Numerical modeling of hydrodynamic processes with sharp discontinuities and high Mach number

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Basic hydrodynamics

Basic (magneto)hydrodynamics in the conservative form:

- Continuity equation (mass conservation law)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{V} = 0$$

- Equation of motion (conservation of momentum, angular momentum)

$$\frac{\partial \rho \vec{V}}{\partial t} + \vec{\nabla} \cdot \rho \vec{V} \vec{V} = -\vec{\nabla} P - \rho \vec{\nabla} \Phi + \frac{1}{\mu} (\vec{\nabla} \times \vec{B}) \times \vec{B} \quad (+\vec{f}_{visc} \ldots .)$$

- Energy equation

$$\frac{\partial E}{\partial t} + \vec{\nabla} \cdot E \vec{V} = -\vec{\nabla} \cdot P \vec{V} \ldots ., \qquad E = \left(\rho \epsilon + \frac{\rho V^2}{2} + \frac{B^2}{2\mu} \right)$$

- Induction equation

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\vec{V} \times \vec{B} \right)$$

- Adiabatic and isothermal forms of the equation of state

$$P = (\gamma - 1) \left(E - \frac{\rho V^2}{2} - \frac{B^2}{2\mu} \right), \quad P = \rho a^2$$

Basic hydrodynamics

Determining equations for the viscous effects in astrophysical problems

- Integrated column (surface) density $\Sigma = \int_{-\infty}^{\infty} \rho \, \mathrm{d}z$
- Shear viscous stress

$$\sigma_{R\phi} \approx \eta \frac{\mathrm{d}V_\phi}{\mathrm{d}R} \approx \nu \Sigma \frac{\mathrm{d}V_\phi}{\mathrm{d}R} \approx \alpha a \lambda \Sigma \frac{\mathrm{d}V_\phi}{\mathrm{d}R}, \quad \eta = f \rho \lambda V_{\mathrm{turb}}$$

- Kinematic viscosity ν, viscosity parameter α (Shakura & Sunyaev, *1972*)

$$\nu = \eta/\rho \sim \lambda V_{\mathrm{turb}} \approx \alpha a \lambda, \quad \alpha = V_{\mathrm{turb}}/a,$$

Basic hydrodynamics

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- Cylindrical or spherical parameterization of temperature and viscosity:

$$T = T_\star \left(\frac{R_\star}{R}\right)^p, \quad \alpha = \alpha_\star \left(\frac{R_\star}{R}\right)^n,$$

- The full second order ϕ component of viscosity ($\partial/\partial\phi = 0$, $\partial/\partial z = 0$):

$$\sigma_{R\phi} = -\frac{1}{R^2} \frac{\partial}{\partial R} \left(\alpha a^2 R^3 \Sigma \frac{\partial \ln V_\phi}{\partial R} - \alpha a^2 R^2 \Sigma \right)$$

"Operator split" hd numerical code

- Numerical schema of 1D form of hd equations (Norman & Winkler *1982*)
- Fits well for relatively smooth hydrodynamic processes

Explicit numerical form:

$$\frac{d\rho}{dt} = 0$$

$$\frac{d\Pi}{dt} = -\frac{\partial p}{\partial R} - \rho\frac{\partial \Phi}{\partial R} - \frac{\partial Q}{\partial R} + \rho\frac{V_\phi^2}{R}$$

$$\frac{dJ}{dt} = G_{\rm visc}$$

Operator split procedure:

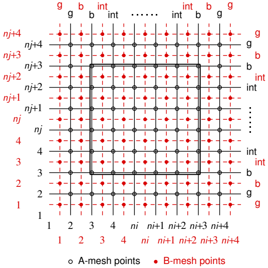
$$\begin{aligned}
(f^1 - f^0)/\Delta t &= L_1(f^0) \\
(f^2 - f^1)/\Delta t &= L_2(f^1) \\
&\vdots \\
(f^m - f^{m-1})/\Delta t &= L_m(f^{m-1})
\end{aligned}$$

m is number of source (RHS) terms

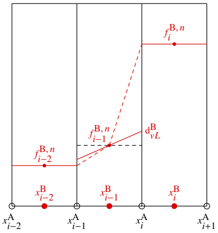
- $\Pi = \rho V_R$ is radial momentum density
- $J = \rho R V_\phi$ is angular momentum density
- $G_{\rm visc}$ is viscous torque
- $Q_i = \rho_i(V_{R\,i+1} - V_{R\,i})[-C_1 a + C_2 \min(V_{R\,i+1} - V_{R\,i}, 0)]$ is the artificial viscosity

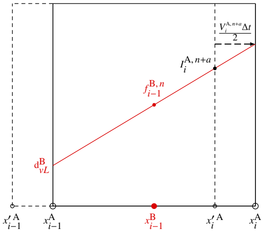
“Operator split” hd numerical code

Schematic chart of the “staggered mesh”



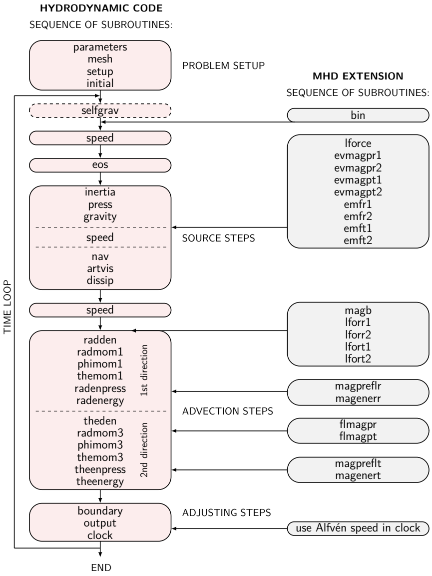
Principles of the van Leer solver:





“Operator split” hd numerical code

Schematic chart of the code



“Operator split” hd numerical code

Example of test problem

- density and pressure field of nonmagnetized gas:
  $\rho_{\text{left}} \sim 1.0\,\mathrm{e}^{-y^2}$, $\rho_{\text{right}} \sim 0.125\,\mathrm{e}^{-y^2}$, $P_{\text{left}} \sim 1.0\,\mathrm{e}^{-y^2}$, $P_{\text{right}} \sim 0.1\,\mathrm{e}^{-y^2}$

Video otevřete kliknutím na následující odkaz:

Riemann-Sod_shock_tube_density.mp4

- density and pressure field of magnetized gas:
  $\rho_{\text{left}} = 1.0$, $\rho_{\text{right}} = 0.125$, $P_{\text{left}} =\sim 1.0$, $P_{\text{right}} = 0.1$
- internal magnetic field: $B_x = 0,75\,\mathrm{G}$, $B_y(\text{left}) = 1.5\,\mathrm{G}$, $B_y(\text{right}) = -1.5\mathrm{G}$

Video otevřete kliknutím na následující odkaz:

Riemann-Sod_shock_tube_density_magnetized.mp4

"Operator split" hd numerical code

Example of test problem

- 2D Riemann-Sod shock tube ("solid walls" boundaries)
- density contours coloured map
- nonzero initial velocity in y direction

Video otevřete kliknutím na následující odkaz:

Riemann-Sod_shock_tube_density_color.mp4

1-D hydrodynamic modeling of circumstellar viscous disks

- Time-dependent 1-D hydrodynamic calculations using own MHD code (Kurfürst, Feldmeier & Krtička *2014*)
- In the models we recognize the wave that converges the initial state to the final stationary state

Left panel: disk of classical Be (B0-type) star (Harmanec 1988), M=14.5 $M_\odot$, R=5.8 $R_\odot$, $T_\mathrm{eff} = 30\,\mathrm{kK}$

To open the video - click on the following link:

Be_evolution.mp4

Right panel: disk of popIII star (Marigo+ 2001), M=50 $M_\odot$, R=30 $R_\odot$, $T_\mathrm{eff} = 30\,\mathrm{kK}$

To open the video - click on the following link:

B[e]_evolution.mp4

- In supersonic region - a shock wave with propagation speed $D = a\sqrt{\Sigma_1/\Sigma_0}$
- The shock propagation time $t_\mathrm{dyn} \approx R/D = 0.3R/a$ - the disk evolution time
- Corresponding disk viscous time $t_\mathrm{visc} = \int_{R_\mathrm{eq}}^{R} V_\phi\,\mathrm{d}R/(\alpha a^2)$

2-D hydrodynamic modeling of circumstellar viscous disks

- Time-dependent 2-D calculations

- 2-D calculation of disk density structure up to 100 stellar radii
- conical computational grid (R - z plane)
- vertical hydrostatic equilibrium

To open the video - click on the following link:

disk2Ddensity.mp4

Processes with large discontinuities - “unsplit” method

- The principles of Roe’s method
- The vectors of primitive and conservative (adiabatic) hydrodynamic variables, $\boldsymbol{W}$ and $\boldsymbol{U}$:

$$\boldsymbol{W} = \begin{pmatrix} \rho \\ v_x \\ v_y \\ v_z \\ P \end{pmatrix}, \quad \boldsymbol{U} = \begin{pmatrix} \rho \\ M_x \\ M_y \\ M_z \\ E \end{pmatrix},$$

- The 1D compact Cartesian form of conservation laws

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}}{\partial x} = 0, \tag{1}$$

- $\boldsymbol{F}$ is the vector of fluxes whose components are

$$\boldsymbol{F} = \begin{bmatrix} \rho v_x \\ \rho v_x^2 + P \\ \rho v_x v_y \\ \rho v_x v_z \\ (E+P) v_x \end{bmatrix}.$$

Processes with large discontinuities - "unsplit" method

- The principles of Roe's method
- The linearised form of conservation law can be written as

$$\frac{\partial \boldsymbol{q}}{\partial t} + \boldsymbol{A}(\boldsymbol{q})\frac{\partial \boldsymbol{q}}{\partial x} = 0, \qquad (2)$$

- $\boldsymbol{A}(\boldsymbol{q})$ is the Jacobian matrix $\partial \boldsymbol{f}/\partial \boldsymbol{q}$
- The system of hd equations in primitive variables $\boldsymbol{W} = (w_1, \ldots, w_5)$ is

$$\frac{\partial \rho}{\partial t} + v_x\frac{\partial \rho}{\partial x} + \rho\frac{\partial v_x}{\partial x} = 0,$$

$$\frac{\partial v_x}{\partial t} + v_x\frac{\partial v_x}{\partial x} + \frac{1}{\rho}\frac{\partial P}{\partial x} = 0,$$

$$\frac{\partial P}{\partial t} + \gamma P\frac{\partial v_x}{\partial x} + v_x\frac{\partial P}{\partial x} = 0.$$

- The explicit Jacobian matrix $\boldsymbol{A}$ in primitive variables is

$$\boldsymbol{A}_{\rm prim} = \begin{bmatrix} v_x & \rho & 0 & 0 & 0 \\ 0 & v_x & 0 & 0 & 1/\rho \\ 0 & 0 & v_x & 0 & 0 \\ 0 & 0 & 0 & v_x & 0 \\ 0 & \rho a^2 & 0 & 0 & v_x \end{bmatrix}, \ \lambda = (v_x - a, v_x, v_x, v_x, v_x + a).$$

Processes with large discontinuities - “unsplit” method

- The principles of Roe’s method
- The Roe fluxes are

$$f^{\text{Roe}}_{i-1/2} = \frac{1}{2}\left(f_{L,i-1/2} + f_{R,i-1/2} + \sum_{\alpha} \boldsymbol{L}^{\alpha} \cdot \left(q_{L,i-1/2} - q_{R,i-1/2}\right) |\lambda^{\alpha}| \boldsymbol{R}^{\alpha}\right),$$

- The corresponding left and right eigenvectors are the columns and rows of the matrices

$$\boldsymbol{L}_{\text{prim}} = \begin{bmatrix} 0 & -\rho/(2a) & 0 & 0 & 1/(2a^2) \\ 1 & 0 & 0 & 0 & -1/a^2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \rho/(2a) & 0 & 0 & 1/(2a^2) \end{bmatrix}, \; \boldsymbol{R}_{\text{prim}} = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 \\ -a/\rho & 0 & 0 & 0 & a/\rho \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ a^2 & 0 & 0 & 0 & a^2 \end{bmatrix}.$$

Processes with large discontinuities - "unsplit" method

- The principles of Roe's method
- The matrix $\boldsymbol{F}$ in conservative variables $\boldsymbol{U} = (u_1, \ldots, u_5)$ is

$$\boldsymbol{F}(\boldsymbol{u}) = \begin{pmatrix} u_2 \\ \dfrac{u_2^2}{u_1} + (\gamma - 1)\left[u_5 - \dfrac{u_2^2 + u_3^2 + u_4^2}{2u_1}\right] \\ u_2 u_3 / u_1 \\ u_2 u_4 / u_1 \\ \left[\gamma u_5 - (\gamma - 1)\dfrac{u_2^2 + u_3^2 + u_4^2}{2u_1}\right] \dfrac{u_2}{u_1} \end{pmatrix}.$$

- The Jacobian matrix $\boldsymbol{A} = \partial \boldsymbol{F}(\boldsymbol{u}) / \partial \boldsymbol{u}$, $\boldsymbol{U} = (\rho, M_x, M_y, M_z, E)$, is

$$\boldsymbol{A}_{\text{cons}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -v_x^2 + \gamma_1 \dfrac{v^2}{2} & -\gamma_3 v_x & -\gamma_1 v_y & -\gamma_1 v_z & \gamma_1 \\ -v_x v_y & v_y & v_x & 0 & 0 \\ -v_x v_z & v_z & 0 & v_x & 0 \\ -v_x H + \gamma_1 \dfrac{v_x v^2}{2} & -\gamma_1 v_x^2 + H & -\gamma_1 v_x v_y & -\gamma_1 v_x v_z & \gamma v_x \end{bmatrix},$$

where $\gamma_1 = (\gamma - 1)$, $\gamma_3 = (\gamma - 3)$ and H is the enthalpy, $H = (E + P)/\rho$.

Processes with large discontinuities - "unsplit" method

- The principles of Roe's method
- The eigenvalues of the matrix $\boldsymbol{A}_{\rm cons}$ are $\lambda = (v_x - a, v_x, v_x, v_x, v_x + a)$, where $a^2 = (\gamma - 1)(H - v^2/2) = \gamma P/\rho$ is the adiabatic speed of sound.
- The corresponding left and right eigenvectors are the columns and rows of the matrices

$$\boldsymbol{L}_{\rm cons} = \begin{bmatrix} \gamma_1 v_+^2/(2a^2) & -\gamma_1 v_+/(2a^2) & -\gamma_1 v_y/(2a^2) & -\gamma_1 v_z/(2a^2) & \gamma_1(2a^2) \\ -v_y & 0 & 1 & 0 & 0 \\ -v_z & 0 & 0 & 1 & 0 \\ 1_\gamma^- & \gamma_1 v_x/a^2 & \gamma_1 v_y/a^2 & \gamma_1 v_z/a^2 & -\gamma_1/a^2 \\ \gamma_1 v_-^2/(2a^2) & -\gamma_1 v_-/(2a^2) & -\gamma_1 v_y/(2a^2) & -\gamma_1 v_z/(2a^2) & \gamma_1/(2a^2) \end{bmatrix},$$

$$\boldsymbol{R}_{\rm cons} = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ v_x - a & 0 & 0 & v_x & v_x + a \\ v_y & 1 & 0 & v_y & v_y \\ v_z & 0 & 1 & v_z & v_z \\ H - v_x a & v_y & v_z & v^2/2 & H + v_x a \end{bmatrix},$$

where $\gamma_1 v_+^2 = (\gamma_1 v^2/2 + v_x a)$, $\gamma_1 v_-^2 = (\gamma_1 v^2/2 - v_x a)$, $\gamma_1 v_+ = (\gamma_1 v_x + a)$, $\gamma_1 v_- = (\gamma_1 v_x - a)$, and $1_\gamma^- = [1 - \gamma_1 v^2/(2a^2)]$.

Processes with large discontinuities - “unsplit” method

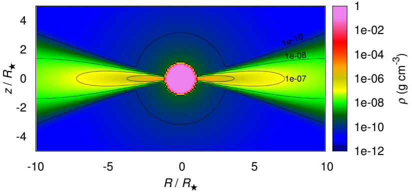
- Analytical 1-D expansion of SN ejecta into circumstellar medium
- SN progenitor: sgB[e] star, M=40 $M_{\odot}$, R=75 $R_{\odot}$

Video otevřete kliknutím na následující odkaz:

SN_CSM_interaction_semianalytical_solution.mp4

Processes with large discontinuities - “unsplit” method

- Time-dependent 2-D calculation of adiabatic interaction between SN ejecta and circumstellar disk, $\dot{M}_{\mathrm{disk}} = 10^{-5}\, M_{\odot}\, \mathrm{yr}^{-1}$ (Kraus+ 2007, Kurfürst & Krtička in prep.)
- SN progenitor: sgB[e] star, M=40 $M_{\odot}$, R=75 $R_{\odot}$, time of simulation: 50 hrs



initial density profile

Video otevřete kliknutím na následující odkaz:

SN_CSM_interaction_density.mp4

Video otevřete kliknutím na následující odkaz:

SN_CSM_interaction_velocity.mp4

Video otevřete kliknutím na následující odkaz:

SN_CSM_interaction_temperature.mp4

Conclusions

- We have developed two basic types of our own hydrodynamic (MHD) codes based on different principles
- Each of them fits to different nature of hydrodynamic processes, operator split code fits to rather smooth hydrodynamics, while the unsplit code well calculates the processes with sharp discontinuities
- Most of the astrophysical processes are successfully calculated on the both types of the codes, it provides a comparison of results
- Both the codes use the full Navier-Stokes viscosity terms to model the realistic viscous and dissipative effects
- Both the types of the codes are written in all basic orthogonal geometries, i.e., Cartesian, cylindrical, spherical as well as in the special "conical" non-orthogonal geometry, which best fits to calculations of accretion/decretion astrophysical disks

Thank you for your attention