Solved example 9.5 f) :

$$\lim_{x\to 0} \frac{\ln(1+x\operatorname{arctg} x)+1-\mathrm{e}^{x^2}}{\sqrt{1+2x^4}-1}$$

1) For example, we can "pre-calculate" expansion of the logarithmic function, where we substitute

$$x\operatorname{arctg} x = u.$$

If nothing prevents the development of the function at point 0 (here it is also given by the limit), it is the simplest, sometimes, for example with the polynomial function $1/x$ it is not possible, then it is easiest to expand it at point 1. The expansion of the logarithmic function at the point 0 then it will be

$$\ln(1+u) = 0 + \left.\frac{\partial\left[\ln(1+u)\right]}{\partial u}\right|_{u=0} u + \frac{1}{2!}\left.\frac{\partial^2\left[\ln(1+u)\right]}{\partial u^2}\right|_{u=0} u^2 + \ldots =$$
$$= u - \frac{u^2}{2} + \ldots$$

2) We will expand the function $\operatorname{arctg} x$ separately, again at the point 0:

$$\operatorname{arctg} x = 0 + \left.\frac{1}{1+x^2}\right|_{x=0} x - \frac{1}{2!}\left.\frac{2x}{(1+x^2)^2}\right|_{x=0} x^2 - \frac{1}{3!}\left.\left[\frac{2}{(1+x^2)^2} - \frac{8x^2}{(1+x^2)^3}\right]\right|_{x=0} x^3 + \ldots$$
$$= x - \frac{x^3}{3} + \ldots,$$

after substituting for u (we must not forget the expanded function $\operatorname{arctg} x$ yet multiply by x) we get

$$\ln(1+x\operatorname{arctg} x) = x^2 - \frac{x^4}{3} + \ldots - \frac{1}{2}\left(x^2 - \frac{x^4}{3} + \ldots\right)^2 + \ldots = x^2 - \frac{x^4}{3} - \frac{x^4}{2} + \ldots = x^2 - \frac{5x^4}{6} + \ldots$$

3) We can handle the other functions in a similar way:

$$1-\mathrm{e}^{x^2} \quad (\text{kde } x^2 = z) \quad = 1 - \left(1 + z + \frac{z^2}{2} + \ldots\right) = -x^2 - \frac{x^4}{2} - \ldots,$$
$$\sqrt{1+2x^4}-1 \quad (\text{kde } 2x^4 = t) \quad = \left(1 + \left.\frac{1}{2\sqrt{1+t}}\right|_{t=0} t - \frac{1}{2!}\left.\frac{1}{4(1+t)^{3/2}}\right|_{t=0} t^2 + \ldots\right) - 1 =$$
$$= x^4 - \frac{x^8}{2} + \ldots$$

4) We substitute all the partial expansions into the given expression:

$$\lim_{x\to 0} \frac{\ln(1+x\operatorname{arctg} x)+1-\mathrm{e}^{x^2}}{\sqrt{1+2x^4}-1} = \lim_{x\to 0} \frac{x^2 - \frac{5x^4}{6} + \ldots - x^2 - \frac{x^4}{2} + \ldots}{x^4 - \ldots} =$$

(where all unspecified expressions, symbolized by dots, contain higher powers of x than the fourth, so after rearrangement we get)

$$= \lim_{x\to 0} \frac{x^4\left[\left(-\frac{5}{6} - \frac{1}{2}\right) + \ldots\right]}{x^4(1-\ldots)} = \frac{\left.-\frac{4}{3} + \ldots\right|_{x=0}}{\left.1-\ldots\right|_{x=0}} = -\frac{4}{3}.$$