(7.137)

To find the transformation of absorption coefficient  $\alpha_{\nu}$  we assume material in  $\mathcal{K}$  streaming with velocity v between two planes parallel to the x-axis, while  $\mathcal{K}'$  is the rest frame of the material. The optical depth  $\tau$  along the ray must be an invariant, since  $e^{-\tau}$  gives the fraction of photons flowing through the material, which involves direct counting. Denoting l, l' the distance of the two planes and  $\theta$ ,  $\theta'$  the deviation angle of the ray from x-axis,

$$\tau = \frac{l\alpha_{\nu}}{\sin\theta} = \frac{l}{\nu\sin\theta} \,\nu\alpha_{\nu} = \text{Lorentz invariant.}$$
(7.138)

since  $\nu \sin \theta$  is proportional to the y component of the photon four-momentum  $k_y$ . But  $k_y = k'_y$ and l = l', being perpendicular to the motion. Therefore

$$\nu \alpha_{\nu} = \text{Lorentz invariant.}$$
(7.139)

The transformation of the emission coefficient  $j_{\nu} = \alpha_{\nu} S_{\nu}$  from Eqs. (7.137) and (7.139) is

$$\frac{j_{\nu}}{\nu^2} = \text{Lorentz invariant},$$
 (7.140)

where the derivation of Eq. (7.140) can be based also on Eq. (7.120). The emission coefficient can be written as

$$j_{\nu} = n \frac{\mathrm{d}P_{\mathrm{e}}}{\mathrm{d}\Omega \,\mathrm{d}\nu},\tag{7.141}$$

where n is the density of emitters (particles/m<sup>3</sup>). Since Eq. (7.40) gives  $d\nu = d\nu'\gamma(1 + \beta\mu')$ , and also  $n = \gamma n'$  by Lorentz contraction along the motion,

$$j_{\nu} = \gamma^2 (1 + \beta \mu')^2 n' \frac{\mathrm{d}P'}{\mathrm{d}\Omega' \,\mathrm{d}\nu'} = \left(\frac{\nu}{\nu'}\right)^2 j'_{\nu},\tag{7.142}$$

It is often convenient to determine the quantities  $\alpha_{\nu}$ ,  $j_{\nu}$ ,  $S_{\nu}$ , in the rest frame of the material. By the above results we can then find them in any frame. Because the transformation of  $\nu$  involves the direction  $\theta$  of the ray, these quantities will not, in general, be isotropic, even if they are isotropic in the rest frame. The observed nonisotropy of the cosmic microwave background can be used to find the velocity of the earth through the background.

## Chapter 8

# Free-Free Transitions (Bremsstrahlung)

(Rybicki & Lightman 1979; Mihalas & Mihalas 1984): We call *free-free emission* or *bremsstrahlung* the radiation produced by a charge acceleration in the Coulomb field. A classical treatment is justified in some regimes, therefore, we first give a classical treatment and in Sect. 8.4 we add the quantum corrections (Gaunt factors).

The bremsstrahlung due to the collision of identical particles (electron-electron, protonproton) is zero in the dipole approximation, because the dipole moment  $\sum_i e_i \mathbf{r}_i$  is simply proportional to the (symmetric) center of mass  $\sum_i m_i \mathbf{r}_i = \mathbf{0}$ . In electron-ion bremsstrahlung the electrons are the primary radiators, since the relative accelerations are inversely proportional to the masses, while the charges are roughly equal. Since the ions are comparatively massive, we may treat the electron as moving in a fixed Coulomb field of the ion.

## 8.1 Emission from a Single Electron

Let us assume that the electron moves rapidly enough so that the deviation of its path from a straight line is negligible. This is the *small-angle scattering* regime. This approximation is not necessary, however, it is highly instructive because it simplifies the analysis and leads to equations of the correct form. Consider an electron of charge -e moving past an ion of charge Ze with impact parameter b (see Fig. 5.1). The dipole moment is  $\mathbf{d} = -e\mathbf{R}$  and its second derivative is  $\mathbf{\ddot{d}} = -e\mathbf{\dot{v}}$ , where  $\mathbf{v}$  is the velocity of the electron. The Fourier transform of this equation (cf. Eq. (6.28)) is

$$-\omega^2 \hat{\boldsymbol{d}}(\omega) = -e \int_{-\infty}^{\infty} \dot{\boldsymbol{v}} e^{-i\omega t} dt.$$
(8.1)

We derive expressions for  $\hat{\boldsymbol{d}}(\omega)$  in the asymptotic limits of large and small frequencies. The electron is in close interaction with the ion over a *collision time* interval, which is of order

$$\tau = \frac{b}{v}.\tag{8.2}$$

For  $\omega \tau \gg 1$  the exponential in the integral oscillates rapidly, and the integral is small. For  $\omega \tau \ll 1$  the exponential is essentially unity, so we may write

$$\hat{\boldsymbol{d}}(\omega) = \begin{cases} \frac{e}{\omega^2} \Delta \boldsymbol{v}, & \omega \tau \ll 1, \\ 0, & \omega \tau \gg 1, \end{cases}$$
(8.3)

where  $\Delta v$  is the change of velocity during the collision. Using Eqs. (6.30) and (8.3) gives

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \begin{cases} \frac{1}{6\pi^2\epsilon_0} \frac{e^2}{c^3} |\Delta \mathbf{v}|^2, & \omega\tau \ll 1, \\ 0, & \omega\tau \gg 1. \end{cases}$$
(8.4)

Let us now estimate  $\Delta v$ . Since the path is almost linear, the change in velocity is predominantly normal to the path and we integrate the normal component of acceleration,

$$\Delta v = \frac{1}{4\pi\epsilon_0} \frac{Ze^2}{m_{\rm e}} \int_{-\infty}^{\infty} \frac{b\,{\rm d}t}{(b^2 + v^2t^2)^{3/2}} = \frac{1}{2\pi\epsilon_0} \frac{Ze^2}{m_{\rm e}bv}.$$
(8.5)

For small angle scatterings, the emission from a single collision (Eq. (8.4)) is

$$\frac{\mathrm{d}W(b)}{\mathrm{d}\omega} = \begin{cases} \frac{1}{24\pi^4\epsilon_0^3} \frac{Z^2 e^6}{m_{\mathrm{e}}^2 c^3 b^2 v^2}, & b \ll \frac{v}{\omega}, \\ 0, & b \gg \frac{v}{\omega}. \end{cases}$$
(8.6)

We determine the total spectrum for a medium with ion density  $n_{\rm ion}$ , electron density  $n_{\rm e}$ and for a fixed electron speed v. Note that the flux of electrons (electrons per unit area per unit time) incident on one ion is  $n_{\rm e}v$ . The element of area is  $2\pi b \, db$  about a single ion. The total emission per unit time per unit volume per unit frequency range is then  $dW/(d\omega \, dV \, dt)$ , which we explicitly express as  $(dW \, dV)/[d\omega \, (dV)^2 \, dt]$ , where  $dV/(dV)^2$  we expand as  $(v \, dt \, dS) \, (n_e n_{\rm ion})$ where the first bracket is dV of incident electron while the second bracket we may regard as  $(n_e n_{\rm ion} \, dm_e \, dm_{\rm ion})/(dm_e \, dm_{\rm ion}) = \rho^2/(\rho \, dV)^2$ . We write

$$\frac{\mathrm{d}W}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} = n_{\mathrm{e}}n_{\mathrm{ion}}2\pi v \int_{b_{\mathrm{min}}}^{\infty} \frac{\mathrm{d}W(b)}{\mathrm{d}\omega} b\,\mathrm{d}b,\tag{8.7}$$

where  $b_{\min}$  is some minimum value of impact parameter whose choice we discuss below.

It would seem that the asymptotic limits in Eq. (8.4) are insufficient to evaluate the integral in Eq. (8.7), which requires values of  $dW(b)/d\omega$  for a full range of impact parameters. However, a very good approximation can be achieved using only its low frequency asymptotic limit. We substitute the case  $b \ll v/\omega$  from Eq. (8.6) into Eq. (8.7). This gives

$$\frac{\mathrm{d}W}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} = \frac{1}{12\pi^3\epsilon_0^3} \frac{e^6}{m_{\rm e}^2 c^3 v} n_{\rm e} n_{\rm ion} Z^2 \int_{b_{\rm min}}^{b_{\rm max}} \frac{\mathrm{d}b}{b} = \frac{1}{12\pi^3\epsilon_0^3} \frac{e^6}{m_{\rm e}^2 c^3 v} n_{\rm e} n_{\rm ion} Z^2 \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right), \quad (8.8)$$

where  $b_{\text{max}}$  is some value of *b* beyond which the contribution of  $b \ll v/\omega$  limit to the integral becomes negligible. The (uncertain) value of  $b_{\text{max}}$  is of order  $v/\omega$ . Since  $b_{\text{max}}$  is in the logarithm, its precise value is of low importance, we take  $b_{\text{max}} \equiv v/\omega$  and make a small error. The use of the asymptotic forms (8.6) is justified, because equal intervals in the logarithm of *b* contribute equally to the emission, and over most of these intervals the emission is determined by its low frequency asymptotic limit.

We can estimate  $b_{\min}$  in two ways. First we regard the value at which the straight-line approximation ceases to be valid. Since this occurs when  $\Delta v \sim v$ , we take from Eq. (8.5)  $b_{\min}^{(1)} = Ze^2/(4\pi\epsilon_0 m_e v^2)$ , where the half-factor comes due to integration from 0 to  $-\infty$ . A second way to find  $b_{\min}$  is quantum in nature and is based on treating the collision process in terms of classical orbits. From the uncertainty principle  $\Delta x \Delta p \leq \hbar$ ; taking  $\Delta x \sim b$  and  $\Delta p \sim m_e v$  we have  $b_{\min}^{(2)} = h/(m_e v)$ . When  $b_{\min}^{(1)} \gg b_{\min}^{(2)}$ , a classical approach is valid and we use  $b_{\min} = b_{\min}^{(1)}$ . This occurs when  $\frac{1}{2}mv^2 \ll Z^2\mathcal{R}y$ , where  $\mathcal{R}y = me^4/(2\hbar^2)$  is the Rydberg energy for the hydrogen atom. When  $b_{\min}^{(1)} \ll b_{\min}^{(2)}$ , or, equivalently,  $\frac{1}{2}mv_e^2 \gg Z^2\mathcal{R}y$ , the uncertainty principle plays an important role, and the classical calculation cannot strictly be used. Nonetheless, results of the correct order of magnitude are obtained by simply setting  $b_{\min} = b_{\min}^{(2)}$ . For any regime, we may state the exact results in terms of a Gaunt factor  $g_{\mathrm{ff}}(v,\omega)$ , using

For any regime, we may state the exact results in terms of a Gaunt factor  $g_{\rm ff}(v,\omega)$ , using which Eq. (8.8) is

$$\frac{\mathrm{d}W}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} = \frac{1}{12\sqrt{3}\pi^2\epsilon_0^3} \frac{e^6}{m_{\rm e}^2 c^3 v} n_{\rm e} n_{\rm ion} Z^2 g_{\rm ff}(v,\omega).$$
(8.9)

Comparison of Eqs. (8.8) and (8.9) gives

$$g_{\rm ff}(v,\omega) = \frac{\sqrt{3}}{\pi} \ln\left(\frac{b_{\rm max}}{b_{\rm min}}\right). \tag{8.10}$$

The Gaunt factor is a function of the electron energy and of the frequency of the emission. Extensive tables and graphs of it exist in the literature.

#### 8.2 Thermal Free-Free Emission:

We apply these formulas to thermal bremsstrahlung by averaging the above single-speed expressions over a thermal distribution of speeds. A particle has a Maxwellian speed range  $\langle v, v + dv \rangle$ as  $dP \propto v^2 \exp[-mv^2/(2kT)] dv$ . Now we integrate Eq. (8.8) over this probability function, setting  $0 \leq v < \infty$ . At frequency  $\nu$ , the incident velocity must be at least  $h\nu \leq \frac{1}{2}m_ev^2$ , because otherwise a photon of energy  $h\nu$  could not be created. This lower limit cutoff over electron velocities is called a *photon discreteness effect*. From the integral

$$\frac{\mathrm{d}W(T,\omega)}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} = \frac{\int_{v_{\min}}^{\infty} \frac{\mathrm{d}W(v,\omega)}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} v^2 \,\mathrm{e}^{-\frac{m_e v^2}{2kT}}\,\mathrm{d}v}{\int_{0}^{\infty} v^2 \,\mathrm{e}^{-\frac{m_e v^2}{2kT}}\,\mathrm{d}v},\tag{8.11}$$

where  $v_{\min} = (2h\nu/m_e)^{1/2}$  and  $d\omega = 2\pi d\nu$ , by substituting Eq. (8.9) we obtain

$$\frac{\mathrm{d}W(T,\nu)}{\mathrm{d}\nu\,\mathrm{d}V\,\mathrm{d}t} = \frac{1}{6\sqrt{3}\pi\epsilon_0^3} \frac{e^6}{m_{\rm e}^2 c^3} n_{\rm e} n_{\rm ion} Z^2 \frac{\int_{v_{\rm min}}^{\infty} g_{\rm ff}\left(v,\nu\right) v \,\mathrm{e}^{-\frac{m_{\rm e}v^2}{2kT}} \,\mathrm{d}v}{\int_0^{\infty} v^2 \,\mathrm{e}^{-\frac{m_{\rm e}v^2}{2kT}} \,\mathrm{d}v}.$$
(8.12)

Integration of the denominator gives  $\sqrt{\pi/2} (kT/m_{\rm e})^{3/2}$ , we integrate the numerator by substituting  $m_{\rm e}v^2/(2kT) = x$ , from  $m_{\rm e}v_{\rm min}^2/(2kT)$  to  $\infty$ , giving  $\bar{g}_{\rm ff}(T,\nu) \frac{kT}{m_{\rm e}} \exp(-h\nu/kT)$ . The integral (8.12) becomes

$$\frac{\mathrm{d}W(T,\nu)}{\mathrm{d}\nu\,\mathrm{d}V\,\mathrm{d}t} = \frac{e^6}{6\pi\epsilon_0^3 c^3 m_\mathrm{e}} \left(\frac{2}{3\pi k m_\mathrm{e}}\right)^{1/2} T^{-1/2} n_\mathrm{e} n_\mathrm{ion} Z^2 \,\mathrm{e}^{-\frac{h\nu}{kT}} \,\bar{g}_\mathrm{ff}(T,\nu). \tag{8.13}$$

Evaluation of Eq. (8.13) in SI units (radiation energy density  $[J s^{-1} m^{-3} H z^{-1}]$ ) gives

$$\epsilon_{\nu}^{\rm ff} \approx 6.78 \times 10^{-51} \, T^{-1/2} \, n_{\rm e} n_{\rm ion} Z^2 \, {\rm e}^{-\frac{h\nu}{kT}} \, \bar{g}_{\rm ff} \, (T,\nu), \tag{8.14}$$

where  $\bar{g}_{\rm ff}(T,\nu)$  is a velocity averaged *Gaunt factor*. In cgs units the numerical factor in Eq. (8.14) is ~ 6.83 × 10<sup>-38</sup>, noting that the value of elementary charge  $e \approx 4.803 \times 10^{-10}$  statC,  $\epsilon_0 \equiv 1$ , and Eq. (8.14) must be multiplied by  $(4\pi)^3$ .

The values of  $\bar{g}_{\rm ff}$  for  $h\nu/(kT) \gg 1$  are not important, since the spectrum here cuts off.  $\bar{g}_{\rm ff}$  is of order unity for  $h\nu/(kT) \sim 1$  and ranges within 1 - 5 for  $10^{-4} < h\nu/(kT) < 1$ . Good order of magnitude estimates are made by setting  $\bar{g}_{\rm ff}$  to unity. The spectrum of bremsstrahlung is rather "flat" in a log-log plot up to its cutoff at about  $h\nu \sim kT$  (this is however true only for optically thin sources, we have not yet considered absorption of photons by free electrons). To obtain the formulas for non-thermal bremsstrahlung, we need to know the distribution of velocities, the emission formula from a single-speed electron must be averaged over that distribution, and one also must have the appropriate Gaunt factors.

We give formulas for the total power per unit volume emitted by thermal bremsstrahlung, that we obtain by integrating Eq. (8.13) over frequency:

$$\frac{\mathrm{d}W(T)}{\mathrm{d}V\,\mathrm{d}t} = \frac{e^6}{6\pi\epsilon_0^3 h c^3 m_\mathrm{e}} \left(\frac{2k}{3\pi m_\mathrm{e}}\right)^{1/2} T^{1/2} n_\mathrm{e} n_\mathrm{ion} Z^2 \,\bar{g}_B(T),\tag{8.15}$$

or, evaluated in SI (radiation power density  $[J s^{-1} m^{-3}]$ ),

$$\epsilon^{\rm ff} \approx 1.42 \times 10^{-40} T^{1/2} n_{\rm e} n_{\rm ion} Z^2 \bar{g}_B (T).$$
 (8.16)

The frequency (and velocity) averaged Gaunt factor  $\bar{g}_B(T)$  is in the range 1.1 - 1.5, setting the value 1.2 will give a 20% accuracy.

### 8.3 Thermal Free-Free Absorption:

It is possible to relate the (thermal) absorption of radiation to the bremsstrahlung emission process. In that case we have the Kirchhoff's law,

$$j_{\nu}^{\rm ff} = \alpha_{\nu}^{\rm ff} B_{\nu}(T), \qquad (8.17)$$

where  $B_{\nu}(T)$  is the Planck function,  $\alpha_{\nu}^{\text{ff}}$  is the free-free absorption coefficient, and  $j_{\nu}^{\text{ff}}$  is related to the emission formula (8.13) by

$$\frac{\mathrm{d}W}{\mathrm{d}\nu\,\mathrm{d}V\,\mathrm{d}t} = 4\pi j_{\nu}^{\mathrm{ff}}.\tag{8.18}$$

Using the Planck function  $B_{\nu}(T) = 2h\nu^3 c^{-2} \left[ \exp\left(\frac{h\nu}{kT}\right) - 1 \right]^{-1}$ , Eqs. (8.17) and (8.18) give

$$\alpha_{\nu}^{\rm ff} = \frac{e^6}{48\pi^2\epsilon_0^3 hm_{\rm e}c} \left(\frac{2}{3\pi km_{\rm e}}\right)^{1/2} T^{-1/2} n_{\rm e} n_{\rm ion} Z^2 \nu^{-3} \left(1 - e^{-\frac{h\nu}{kT}}\right) \bar{g}_{\rm ff} (T,\nu).$$
(8.19)

Evaluating Eq. (8.19) in SI units,  $\alpha_{\nu}^{\rm ff}({\rm m}^{-1})$  is:

$$\alpha_{\nu}^{\rm ff} \approx 3.69 \times 10^{-2} \, T^{-1/2} \, n_{\rm e} n_{\rm ion} Z^2 \, \nu^{-3} \left( 1 - {\rm e}^{-\frac{h\nu}{kT}} \right) \bar{g}_{\rm ff} \, (T, \nu). \tag{8.20}$$

For  $h\nu \gg kT$  the exponential is negligible and  $\alpha_{\nu}^{\rm ff} \sim \nu^{-3}$ . In the Rayleigh-Jeans regime,  $h\nu \ll kT$ , Eq. (8.19) becomes

$$\alpha_{\nu}^{\rm ff} = \frac{e^6}{48\pi^2\epsilon_0^3 km_{\rm e}c} \left(\frac{2}{3\pi km_{\rm e}}\right)^{1/2} T^{-3/2} n_{\rm e} n_{\rm ion} Z^2 \nu^{-2} \bar{g}_{\rm ff} (T,\nu), \tag{8.21}$$

or, with numerically evaluated constant factor,

$$\alpha_{\nu}^{\rm ff} \approx 1.77 \times 10^{-12} \, T^{-3/2} \, n_{\rm e} n_{\rm ion} Z^2 \, \nu^{-2} \, \bar{g}_{\rm ff} \, (T, \nu). \tag{8.22}$$

We find the Rosseland mean  $\alpha_R^{\rm ff}$  of the free-free absorption by

$$\frac{1}{\alpha_R^{\rm ff}} = \frac{\int_0^\infty (\alpha_\nu^{\rm ff} + \sigma_\nu^{\rm ff})^{-1} \frac{\partial B_\nu(T)}{\partial T} \,\mathrm{d}\nu}{\int_0^\infty \frac{\partial B_\nu(T)}{\partial T} \,\mathrm{d}\nu},\tag{8.23}$$

where  $\sigma_{\nu}^{\rm ff}$  is the scattering coefficient. We find the temperature derivative of the Planck function,

$$\frac{\partial B_{\nu}(T)}{\partial T} = \frac{2h^2\nu^4}{c^2kT^2} \left(e^{\frac{h\nu}{kT}} - 1\right)^{-2} e^{\frac{h\nu}{kT}},\tag{8.24}$$

integration over  $\nu$  gives the denominator equal to  $8\pi^4 k^4 T^3/(15c^2h^3)$ . Neglecting  $\sigma_{\nu}^{\rm ff}$  and using Eq. (8.19), the integral (8.23) is

$$\frac{1}{\alpha_R^{\rm ff}} = \frac{15\epsilon_0^3 c}{h^2 e^6} \left(\frac{6m_{\rm e}}{\pi}\right)^{3/2} \frac{(kT)^{7/2}}{n_{\rm e} n_{\rm ion} Z^2} \int_0^\infty \bar{g}_{\rm ff}^{-1} \frac{x^7 \,{\rm e}^x \,{\rm d}x}{\left({\rm e}^x - 1\right)^2 \left(1 - {\rm e}^{-x}\right)}.$$
(8.25)

The integral in Eq. (8.25) is  $\frac{8}{3}(945\,\zeta(7) + \pi^6)\,\bar{g}_R^{-1} \approx 5104.74\,\bar{g}_R^{-1}$ ,  $\zeta$  is Riemann zeta function, and  $\bar{g}_R$  is a (order of unity) weighted frequency average of  $\bar{g}_{\rm ff}$ . We evaluate Eq. (8.25) in SI units,

$$\alpha_R^{\rm ff} \approx 2.07 \times 10^{-35} \, T^{-7/2} \, n_{\rm e} n_{\rm ion} Z^2 \, \bar{g}_R, \tag{8.26}$$

while in cgs units the numerical factor in Eq. (8.26) is  $\sim 2.07 \times 10^{-25}$ , noting that the value of elementary charge  $e \approx 4.803 \times 10^{-10}$  statC,  $\epsilon_0 \equiv 1$ , and Eq. (8.25) must be divided by  $(4\pi)^3$  (RL introduce the numerical factor  $\sim 1.7 \times 10^{-25}$  in Eq. (8.26), they likely take into account also the scattering coefficient and numerical adjustment of Gaunt factor).

#### 8.4 Relativistic Bremsstrahlung

We now treat relativistic particles by the method of *virtual quanta*, a full understanding would require quantum electrodynamics. Consider the collision between an electron and a heavy ion of charge Ze. Normally, the ions move slowly in comparison to the electrons (in the rest frame of the medium as a whole), but it is possible to view the process in a frame of reference in which the electron is initially at rest. In that case the ion moves rapidly toward the electron along the x axis with velocity v while the electron is initially at rest on the y axis, in a distance b from the origin. We recall from Sect. 7.5 that the electrostatic field of the ion is transformed into an essentially transverse pulse with  $|\mathbf{E}| \sim |\mathbf{B}|$ , which the electron "sees" as a pulse of electromagnetic radiation. This radiation then Compton scatters off the electron to produce emitted radiation. Transforming back to the rest frame of the ion (lab frame), we obtain the bremsstrahlung emission of the electron. The relativistic bremsstrahlung can be regarded as the Compton scattering of the virtual quanta of the ion's electrostatic field as "seen" in the electron's frame. In the (primed) electron rest frame, setting v = c in the ultrarelativistic limit, the spectrum of the virtual quanta pulse (cf. Eq. (7.88)) is

$$\frac{\mathrm{d}W'}{\mathrm{d}A'\,\mathrm{d}\omega'} = \frac{(Ze)^2}{16\pi^4\epsilon_0^2 {b'}^2 c} \left(\frac{\omega'b'}{\gamma c}\right)^2 K_1^2\left(\frac{\omega'b'}{\gamma c}\right). \tag{8.27}$$

In the primed frame the virtual quanta are scattered by the electron according to the Thomson cross section for  $\hbar\omega' \leq m_{\rm e}c^2$ , and according to the Klein-Nishina cross section for  $\hbar\omega' > m_{\rm e}c^2$  (see Chapter 9). In the low-frequency limit, where we may regard  $d\omega'$  as the single Thomson cross section  $\sigma_T$ , the scattered radiation is

$$\frac{\mathrm{d}W'}{\mathrm{d}\omega'} = \sigma_T \frac{\mathrm{d}W'}{\mathrm{d}A'\,\mathrm{d}\omega'}.\tag{8.28}$$

Since energy and frequency are Lorentz transformed identically, the energy per frequency emitted in the lab frame is  $dW/d\omega = dW'/d\omega'$ . To write  $dW/d\omega$  as a function of b and  $\omega$ , rather than b' and  $\omega'$ , we note that transverse lengths are unchanged, b = b', and that  $\omega = \gamma \omega'(1 + \beta \cos \theta')$ , (cf. Eq. (7.40)), where  $\theta'$  is the scattering angle in the electron rest frame. Because such scattering is forward-backward symmetric, we have the averaged relation  $\omega = \gamma \omega'$ . Thus the emission in the lab frame is

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} = \frac{Z^2 e^6}{96\pi^5 \epsilon_0^4 b^2 m_\mathrm{e}^2 c^5} \left(\frac{\omega b}{\gamma^2 c}\right)^2 K_1^2 \left(\frac{\omega b}{\gamma^2 c}\right). \tag{8.29}$$

Equation (8.29) is the energy per unit frequency emitted by the collision of an ion and a relativistic electron at impact parameter b. For a plasma with electron and ion densities  $n_{\rm e}$  and  $n_{\rm ion}$ , respectively, we repeat the arguments leading to Eq. (8.7), where v is replaced by c and where  $b_{\rm min} \sim h/(m_{\rm e}c)$  according to the uncertainty principle. The integral in Eqs. (8.7) and (8.29) is identical to that in Eq. (7.91), except for an additional factor  $\gamma$  in the argument. Thus we have the low-frequency limit,  $h\omega < \gamma m_{\rm e}c^2$ ,

$$\frac{\mathrm{d}W}{\mathrm{d}\omega\,\mathrm{d}V\,\mathrm{d}t} \sim \frac{Z^2 e^6 n_{\rm e} n_{\rm ion}}{48\pi^4 \epsilon_0^4 m_{\rm e}^2 c^4} \ln\left(\frac{0.68\gamma^2 v}{\omega b_{\rm min}}\right).\tag{8.30}$$

At higher frequencies Klein-Nishina corrections must be used.

For a thermal distribution of electrons, a useful approximate expression for the frequency integrated power (J s<sup>-1</sup> m<sup>-3</sup>) in SI units is (see Novikov and Thorne 1973, cf. Eq. (8.15))

$$\frac{\mathrm{d}W}{\mathrm{d}V\,\mathrm{d}t} = 1.4 \times 10^{-40} T^{1/2} Z^2 n_{\mathrm{e}} n_{\mathrm{ion}} \bar{g}_B \left(1 + 4.4 \times 10^{-10} T\right).$$
(8.31)

The second term in brackets is a relativistic correction to Eq. (8.16).

## Chapter 9

# Compton scattering

### 9.1 The Kompaneets Equation

(Rybicki & Lightman 1979; Mihalas & Mihalas 1984): We make a short excursion to Thomson and Compton scattering basics to explain following principles. Thomson scattering, or the scattering of a photon by an electron at rest, strictly only applies at low photon energy, i.e., when  $h\nu \ll m_ec^2$ . If the photon energy is comparable to or greater than the electron energy, non-classical effects must be taken into account, and the process is called Compton scattering. A further interesting situation develops when the electron is moving - in this case energy can be transferred to the photon, and the process is called inverse Compton scattering. This last process is an important mechanism in high energy astrophysics.

In Thomson scattering, we have (in cgs)

$$\frac{\mathrm{d}\sigma_T}{\mathrm{d}\Omega} = \left(\frac{e^2}{m_{\rm e}c^2}\right)^2 \frac{\left(1 + \cos^2\chi\right)}{2},\tag{9.1}$$

where e now denotes the electron charge,  $\Omega$  is solid angle and  $\chi$  is an angle of scattering. In Thomson scattering the incident photon and scattered photon have the same wavelength or energy, so this scattering is also called coherent or elastic. If we now move to photons of energy  $h\nu \cong m_{\rm e}c^2$ , the scattering is modified by the appearence of quantum effects, through a change in the kinematics of the collision, and an alteration of the cross-section.

To do the kinematics of the collision correctly at high photon energy, momentum and energy must be conserved. Let the incident photon have energy  $h\nu_i$  and momentum  $h\nu_i/c$ , the scattered photon have energy  $h\nu_f$  and momentum  $h\nu_f/c$ , and the electron (initially at rest with rest energy  $m_ec^2$ ) acquires energy  $E = \sqrt{\gamma^2 m_e^2 v_e^2 c^2 + m_e^2 c^4}$  (where  $\gamma$  is the Lorentz factor) and momentum  $\boldsymbol{p}_e = \gamma m_e \boldsymbol{v}_e$ . The photon scattering angle is again  $\chi$ . From equality of energy and momenta components before and after scattering we obtain

$$\lambda_{\rm f} - \lambda_{\rm i} = \frac{h}{m_{\rm e}c} \left(1 - \cos\chi\right),\tag{9.2}$$

where  $\lambda_{\rm f}$  and  $\lambda_{\rm i}$  are the wavelength of scattered and incident photon, respectively, and the Compton wavelength  $\lambda_C \equiv h/(m_{\rm e}c) \approx 0.02426$  Å. The energy of a scattered photon (expressed using frequency  $\nu$  instead of wavelength  $\lambda$ ) is

$$h\nu_{\rm f} = \frac{h\nu_{\rm i}}{1 + \frac{h\nu_{\rm i}}{m_{\rm e}c^2} \left(1 - \cos\chi\right)},\tag{9.3}$$

The compton wavelength can be regarded as a wavelength change  $\Delta \lambda$  in the incident photon. Note that for  $\lambda \gg \lambda_C$  the change is negligible and we get back to the Thomson scattering. In full treatment of the problem yields the Klein-Nishina formula for the scattering cross-section:

$$\frac{\mathrm{d}\,\sigma}{\mathrm{d}\Omega} = \frac{1}{2} r_{\mathrm{e}}^2 \left(\frac{\nu_{\mathrm{f}}}{\nu_{\mathrm{i}}}\right)^2 \left(\frac{\nu_{\mathrm{f}}}{\nu_{\mathrm{i}}} + \frac{\nu_{\mathrm{i}}}{\nu_{\mathrm{f}}} - \sin^2\chi\right),\tag{9.4}$$

where  $r_{\rm e} = e^2/(m_{\rm e}c^2) = \alpha \hbar/(m_{\rm e}c) \approx 2.81794 \times 10^{-13} \,\mathrm{cm} \ (\alpha \approx 1/137 \,\mathrm{is} \,\mathrm{the} \,\mathrm{fine} \,\mathrm{structure}$ constant) is the classical electron radius. Note that in case  $\nu_{\rm f} \simeq \nu_{\rm i}$  (for  $h\nu \ll m_{\rm e}c^2$ , see Eq. (9.3)), Eq. (9.4) approaches the Thomson regime:

$$\frac{\mathrm{d}\,\sigma}{\mathrm{d}\Omega} = \frac{3}{16\pi}\sigma_T \left(1 + \cos^2\chi\right),\tag{9.5}$$

where the Thomson cross-section  $\sigma_T = (8\pi/3) r_e^2 \approx 6.652 \times 10^{-25} \text{ cm}^2$ .

Integration over solid angle gives total cross-section (where  $x = h\nu_{\rm i}/(m_{\rm e}c^2)$ ),

$$\sigma = 2\pi \int_0^\pi \frac{\mathrm{d}\,\sigma}{\mathrm{d}\Omega} \sin\theta \,\mathrm{d}\theta$$
  
=  $\frac{3}{4}\sigma_T \left\{ \frac{1+x}{x^3} \left[ \frac{2x\,(1+x)}{1+2x} - \ln\,(1+2x) \right] + \frac{1}{2x}\ln\,(1+2x) - \frac{1+3x}{(1+2x)^2} \right\}.$  (9.6)

Equation (9.6) can be expanded in the limits  $x \to 0$  and  $x \to \infty$  to yield the following formulae for the total cross-section

$$\sigma \simeq \sigma_T \left( 1 - 2x + \frac{26x^2}{5} + \dots \right) \qquad \text{for } x \ll 1, \tag{9.7}$$

$$\sigma \simeq \frac{3}{8}\sigma_T \frac{1}{x} \left( \ln 2x + \frac{1}{2} + \dots \right) \qquad \text{for } x \gg 1 \qquad (9.8)$$

for the non-relativistic and ultra-relativistic cases, respectively. The main effect is thus to reduce the cross-section at high photon energies, i.e., the scattering of the photons becomes less efficient.

An important case arises when the electrons are no longer considered to be at rest. In inverse scattering, energy is transferred from the electrons to the photons, i.e., it is the opposite of Compton scattering, in which the photons transfer energy to the electrons. Inverse Compton scattering can produce substantial fluxes of photons in the optical to X-ray region. Fig. 7.1 describes the frequency relation between that received by observer (in a rest frame  $\mathcal{K}$ ) and in a frame  $\mathcal{K}'$  emitted radiation, taking travel time into account. In a rest frame  $\mathcal{K}$  of observer, emitting source moves from 1 to 2 with velocity v. Photons emitted in interval dt' in  $\mathcal{K}'$  are separated in  $\mathcal{K}$  by (time-dilation)

$$\mathrm{d}t = \gamma \,\mathrm{d}t'.\tag{9.9}$$

By that time, a source has moved in  $\mathcal{K}$  to distance  $l = v \, dt$ , and  $d = v \, dt \cos \theta$  towards observer. Difference in arrival times for radiation emitted at 1 and 2 as seen by observer then is

$$\Delta t = t_2 - t_1 = dt + l_2 - l_1 = dt - \frac{d}{c} = dt - \frac{v \, dt \cos \theta}{c} = \gamma \left(1 - \beta \cos \theta\right) dt', \tag{9.10}$$

where  $l_1$ ,  $l_2$  are distances between the points 1, 2 and observer, and  $\beta$  is v/c. From the definition of frequencies  $\nu = 1/\Delta t$  and  $\nu' = 1/dt'$  we obtain

$$\nu = \frac{\nu'}{\gamma \left(1 - \beta \cos \theta\right)} = D\nu' \tag{9.11}$$

with D = Doppler factor. Note that the Doppler factor depends on angle between observer and direction of source motion and it can be very large, e.g., for large speeds  $v \to c$  and for head-on motion ( $\theta = 0$ )

$$D = \frac{1}{\gamma (1 - \beta)} = \frac{1 + \beta}{\gamma (1 - \beta^2)} \simeq 2\gamma.$$
(9.12)

We can use Eq. (9.11) also for photon energies  $h\nu'$  and  $h\nu$  in  $\mathcal{K}'$  and  $\mathcal{K}$ . We denote  $\epsilon'_i = h\nu'_i$ and  $\epsilon'_f = h\nu'_f$  the energies of incident and scattered photon in emitted radiation frame  $\mathcal{K}'$ , and  $\epsilon_i = h\nu_i$  and  $\epsilon_f = h\nu_f$  the energies of incident and scattered photon in observers rest frame  $\mathcal{K}$ . We also have to note that Eq. (9.2) may be now split into the two frames as (while in the observer's frame  $\mathcal{K}$  this will be analogous)

$$\epsilon_{\rm f}' = \frac{\epsilon_{\rm i}'}{1 + \frac{\epsilon_{\rm i}'}{m_{\rm e}c^2} \left(1 - \cos\chi'\right)},\tag{9.13}$$

where the angle  $\cos \chi'$ , regarding the radiation rest frame  $\mathcal{K}'$ , i.e., the frame in which the scatterin electron is at rest (or analogously  $\cos \chi$ ), is defined as  $\cos \chi' = \mathbf{n}'_i \cdot \mathbf{n}'_f$ , which in spherical coordinates (where  $\mathbf{n} = \sin \theta \cos \phi$ ,  $\sin \theta \sin \phi$ ,  $\cos \theta$ ) gives

$$\cos \chi' = \cos \theta'_{\rm i} \cos \theta'_{\rm f} + \sin \theta'_{\rm i} \sin \theta'_{\rm f} \cos \left( \phi'_{\rm f} - \phi'_{\rm i} \right). \tag{9.14}$$

For further solution we need to employ the aberration formula (see Sect. 7.2.4) that results in the following relations:

$$\epsilon_{i} = \frac{\epsilon'_{i}}{\gamma \left(1 - \beta \cos \theta_{i}\right)} \quad \Leftrightarrow \quad \epsilon'_{i} = \epsilon_{i} \gamma \left(1 - \beta \cos \theta_{i}\right), \tag{9.15}$$

$$\epsilon_{\rm f} = \frac{\epsilon_{\rm f}'}{\gamma \left(1 - \beta \cos \theta_{\rm f}\right)} = \epsilon_{\rm f}' \gamma \left(1 + \beta \cos \theta_{\rm f}'\right). \tag{9.16}$$

In Thomson regime where  $\epsilon'_i \ll m_e c^2$ ,  $\epsilon'_i \approx \epsilon'_f$ , employing Eqs. (9.15) and (9.16) gives

$$\epsilon_{\rm f} = \epsilon_{\rm i} \gamma^2 \left(1 - \beta \cos \theta_{\rm i}\right) \left(1 + \beta \cos \theta_{\rm f}'\right) = \epsilon_{\rm i} \frac{1 - \beta \cos \theta_{\rm i}}{1 - \beta \cos \theta_{\rm f}}.$$
(9.17)

For head-on scattering with maximum energy gain (where  $\theta_i = \pi$  and  $\theta_f = 0$ , scattered photons thus turn around) and for large speeds  $v \to c$  we obtain

$$\epsilon_{\rm f,max} = \epsilon_{\rm i} \, \frac{1+\beta}{1-\beta} = \epsilon_{\rm i} \, \gamma^2 (1+\beta)^2 \simeq 4\gamma^2 \epsilon_{\rm i}. \tag{9.18}$$

This analysis shows that the mean frequency of the photons after the collision may increase up to a factor  $\gamma^2$ , so that high frequency radio photons in collisions with relativistic electrons for which  $\gamma$  is of order 10<sup>3</sup> to 10<sup>4</sup> can be boosted in the UV and X-ray regions. There is however a practical limit to the amount of boosting possible beyond the Thomson limit (Klein-Nishina limit where  $\epsilon'_i \gg m_e c^2$  thus  $\epsilon_i \gg m_e c^2/\gamma$ ), which can be seen from the conservation of energy

$$\epsilon_{\rm f,max} \lesssim \gamma m_{\rm e} c^2 + \epsilon_{\rm i},$$
(9.19)

scattered photon energies (in the lab frame) are thus limited to  $\gamma m_{\rm e}c^2$ ,

It is convenient to define the scattering cross-section as the equivalent area of the incident wavefront that delivers the same power as that re-radiated by the particle: that is,  $\sigma = \text{total re-radiated power}/\langle u \rangle$ , where  $\langle u \rangle$  is the time-averaged radiative flux of incident photons. The power emitted in a single scattering in case of an isotropic distribution of photons in Thomson regime in an electron rest frame  $\mathcal{K}'$  is given as

$$\frac{\mathrm{d}E'}{\mathrm{d}t'} = \sigma_T \langle u' \rangle = \sigma_T c U'_{\mathrm{rad}} = \sigma_T c \int n'_{\mathrm{ph}}(\epsilon'_{\mathrm{i}}) \,\epsilon'_{\mathrm{f}}(\epsilon'_{\mathrm{i}}, \Omega') \,\mathrm{d}\epsilon'_{\mathrm{i}} \,\mathrm{d}\Omega', \qquad (9.20)$$

where  $U'_{\rm rad}$  is the radiative energy density and  $n'_{\rm ph}(\epsilon'_i) d\epsilon'_i$  is number density of incident photons with energy in the interval  $\langle \epsilon'_i, \epsilon'_i + d\epsilon'_i \rangle$ , and  $\Omega'$  is the solid angle. The emitted power dE'/dt' = dE/dt, the phase space volume  $d^3\mathbf{x}'d^3\mathbf{p}' = d^3\mathbf{x} d^3\mathbf{p}$ , are the Lorentz invariants between inertial frames. The number of particles within phase volume element, dN is countable (conserved) quantity, thus dN' = dN and for the same reason also the phase space distribution  $f'(\mathbf{x}', \mathbf{p}') = dN'/(d^3\mathbf{x}'d^3\mathbf{p}') = f(\mathbf{x}, \mathbf{p}) = dN/(d^3\mathbf{x} d^3\mathbf{p})$  is also the Lorentz invariant between inertial frames.

We are now interested in the rate of arrival of photons at the origin of the frame  $\mathcal{K}'$ from the direction  $\theta$ . Let us consider two photons which arrive there at times  $t'_1$  and  $t'_2$ (see Fig. 7.1). The coordinates of these events in  $\mathcal{K}$  are  $[x_1, 0, 0, t_1] = [\gamma v t'_1, 0, 0, \gamma t'_1]$  and  $[x_2, 0, 0, t_2] = [\gamma v t'_2, 0, 0, \gamma t'_2]$ , respectively. This calculation makes the important point that the photons in the beam are propagated along parallel but separate trajectories in  $\mathcal{K}$  as illustrated by Fig. 7.1. From the geometry of the figure, it is apparent that the time difference when the photons arrive at a plane perpendicular to their direction of propagation in  $\mathcal{K}$  (to the steeply dropping line at the lower left end of the image) is

$$\Delta t = t_2 + \frac{x_2 - x_1}{c} \cos \theta - t_1 = (t'_2 - t'_1) \gamma (1 + \beta \cos \theta), \qquad (9.21)$$

that is, the time interval between the arrival of photons from the direction  $\theta$  is shorter by a factor  $\gamma [1 + (v/c) \cos \theta]$  in  $\mathcal{K}'$  than it is in  $\mathcal{K}$ . Thus, the rate of arrival of photons, and correspondingly their number density, is greater by this factor  $\gamma [1 + \beta \cos \theta]$  in  $\mathcal{K}'$  as compared with  $\mathcal{K}'$ . This is exactly the same factor by which the energy of the photon has increased in Eq. (9.20). On reflection, we should not be surprised by this result because these are two different aspects of the same relativistic transformation between the frames  $\mathcal{K}$  and  $\mathcal{K}'$ , in one case the frequency interval and, in the other, the time interval.

Thus, as observed in  $\mathcal{K}'$ , the energy density of the beam is therefore

$$U_{\rm rad}' = \left[\gamma \left(1 + \beta \cos \theta\right)\right]^2 U_{\rm rad},\tag{9.22}$$

Now, this energy density is associated with the photons incident at angle  $\theta$  in the frame  $\mathcal{K}$  and it consequently arrives within solid angle  $2\pi \sin \theta \, d\theta$  in  $\mathcal{K}$ . We assume that the radiation field in  $\mathcal{K}$  is isotropic and therefore we can now work out the total energy density seen by the electron in  $\mathcal{K}'$  by integrating over solid angle (steradian) in  $\mathcal{K}$ , that is,

$$U_{\rm rad}' = U_{\rm rad} \int_0^{\pi} \gamma^2 \left(1 + \beta \cos \theta\right)^2 \frac{1}{2} \sin \theta \, \mathrm{d}\theta = \gamma^2 \left(1 + \frac{\beta^2}{3}\right) U_{\rm rad},\tag{9.23}$$

Therefore, substituting directly into Eq. (9.20), we find the average energy  $\langle E_{\gamma} \rangle$  gained by the photon field due to the scattering of the low energy photons as  $dE/dt = \sigma_T c (U'_{rad} - U_{rad})$ . We have therefore to subtract the energy of these photons to find the total energy gain to the photon field in S. The rate at which energy is removed from the low energy photon field is  $\sigma_T c U_{rad}$  and therefore, subtracting, we find an emitted power for a single scattering

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \sigma_T c \left[ \gamma^2 \left( 1 + \frac{\beta^2}{3} \right) - 1 \right] U_{\mathrm{rad}} = \frac{4}{3} \sigma_T c \,\beta^2 \gamma^2 U_{\mathrm{rad}}. \tag{9.24}$$

We can now calculate the total Compton power per unit volume, for a medium of relativistic electrons. Let  $N_{\rm e}(\gamma) \, \mathrm{d}\gamma$  be the number of electrons per unit volume with  $\gamma$  in the interval  $\langle \gamma, \gamma + \mathrm{d}\gamma \rangle$ . Then total Compton power is given as

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{tot}} = \int \frac{\mathrm{d}E}{\mathrm{d}t} N_{\mathrm{e}}(\gamma) \,\mathrm{d}\gamma. \tag{9.25}$$

The total Compton power can be thus calculated, provided the distribution of the electrons is known (see RL).

However, we can now calculate the average power of the photon field gained from the electrons as follows. For thermal distribution of non-relativistic electrons,  $\langle \beta^2 \rangle = 3kT/(m_{\rm e}c^2)$ ,  $\gamma \simeq 1$ , Eq. (9.24) becomes

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{tot}} = \left(\frac{4kT}{m_{\mathrm{e}}c^2}\right)\sigma_T c \,N_{\mathrm{e}}U_{\mathrm{rad}},\tag{9.26}$$

where  $N_{\rm e}$  is total number of electrons. Hence, if  $4kT_{\rm e} > \langle E_{\gamma} \rangle$ , the net energy is from electrons to photons (inverse Compton scattering) and gas heats up, while if  $4kT_{\rm e} < \langle E_{\gamma} \rangle$  the net energy transfer is from photons to electrons and gas cools down. In other words, we may say that in a typical collision between an electron and a photon, the electron energy changes by  $[4kT_{\rm e}/(m_{\rm e}c^2)] \langle E_{\gamma} \rangle$ .

Compton y-parameter gives the condition for a significant change of energy of photon due to repeated scattering, that is,

$$y \equiv (\langle \Delta(h\nu) \rangle$$
 of a photon per scattering) × (mean # of scatterings). (9.27)

When electrons and photons co-exist in a region of size l, the repeated scattering of photons by the electrons will distort the original spectrum of the photons (i.e., Comptonization). The mean free path of the photon due to Thomson scattering is  $\lambda_{\gamma} = (n_{\rm e}\sigma_T)^{-1}$ . If the size of the region l is such that  $l/\lambda_{\gamma} \gg 1$ , then the photon will undergo several collisions in this region. On the other hand, if  $l/\lambda_{\gamma} \ll 1$ , then we may expect only few collisions. Therefore let us define optical depth as  $\tau_{\rm e} \equiv l/\lambda_{\gamma} = n_{\rm e}\sigma_T l$ , so that  $\tau_{\rm e} \gg 1$  implies strong scattering. If  $\tau_{\rm e} \gg 1$ , then the photon undergoes  $N_{\rm sc}(\gg 1)$  collisions in traveling a distance l. From standard random-walk arguments, we have  $N_{\rm sc} = \tau_{\rm e}^2$ . On the other hand, if  $\tau_{\rm e} \leq 1$ , then  $N_{\rm sc} \simeq \tau_{\rm e}$ . Therefore an estimate for the number of scattering is  $N_{\rm sc} \simeq \max(\tau_{\rm e}, \tau_{\rm e}^2)$ . The average fractional change in the photon energy per collision (see Eq. (9.26)) is given by the term  $4kT/(m_{\rm e}c^2)$ . Hence the condition for a significant change of energy is

$$y \simeq N_{\rm sc} \left(\frac{4kT}{m_{\rm e}c^2}\right) = \left(\frac{4kT}{m_{\rm e}c^2}\right) \max\left(\tau_{\rm e}, \tau_{\rm e}^2\right).$$
 (9.28)

The photon must thus undergo  $\sim m_{\rm e}c^2/(4kT)$  collisions to significantly increase its energy.

Note that the Compton *y*-parameter is generally given by Kompaneets equation of the spectrum for Comptonization, whose detailed derivation we do not introduce here since it is difficult and is obtained by solving the non-relativistic diffusion equation for the motion of photons through phase-space (Kompaneets, 1957). Kompaneets equation is the specialized form of a Fokker-Planck equation with its general form

$$\frac{\partial n(\nu,t)}{\partial t} = n_{\rm e} \sigma_T \frac{h}{m_{\rm e} c} \frac{1}{\nu^2} \frac{\partial}{\partial \nu} \left[ \nu^4 \left( \frac{kT}{h} \frac{\partial n(\nu,t)}{\partial \nu} + n(\nu,t) + n^2(\nu,t) \right) \right],\tag{9.29}$$

where  $n(\nu, t)$  is the time-dependent photon distribution function (photon number density. Equation (9.29) is usually written in the form

$$\frac{\partial n}{\partial y} = \frac{1}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial n}{\partial x} + n + n^2 \right) \right]$$
(9.30)

where we substitute  $dy = [kT/(m_e c)] n_e \sigma_T dt$  and  $x = h\nu/(kT)$ .

Within the thermal Comptonization in far downstream (Katz et al. 2010; Nakar & Sari 2010) which relate  $N_{\rm sc} = d^2/\lambda^2 = ct/\lambda$  to thermalization length  $L_T$  and  $\beta_{\rm d}$  in case of thermally determined distribution of free electrons, yielding  $N_{\rm sc} = L_T/(\beta_{\rm d}\lambda) = L_T n_{\rm d}\sigma_T/\beta_{\rm d}$ ) the relevant Compton y-parameter in our case is

$$y = \frac{4kT}{m_{\rm e}c^2} L_T n_{\rm d} \,\sigma_T \,\beta_{\rm d}^{-1} = 49 \frac{4kT}{m_{\rm e}c^2} \left(L_T n_{\rm u} \,\sigma_T \,\beta_{\rm s}\right) \beta_{\rm s}^{-2} \sim 4 \times 10^{-4} \frac{kT}{\rm eV} \left(L_T n_{\rm u} \,\sigma_T \,\beta_{\rm s}\right) \,\beta_{\rm s}^{-2}, \tag{9.31}$$

hence it has 4 times higher value than the Kompaneets y-parameter. It is much larger than unity for  $L_T n_{\rm u} \sigma_T \beta_{\rm s} \gg 2.5 \times 10^3 (kT/{\rm eV})^{-1} \beta_{\rm s}^2$ . Photon effective generation rate  $Q_{\gamma,\rm eff}$  includes, therefore, all photons produced down to an energy that allows them to be upscattered to  $kT_{\rm d}$ . For bremsstrahlung emission, which we assume to be the main source of photons, the number of photons generated diverges logarithmically at low energy, so that  $Q_{\gamma,\rm eff}$  may be significantly larger than the bremsstrahlung generation rate of photons at  $kT_{\rm d}$ . In order for the photon energy to significantly increase by scattering, it must be scattered  $\sim m_{\rm e}c^2/(4kT)$  times before getting re-absorbed.

We need also the standard equation for the mean number of photons  $\overline{N}$  per unit volume (given by statistical physics basics, where  $n(\nu, t)$  has the form  $n(\nu) = [e^x - 1]^{-1}$  with  $x = h\nu/(kT)$  in case of the equilibrium Planck distribution)

$$\bar{N} = \frac{8\pi}{c^3} \int_0^\infty n(\nu, t) \,\nu^2 \,\mathrm{d}\nu.$$
(9.32)

Let us now assume that  $n(\nu, 0)$  corresponds to a bremsstrahlung spectrum and that the electron temperature is constant in time. If we neglect the variation with photon energy of the Gaunt factor, this means that we can write  $n(\nu, 0) = n_0$ , where

## 9.2 Cutoff in the Bremsstrahlung Spectrum

In order to study the up-scattering of the low-energy bremsstrahlung photons, we assume here that the cutoff in the bremsstrahlung spectrum due to self-absorption occurs at frequency  $\nu_{\min}$ such that  $h\nu_{\min} \ll kT_{\rm e}$  which implies  $n(\nu, t) \gg 1$  (from Eq. (9.32), since  $C \{\exp[h\nu/(kT)] - 1\}^{-1} \approx kT/(h\nu)$ . However, when  $n(\nu, t) \gg 1$ , the Kompaneets equation (9.29) can be approximated by the equation

$$\frac{\partial n(\nu,t)}{\partial t} = n_{\rm e} \sigma_T \frac{h}{m_{\rm e} c} \frac{1}{\nu^2} \frac{\partial}{\partial \nu} \left[ \nu^4 n^2(\nu,t) \right]. \tag{9.33}$$

We consider the situation (see Sunyaev & Zeldovich 1970 for this paragraph) when a slow change of temperature occurs due to a flow of heat. The kinetic equation for photons with  $m_ec^2 \gg kT_e$  including the variation of electron temperature has the form (cf. also Eq. (9.29))

$$\frac{\partial n(x,t)}{\partial x} = \frac{a}{x^2} \frac{\partial}{\partial x} \left[ x^4 \left( \frac{\partial n}{\partial x} + n + n^2 \right) \right] + \frac{\mathcal{K} e^{-x}}{x^3} \left[ 1 - n(e^x - 1) \right] - x \frac{\partial n}{\partial x} \frac{\partial \ln T_e}{\partial t}, \qquad (9.34)$$
where  $x = h\nu/kT_{\rm e}$ . The first term describes the change of frequency due to Compton scattering for which  $a = \sigma_0 cn_{\rm e}kT_{\rm e}/(m_{\rm e}c^2)$  (Kompaneets 1956; Weymann 1965) and the second term bremsstrahlung together with the corresponding reverse and induced processes where

$$\mathcal{K} = \frac{8\pi}{3} \frac{e^6 h^2 g(x) \, n_{\rm e}^2}{m_{\rm e} (6\pi m_{\rm e} k T_{\rm e})^{1/2} (k T_{\rm e})^3} \approx 1.25 \times 10^{-12} \, g(x) \, \frac{n_{\rm e}^2}{T_{\rm e}^{3.5}} \tag{9.35}$$

and g(x) is the Gaunt factor. Finally the third term is connected with the fact that the temperature  $T_{\rm e}$  enters into a determination of the variable x. In fact, if no processes occur among the quanta, than  $\partial n(\nu, t)/\partial t|_{\nu} \equiv 0$  and for n(x) we have

$$\frac{\partial n}{\partial t}\Big|_{x} = \frac{\partial n}{\partial t}\Big|_{\nu} + \frac{\partial n}{\partial x}\frac{\partial x}{\partial t}\Big|_{\nu} = \frac{\partial n}{\partial t}\Big|_{\text{phys}} - \frac{\partial n}{\partial x}\frac{x}{T}\frac{\mathrm{d}T}{\mathrm{d}t},\tag{9.36}$$

where  $\partial n/\partial t|_{\text{phys}}$  corresponds to first two terms in Eq. (9.34). The general properties of Eq. (9.34) are obvious: the first term vanishes not only for Planck distribution  $n(x) = (e^x - 1)^{-1}$ , but also for a Bose-Einstein equilibrium distribution  $n(x) = (e^{x+\mu} - 1)^{-1}$  with a given number of quanta. The reason is that the Compton effect does not change the number of quanta, although it redistributes the quanta in frequency. The second term vanishes only in true equilibrium  $n = (e^x - 1)^{-1}$ . The third term describes the perturbing influence of the energy supply in the case when this energy is given primarily to the electrons.

Compton scattering will thus tend to depress a bremsstrahlung spectrum when the frequency is near to the self-absorption cutoff (following Chapline & Stevens 1973, who express the time t in units of  $n_e \sigma_T c$ , cf. Eq. (9.34)). If we assume that the rate of change of the bremsstrahlung spectrum due to Compton scattering is greater than the rate of change due to bremsstrahlung emission, then for large t (but not so large that the photons are absorbed) the distribution function will approach a distribution  $n_c(\nu)$  (where the subscript c is abbreviated from "cutoff") such that

$$\frac{h}{m_{\rm e}c^2} \frac{1}{\nu^2} \frac{\partial}{\partial\nu} \left[ \nu^4 n_{\rm c}^2(\nu) \right] = -S(\nu), \qquad (9.37)$$

where  $S(\nu) = [(hc)^3/(8\pi)] [32\pi m_e/(3kT_e)]^{1/2} (e^2/h) n_e \bar{g}_{\rm ff} \exp[-h\nu/(kT)]/(h\nu)^3$  is the contribution of bremsstrahlung emission to  $\partial n/\partial t$  (which comes from Fokker-Planck solution of bremsstrahlung emission equation (8.13), for the explanation see Chapline & Stevens 1973). Comparing the term  $S(\nu)$  with the terms x and  $\mathcal{K}$  from Eq. (9.34), we have  $n_e \sigma_T c S(\nu) = \mathcal{K} e^{-x}/x^3$  [where  $\sigma_T = (8\pi/3) e^4/(m_e^2 c^4)$  in cgs]. Solving equation (9.37) for  $n_c(\nu)$  gives

$$n_{\rm c}(\nu) = \frac{e}{h\nu^2} \left(\frac{m_{\rm e}^3 c^{10}}{6\pi \, kT_{\rm e}}\right)^{1/4} \left(n_{\rm e} \int_{h\nu/kT_{\rm e}}^{\infty} \bar{g}_{\rm ff} \frac{{\rm e}^{-u}}{u} {\rm d}u\right)^{1/2},\tag{9.38}$$

where the solution of the integral approximately is a function  $-\bar{g}_{\rm ff} \operatorname{Ei}(-h\nu/kT_{\rm e}) = -\bar{g}_{\rm ff} \left[\ln(h\nu/kT_{\rm e}) + \gamma - h\nu/kT_{\rm e} + h\nu/4kT_{\rm e} + \ldots\right]$  where  $\gamma$  is the Euler-Mascheroni constant. The range of frequencies for which the photon distribution function has the form given in equation (9.38) is determined by those values of  $h\nu$  for which the rate of change of the bremsstrahlung spectrum due to Compton scattering exceeds the rate of change of the spectrum due to bremsstrahlung emission and absorption. The lower limit of this range,  $h\nu_{\rm min}$ , can be found by equating  $n_{\rm c}(\nu)$  and the Planck function:

$$n_{\rm c}(\nu_{\rm min}) = \left(e^{\frac{h\nu_{\rm min}}{kT_{\rm e}}} - 1\right)^{-1} \approx \frac{kT_{\rm e}}{h\nu_{\rm min}}$$
(9.39)

for small  $h\nu/(kT_{\rm e})$ . The factor in brackets in equation (9.38) is approximately equal to  $n_{\rm e} \bar{g}_{\rm ff} \ln [kT_{\rm e}/(h\nu)]$  for small  $h\nu/(kT_{\rm e})$ . If we assume that the logarithm in this factor is of order unity, then we obtain

$$h\nu_{\rm min} \approx eh\left(\frac{m_{\rm e}^3 c^{10}}{6\pi}\right)^{1/4} (kT_{\rm e})^{-5/4} \left(n_{\rm e}\,\bar{g}_{\rm ff}\right)^{1/2}.$$
 (9.40)

## Chapter 10

# **Radiative Transitions**

### **10.1** Semi-Classical Theory

(Rybicki & Lightman 1979; Mihalas & Mihalas 1984): We investigate the transitions between atomic states. There are two major objectives: the *selection rules* for radiative transitions and the *strengths* of the radiation. The first of these is in some sense a special case of the second, but we shall regard it separately.

We use the so-called *semi-classical theory*, in which the atom is treated quantum mechanically, but the radiation is treated classically. It is known that this approach correctly predicts the induced radiation processes described by Einstein B coefficients, but that it fails to predict the spontaneous process, described by the Einstein A coefficient. However, this is not a great difficulty, because the Einstein coefficients are related, and any one of them can be used to derive the other two. The physical argument used to justify the semi-classical approach is the following: in the classical limit of radiation is the number of photons per photon state large. Thus the induced processes, which are proportional to the number of photons. Because of the linearity of the induced processes in the number of photons, these processes may be extrapolated to small photon numbers, that is, to the quantum regime. The spontaneous rate can then be quantified by the Einstein relations.

#### 10.1.1 The Electromagnetic Hamiltonian

The relativistic kinetic energy for an uncharged particle of rest mass  $m_0$  is  $T = E - E_0$ , where  $E = mc^2$  is the total particle's energy and  $E_0 = m_0c^2$  is its rest energy. Following Eqs. (7.95) - (7.97),  $T = m_0c^2(\gamma - 1)$ , and we may naïvely regard this as the kinetic part of the relativistic Lagrangian L. However, it is not, because its velocity derivative does not give relativistic momentum  $\mathbf{p} = \gamma m_0 \mathbf{v}$ .

The correct Lagrangian we obtain by integrating the relativistic momentum equation

$$\boldsymbol{\rho} = \frac{\partial L}{\partial \boldsymbol{v}}, \quad \text{so that} \quad L = \int \gamma(\boldsymbol{v}) m_0 \boldsymbol{v} \, \mathrm{d} \boldsymbol{v} - V = -m_0 c^2 \gamma^{-1} - V, \tag{10.1}$$

where V is the potential energy and where the arbitrary constant of integration becomes the (calibrated) part of V.

For a particle of charge q with a rest mass  $m_0$  in an electromagnetic field represented by the potentials  $\phi$  and  $\mathbf{A}$ , the relativistic Lagrangian is given by

$$L = -m_0 c^2 \gamma^{-1} + q \mathbf{v} \cdot \mathbf{A} - q\phi, \qquad (10.2)$$

so that the particle's momentum

$$\boldsymbol{M} = \frac{\partial L}{\partial \boldsymbol{v}} = \boldsymbol{p} + q\boldsymbol{A}.$$
 (10.3)

To verify Eq. (10.2), we use the vector identity

$$\mathbf{v} \times (\mathbf{\nabla} \times \mathbf{A}) + \mathbf{A} \times (\mathbf{\nabla} \times \mathbf{v}) = \mathbf{\nabla} (\mathbf{v} \cdot \mathbf{A}) - (\mathbf{v} \cdot \mathbf{\nabla})\mathbf{A} - (\mathbf{A} \cdot \mathbf{\nabla})\mathbf{v}$$
(10.4)

(cf. Sect. 2.3 in (Kurfürst 2017)), noting that  $\mathbf{v}$  does not depend on  $\mathbf{r}$  so the last terms on both sides of Eq. (10.4) drop. The explicit Euler-Lagrange equation  $\partial L/\partial q = \frac{\mathrm{d}}{\mathrm{d}t}(\partial L/\partial \dot{q})$  becomes

$$\frac{\partial L}{\partial \boldsymbol{r}} = q \boldsymbol{\nabla} (\boldsymbol{v} \cdot \boldsymbol{A}) - q \boldsymbol{\nabla} \phi = q \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) + q (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{A} - q \boldsymbol{\nabla} \phi,$$
  
$$\frac{\mathrm{d}}{\mathrm{d}t} \left( \frac{\partial L}{\partial \boldsymbol{v}} \right) = \frac{\mathrm{d}}{\mathrm{d}t} (\boldsymbol{p} + q \boldsymbol{A}) = \frac{\mathrm{d} \boldsymbol{p}}{\mathrm{d}t} + q \left[ \frac{\partial \boldsymbol{A}}{\partial t} + (\boldsymbol{v} \cdot \boldsymbol{\nabla}) \boldsymbol{A} \right], \qquad (10.5)$$

yielding the Lorentz force (Eq. (5.1))

$$\frac{\mathrm{d}\boldsymbol{p}}{\mathrm{d}t} = q \left[ -\boldsymbol{\nabla}\phi - \frac{\partial\boldsymbol{A}}{\partial t} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) \right] = q \left( E + \boldsymbol{v} \times B \right).$$
(10.6)

The relativistic Hamiltonian for a particle in an external electromagnetic field is

$$H = \mathbf{M} \cdot \mathbf{v} - L = \mathbf{p} \cdot \mathbf{v} + m_0 c^2 \gamma^{-1} + q\phi = \left[ (\mathbf{M} - q\mathbf{A})^2 c^2 + m_0^2 c^4 \right]^{1/2} + q\phi,$$
(10.7)

where we used Eq. (10.3). Expanding Eq. (10.7) in the nonrelativistic limit,  $H = p^2/(2m_0) + V$ and ignoring the (constant) rest mass, we obtain

$$H = \frac{(\mathbf{M} - q\mathbf{A})^2}{2m_0} + q\phi = \frac{M^2}{2m_0} - \frac{q}{m_0}\mathbf{M} \cdot \mathbf{A} + \frac{q^2 A^2}{2m_0} + q\phi.$$
(10.8)

Using the Coulomb gauge (Eq. (5.32)) we have  $\boldsymbol{A} = \text{const.}$  and the operators  $\boldsymbol{M}$  and  $\boldsymbol{A}$  commute,

$$-i\hbar \nabla A\psi = -i\hbar A\nabla \psi$$
, so that  $M \cdot A = A \cdot M$ . (10.9)

Regarding the particle is an electron,  $q \equiv e, m_0 \equiv m_e$ , we may estimate the ratio of the two terms in Eq. (10.8) that contain A:

$$\eta = \frac{eMA/m_{\rm e}}{e^2 A^2/2m_{\rm e}} = \frac{2M}{eA} = \frac{ev/c^2}{2\pi\epsilon_0 \alpha^2 a_0 A},\tag{10.10}$$

where  $\alpha \equiv e^2/(4\pi\epsilon_0\hbar c) \approx 1/137$  is the fine-structure constant and  $a_0 \equiv 4\pi\epsilon_0\hbar^2/(m_e e^2) \approx 5.291 \times 10^{-11}$  m is the Bohr radius.

The ratio  $v/c \sim \alpha$  for electrons in atom, which we estimate from Bohr model of H-atom, where we compare  $m_e v^2/a_0 = e^2/(4\pi\epsilon_0 a_0^2)$ . Another ratio  $A/E \sim \lambda/c$ , where E is the electric field and  $\lambda$  is the wavelength, we approximate by comparing the relations  $\nabla \times \mathbf{A}$  and B = E/c. We rewrite Eq. (10.10) as

$$\eta^2 \sim \frac{\hbar\omega}{2\pi^2\epsilon_0 \alpha a_0^2 \lambda E^2}.$$
(10.11)

Since from Bohr model of H-atom, comparing  $m_{\rm e}v^2/2 = E \sim 2\pi n\hbar c/\lambda$ , it follows that  $\lambda \sim a_0/\alpha$ , while the average energy density (5.45) implies the number of photons  $n_{\rm ph} \sim \epsilon_0 E^2/\hbar\omega$ , we have

$$\eta^2 \sim (n_{\rm ph} a_0^3)^{-1} \gg 1$$
 (10.12)

as the condition that the linear term in A in Eq. (10.8) dominates the quadratic one. In other words, the number of photons inside the atom at one time is small. In fact, the term quadratic in A contributes to two-photon processes, which we ignore here under the assumption that the number of photons is sufficiently small. Note that the photon density at which this assumption fails is  $n_{\rm ph} \sim 10^{31} \,\mathrm{m}^{-3}$ , whereas at the sun's surface we have only  $n_{\rm ph} \sim 10^{18} \,\mathrm{m}^{-3}$ . Ordinarily, the neglect of the  $A^2$  term is justified. We now want to apply this to an atomic system of electrons. To do this we regard the sum of terms of the sort  $(-e/m_e c)\mathbf{M} \cdot \mathbf{A}$  as a perturbation to the atomic Hamiltonian, and we use time-dependent perturbation theory to calculate the transition probabilities between the atomic states (we continue to work in the Coulomb gauge  $\nabla \cdot \mathbf{A} = \nabla \phi = 0$ ).

#### 10.1.2 The Transition Probability

We split the Hamiltonian (Eq. (10.8)) into a stationary and a time-dependent piece:

$$H = H^0 + H^I, (10.13)$$

where  $H^0$  is the time-independent atomic Hamiltonian and  $H^I$  is the perturbation due to the external electromagnetic field. The atomic eigenvalues  $E_k$  and eigenfunctions  $\phi_k$  of  $H^0$  are given by

$$H^0\phi_k = E_k\phi_k \tag{10.14}$$

and the zeroth-order time dependent wave functions are  $\phi_k \exp(-iE_k t/\hbar)$ . Resuming the timedependent Schrödinger equation  $i\hbar \partial \psi/\partial t = H\psi$ , we may expand the actual wave function as

$$\psi(t) = \sum_{k} a_k(t)\phi_k \,\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}E_k t}.$$
(10.15)

The Schrödinger equation implies the probability per unit time  $\mathcal{P}_{fi}$  for a transition from state i to state f is given by

$$\mathcal{P}_{fi} = \frac{4\pi^2}{\hbar^2 T} |H_{fi}^I(\omega_{fi})|^2, \qquad (10.16)$$

where

$$H_{fi}^{I}(\omega) = \int_{0}^{T} H_{fi}^{I}(t') e^{-i\omega t'} dt', \quad H_{fi}^{I}(t) = \int \phi_{f}^{*} H^{I} \phi_{i} d^{3}x, \text{ and } \hbar \omega_{fi} = E_{f} - E_{i}.$$
(10.17)

### 10.2 Line Broadening

Since atomic levels are not infinitely sharp, nor are the lines. We introduced in Sect. 4.5 the line profile function  $\phi(\nu)$  to account for the nonzero width of the line. Many physical effects determine the line shape, we describe only a few here (see, e.g., Mihalas 1978).

#### 10.2.1 Doppler Broadening

The simplest mechanism for line broadening is the Doppler effect. An atom is in thermal motion, so that the frequency of emission or absorption in its own frame corresponds to a different frequency for an observer. Each atom has its own Doppler shift, so that the net

effect is to spread the line out, but not to change its total strength. The change in frequency associated with an atom with velocity component  $v_z$ , along the line of sight (say, z-axis) is, to lowest order in v/c, given by Eq. (7.40),

$$\frac{\nu - \nu_0}{\nu_0} = \frac{v_z}{c}, \quad \text{so that} \quad v_z = \frac{c(\nu - \nu_0)}{\nu_0} \quad \text{and} \quad \mathrm{d}v_z = \frac{c}{\nu_0} \,\mathrm{d}\nu. \tag{10.18}$$

where  $\nu_0$  is the frequency in the rest frame of an atom. The number of atoms having velocities in the range  $\nu$ , to  $\nu + d\nu$ , is proportional to the Maxwellian distribution exponential (see the principles in Kurfürst 2017, Sect. 12.2),

$$e^{-\frac{m_{a}v_{z}^{2}}{2kT}} dv_{z} \sim e^{-\frac{m_{a}c^{2}(\nu-\nu_{0})^{2}}{2\nu_{0}^{2}kT}} d\nu, \quad \text{so that} \quad 1 = C \int_{-\infty}^{\infty} e^{-\frac{m_{a}c^{2}(\nu-\nu_{0})^{2}}{2\nu_{0}^{2}kT}} d\nu, \tag{10.19}$$

where C is a constant which is normalized by the condition  $\int \phi(\nu) d\nu = 1$ , and where we integrate over the whole bracket in the exponential. The profile function is

$$\phi(\nu) = \frac{c}{\nu_0} \sqrt{\frac{m_{\rm a}}{2\pi kT}} \,\mathrm{e}^{-\frac{m_{\rm a}c^2(\nu-\nu_0)^2}{2\nu_0^2 kT}} = \frac{1}{\Delta\nu_D\sqrt{\pi}} \,\mathrm{e}^{-(\nu-\nu_0)^2/(\Delta\nu_D)^2},\tag{10.20}$$

where we define the (temperature dependent) Doppler width  $\Delta \nu_D$  by

$$\Delta \nu_D = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m_a}}, \quad \Delta \nu_D \ll \nu_0.$$
(10.21)

The center-of-line cross section for each atom, neglecting stimulated emission, is

$$\sigma_{\nu_0} = B_{12} \frac{h\nu_0}{4\pi} \phi(\nu_0) = \frac{1}{\Delta\nu_D \sqrt{\pi}} \frac{h\nu_0}{4\pi} B_{12} = \frac{1}{\Delta\nu_D \sqrt{\pi}} \frac{e^2}{4\epsilon_0 m_e c} f_{12}$$
(10.22)

in SI units, while in cgs units it is

$$\sigma_{\nu_0} = \frac{1}{\Delta \nu_D \sqrt{\pi}} \frac{\pi e^2}{m_e c} f_{12} \tag{10.23}$$

for the case of Doppler broadening (cf. Eqs. (6.74) and (6.77)). Numerically this is

$$\sigma_{\nu_0} \approx 1.16 \times 10^{-18} \lambda_0 \sqrt{A/T} f_{12} \,\mathrm{m}^2,$$
 (10.24)

where  $\lambda_0$  is in Å, T in K, and A is the atomic weight for the atom.

In addition to thermal motions, we include also macroscopic turbulent velocities. When the turbulent scale is small in comparison with a photon's mean free path (*microturbulence*) these motions are often accounted for by an effective Doppler width

$$\Delta \nu_D = \frac{\nu_0}{c} \left( \frac{2kT}{m_{\rm a}} + v_{\rm mic} \right)^{1/2}, \qquad (10.25)$$

where  $v_{\rm mic}$  is a root mean-square measure of the turbulent velocities. This assumes that the turbulent velocities also have a Gaussian distribution.

#### 10.2.2 Natural Broadening

A certain width of the atomic level is implied by the uncertainty principle,  $\Delta E \Delta t \ge \hbar$ . A rate of the spontaneous decay of an atomic state n is

$$\gamma = \sum_{n'} A_{nn'},\tag{10.26}$$

where we sum over all states n' of lower energy. If radiation is present, we should add the induced rates. The coefficient of the wave function of state n, therefore, is of the form  $e^{-\gamma t/2}$  and leads to a decay of the electric field by the same factor (the energy then decays proportional to  $e^{-\gamma t}$ ). Therefore, we have an emitted spectrum determined by the damped sinusoid (oscillator) type of electric field, as given in Sect. 3.2.1 in Kurfürst (2017). The profile is (cf. Eq. (6.64))

$$\phi(\nu) = \frac{\gamma/(4\pi^2)}{(\nu - \nu_0)^2 + [\gamma/(4\pi)]^2}.$$
(10.27)

This is called a *Lorentz* (or *natural*) profile. Actually, the above result applies to cases in which only the upper state is broadened (e.g., transitions to the ground state). If both the upper and lower state are broadened, then the appropriate definition for  $\gamma$  is

$$\gamma = \gamma_{up} + \gamma_{low}, \tag{10.28}$$

where  $\gamma_{up}$  and  $\gamma_{low}$  are the widths of the upper and lower states involved in the transition. We can have, for example, a weak but broad line if the lower state is broadened substantially.

#### 10.2.3 Collisional Broadening

The Lorentz profile applies more generally to certain types of collisional broadening mechanisms. For example, if the atom collides with other particles during its emission, the phase of the emitted radiation can be suddenly altered. If the phase changes completely randomly at the collision times, then information about the emitting frequencies is lost. If the collisions occur with frequency  $\nu_{col}$ , that is, each atom experiences  $\nu_{col}$  collisions per unit time on the average, then the profile is

$$\phi(\nu) = \frac{\Gamma/(4\pi^2)}{(\nu - \nu_0)^2 + [\Gamma/(4\pi)]^2},$$
(10.29)

where

$$\Gamma = \gamma + 2\nu_{\rm col}.\tag{10.30}$$

#### 10.2.4 Combined Doppler and Lorentz Broadening

An atom very often shows both a Lorentz profile plus the Doppler effect. In these case the profile is as an average of the Lorentz profile over the various velocity states of the atom:

$$\phi(\nu) = \frac{\Gamma}{4\pi^2} \sqrt{\frac{m_{\rm a}}{2\pi kT}} \int_{-\infty}^{\infty} \frac{{\rm e}^{-\frac{m_{\rm a}v_z^2}{2kT}}}{\left(\nu - \nu_0 - \frac{\nu_0 v_z}{c}\right)^2 + \left(\frac{\Gamma}{4\pi}\right)^2} \,{\rm d}v_z,\tag{10.31}$$

Substituting

$$a = \frac{\Gamma}{4\pi\Delta\nu_D}, \quad u = \frac{\nu - \nu_0}{\Delta\nu_D} \quad \text{and} \quad y = \sqrt{\frac{m_a}{2kT}}v_z,$$
 (10.32)

we can write Eq. (10.31) more compactly using the Voigt function H(a, u),

$$\phi(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}}H(a,u), \quad \text{where} \quad H(a,u) = \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{e^{-y^2}\,\mathrm{d}y}{a^2 + (u-y)^2}.$$
(10.33)

For small values of a, the center of the line is dominated by the Doppler profile, whereas the "wings" are dominated by the Lorentz profile.

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