Inequalities for quantum marginal problems with continuous variables

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We consider a mixed continuous-variable bosonic quantum system and present inequalities which must be satisfied between principal values of the covariances of a complete set of observables of the whole system and the principal values of the covariances of a complete set of observables of a subsystem. We use several classical results for the proof: the Courant-Fischer-Weyl min-max theorem for Hermitian operators and its consequence, the Cauchy interlacing theorem, and prove their analogues in the symplectic setting. For the case of passive transformations of Gaussian mixed states we also prove that the obtained inequalities are, in a sense, the best possible. The obtained mathematical results are applied to the system of \( n \) uncorrelated thermal modes of the electromagnetic field. Finally, we present the results of numerical simulations of the problem, suggesting avenues of further research.

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I. INTRODUCTION

Consider a quantum system that can be divided into two or more subsystems. The system as well as the subsystem can be characterized by some quantities, for instance, the eigenvalues of their density matrices. The quantum marginal problem then deals with the following question: what values of these quantities for the subsystems are consistent with given values for the whole system?

For finite-dimensional Hilbert spaces, the quantum marginal problem was solved by Klyachko1 who showed that the conditions for consistency can be formulated in terms of inequalities for the eigenvalues of density matrices of the system and subsystems. For continuous-variable quantum systems, the quantum marginal problem has been considered for Gaussian states, which play an important role, e.g., in quantum-optical realizations of quantum information protocols. The corresponding Gaussian quantum marginal problem was solved by Eisert et al.2 for a particular situation of a multimode Gaussian state that was divided into single-mode substates. Similarly as for finite-dimensional Hilbert spaces, the consistency conditions were formulated in terms of a set of inequalities for the local and global characteristics, which in this case were the so-called symplectic eigenvalues3 rather than density matrix eigenvalues.

Here we consider another case of the Gaussian quantum marginal problem. Instead of dividing an \( n \)-mode Gaussian system \( A \) into single modes, we divide it into two subsystems \( B \) and \( C \), one consisting of \( n-k \) modes and the other one of \( k \) modes, \( k < n \). We then compare the local symplectic eigenvalues of subsystem \( B \) with the global symplectic eigenvalues for the whole system \( A \), ignoring subsystem \( C \). We will show that the necessary conditions for the symplectic eigenvalues of the system \( A \) and the subsystem \( B \) can be formulated in terms of inequalities for them. In addition, these inequalities have a simpler form than those for the marginal problem analyzed in Ref. 2. Our
results represent a continuous-variable analog of the results found previously for finite-dimensional systems.4

The paper is organized as follows. In Sec. II we make a brief introduction into Gaussian states and symplectic eigenvalues, in Sec. III we state and prove our main result, in Sec. IV we give an application to the modes of the electromagnetic field, in Sec. V we present results of numerical simulations of the problem, and conclude in Sec. VI.

II. GAUSSIAN STATES AND SYMPLECTIC EIGENVALUES

Consider a bosonic quantum system with \( n \) degrees of freedom. By the Stone-von Neumann theorem,\(^5\) it is fully characterized by any \( 2n \)-tuple of Hermitian operators on a Hilbert space \( \mathcal{H} \)

\[
(\hat{x}_1, \ldots, \hat{x}_n, \hat{p}_1, \ldots, \hat{p}_n)
\]

fulfilling the canonical commutation relations \([\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0, [\hat{x}_i, \hat{p}_j] = i\delta_{ij}\). The operators \( \hat{x}_i, \hat{p}_i \) are called canonical positions and momenta; in the case of an optical realization of a continuous-variable quantum system, they are the quadratures of the electromagnetic field. The pair \((\hat{x}_i, \hat{p}_j)\) then corresponds to the \( i \)th mode of the field.

All real linear combinations of such operators form a real \( 2n \)-dimensional vector space \( V \) of generalized quadratures. This vector space is equipped with a nondegenerate antisymmetric bilinear form \( \omega \) given by the bilinear extension of the canonical commutation relations by

\[
\omega(v_1\hat{x}_1 + \cdots + v_2n\hat{p}_n, w_1\hat{x}_1 + \cdots + w_2n\hat{p}_n) = v_1w_{n+1} - w_1v_{n+1} + \cdots + v_nw_{2n} - w_nv_{2n}.
\]

The form \( \omega \) expresses, apart from the factor \( i \), the commutator of generalized quadratures, and provides a symplectic structure on \( V \).

Let \( \hat{v} = v_1\hat{x}_1 + \cdots + v_2n\hat{p}_n \). With the help of Weyl operators \( \exp(i\hat{v}) \), we can define the characteristic function

\[
\chi(\hat{v}) = \text{Tr}[\hat{\rho} \exp(i\hat{v})],
\]

which, when known on the whole vector space \( V \), completely characterizes the state \( \hat{\rho} \) (see, e.g., Refs. 3 and 6 and the references therein).

A Gaussian state is a state for which \( \chi(\hat{v}) \) is a Gaussian function of \( v_1, \ldots, v_{2n} \). Equivalently, we can say it is a state for which the logarithm of \( \chi(\hat{v}) \) is at most quadratic. Interestingly, by the theorem of Marcinkiewicz7 Gaussian states can simply be characterized by the condition that the logarithm of the characteristic function is a finite degree polynomial.

For Gaussian states it thus holds

\[
\ln \chi(\hat{v}) = i\alpha(\hat{v}) - \frac{1}{2}\kappa(\hat{v}, \hat{v}),
\]

where \( \alpha \in V^* \) is a linear form on \( V \) and \( \kappa \) is a positive definite symmetric bilinear form on \( V \) called covariance. For non-Gaussian states the expression (1) constitutes the first two terms of an infinite power series, provided the expansion exists at all, i.e., if the radius of convergence of \( \ln \chi \) is nonzero. The first moments \( \alpha \) can always be made zero locally, and are hence not interesting for our purposes here.

For a given basis of \( V \), the bilinear forms \( \alpha \) and \( \kappa \) can be expressed by their corresponding matrices. By a suitable choice of basis, it is possible to put the matrices of both \( \alpha \) and \( \kappa \) into an especially simple form that we will denote by \( J \) and \( K \), respectively. The matrices \( J \) and \( K \) have the block form

\[
J = \begin{pmatrix} 0_n & 1_n \\ -1_n & 0_n \end{pmatrix}, \quad K = \begin{pmatrix} \text{diag} \mu & 0_n \\ 0_n & \text{diag} \mu \end{pmatrix},
\]

where \( 0_n \) and \( 1_n \) denote the \( n \times n \) zero and unit matrix, respectively. The values \( \mu = (\mu_1, \ldots, \mu_n) \) characterize the Gaussian state completely and are called symplectic eigenvalues. In the following, we shall always assume \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \).
We will also make use of a basis in \( V \) derived from the previous one by changing the order of the basis vectors via the permutation
\[
\begin{pmatrix}
1 & 2 & \ldots & n & n+1 & n+2 & \ldots & 2n \\
1 & n+1 & 2 & n+2 & \ldots & \ldots & n & 2n
\end{pmatrix},
\]
which permutes the entries of the matrices representing the forms \( \omega, \kappa \) accordingly. Another important bilinear form is the covariance \( o \) of the \( n \)-mode vacuum state represented by \( 2n \times 2n \) unit matrix \( 1_{2n} \).

Changing the basis in the vector space \( V \) via a transformation represented by the matrix \( A \) changes the matrices representing the forms \( \omega, \kappa \), and \( o \) according to the rule \( J = AA^t \) etc. with \( ^t \) denoting transposition. Among these transformations there are such that preserve the canonical form of the matrix of the form \( \omega \), i.e., \( J = AA^t \); they are called symplectic and the corresponding transformation matrix \( A \) belongs to the symplectic group \( \text{Sp}(2n, \mathbb{R}) \). Transformations which, in addition, preserve the form \( o \) representing the vacuum state, i.e., \( 1_{2n} = AA^t \), are called passive and their matrices are at the same time symplectic and orthogonal. The term passive stems from their possible realization in terms of passive optical elements (beam splitters and phase shifters) for quantum states of light. Symplectic transformations which are not orthogonal are called active as they need active optical elements (squeezers) for their optical realization.

### III. MAIN RESULTS AND THEIR PROOFS

Consider the situation when we have a mixed state characterized by the global symplectic eigenvalues \( \mu = (\mu_1, \ldots, \mu_n) \) and a corresponding state of a \((n-k)\)-mode subsystem characterized by the symplectic eigenvalues \( \nu = (\nu_1, \ldots, \nu_{n-k}) \), \( k \in \{1, \ldots, n-1\} \). We are interested in the relations which have to be satisfied among the symplectic eigenvalues \( \mu \) and \( \nu \).

First, we shall formulate the symplectic eigenvalue problem as an eigenvalue problem for a Hermitian operator. Then we use the Courant-Fischer-Weyl min-max theorem, e.g., and its consequences such as the Cauchy interlacing theorem. Let us define the complement of a vector subspace \( W \) with respect to a bilinear form \( b \) as \( W^b = \{ \hat{u} \in V | b(\hat{u}, \hat{w}) = 0 \text{ for all } \hat{w} \in W \} \). A subspace \( W \subset V \) is symplectic if the restriction of the form \( \omega \) onto \( W \) is nondegenerate.

We are focused here on Gaussian states of a bosonic system which are uniquely determined by a positive symmetric covariance form \( \kappa(u, v) = \kappa(v, u) \) in \( V \). It can be transformed into an operator \( F \) such that \( \kappa(u, v) = \omega(Fu, v) = \omega(Fv, u) \). Writing the last equation in the form \( \omega(Fu, v) + \omega(u, Fv) = 0 \) we infer that \( F \) belongs to Lie algebra \( \text{sp}(V) \) of the (real) symplectic group \( \text{Sp}(V) \). In addition, \( F \) is skew-symmetric with respect to the covariance form \( \kappa(u, Fv) = \omega(Fu, Fv) = -\kappa(Fu, V) \) and hence its spectrum is purely imaginary and symmetric with respect to the real axis. Positive eigenvalues \( 0 < \nu_1 \leq \nu_2 \leq \ldots \leq \nu_n \) of \( iF \) are called symplectic spectrum of the Gaussian state.

**Theorem 1.** Let \( V \) be a vector space, \( \dim V = 2n \), \( \omega \) a symplectic form on \( V \), \( \kappa \) positive definite bilinear form on \( V \), and \( \mu_1 \leq \mu_2 \leq \ldots \leq \mu_n \) the symplectic eigenvalues of \( \kappa \). Let \( W \subset V \) be a symplectic subspace, \( \dim W = 2n - 2 \), and \( \nu_1 \leq \nu_2 \leq \ldots \leq \nu_{n-1} \) the symplectic eigenvalues obtained by restriction of \( \omega \) and \( \kappa \) onto \( W \). Then
\[
\mu_1 \leq \nu_1 \leq \mu_3, \quad \mu_2 \leq \nu_2 \leq \mu_4, \quad \ldots, \quad \mu_{n-2} \leq \nu_{n-2} \leq \mu_{n}, \quad \mu_{n-1} \leq \nu_{n-1}.
\] (3)

**Proof.** Consider the Hermitian operator \( iF : V \to V \) acting in complexification \( V \otimes \mathbb{C} \) with metric given by Hermitian extension of the covariance form \( \kappa(u, v) \). Its eigenvalues occur in pairs \( (\mu_j, -\mu_j) \), the positive member being the corresponding symplectic eigenvalue. The eigenvalues of \( iF \) (ordered non-decreasingly) are thus
\[
-\mu_n \leq -\mu_{n-1} \leq \cdots \leq -\mu_1 \leq \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n.
\]
Now we use the Cauchy interlacing theorem (Ref. 9, Corollary III.1.5) for the subspace \( W \) and obtain inequalities (3). Note that there is no need of defining a complex structure in Theorem 1.

**Theorem 2.** Let \( V \) be a vector space, \( \dim V = 2n, \omega \) a symplectic form on \( V \), \( \kappa \) and \( \sigma \) positive definite bilinear forms on \( V \), such that \( \iota = \omega^{-1} \sigma \) is a complex structure, \( \mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \) the symplectic eigenvalues of \( \kappa \). If \( W^\sigma = W^\omega \), then

\[
\mu_1 \leq v_1 \leq \mu_2 \leq v_2 \leq \cdots \leq \mu_{n-1} \leq v_{n-1} \leq \mu_n.
\]

Conversely, let \( v_1 \leq v_2 \leq \cdots \leq v_n \) be given such that

\[
\mu_1 \leq v_1 \leq \mu_2 \leq v_2 \leq \cdots \leq \mu_{n-1} \leq v_{n-1} \leq \mu_n.
\]

Then there exists a subspace \( W \subset V \) such that \( \dim W = 2n - 2 \) and \( W^\sigma = W^\omega \).

**Proof.** To prove the first claim, we define a complex structure \( \iota = \omega^{-1} \sigma \) on \( V \) and we consider \( V \) as a vector space of dimension \( n \) over the field of complex numbers. The real bilinear form \( \kappa \) is obtained as a realification of a complex Hermitian form \( \ell \) with principal values \( \mu \). The subspace \( W \) in this case is obtained as a \( (n-1) \)-dimensional complex subspace of the complex vector space \( V \).

We then again use theorem (Ref. 9, Corollary III.1.5) for the \( (n-1) \)-dimensional complex subspace \( W \) and we immediately obtain (5).

As for the second claim, in the basis which is orthonormal with respect to \( \kappa \) and adapted to \( \omega^{-1} \) on the subspaces \( W \) and \( W^\sigma \) (see the permuted matrices in (2) applied to \( \omega^{-1} \))

\[
iF|_W = \begin{pmatrix}
0_{n-1} & (\text{diag } v)^{-1} \\
-(\text{diag } v)^{-1} & 0_{n-1}
\end{pmatrix},
iF|_{W^\sigma} = \begin{pmatrix}
0 & \gamma^{-1} \\
-\gamma^{-1} & 0
\end{pmatrix}
\]

our assertion reduces to the claim that the zeros of the characteristic polynomial of \( iF \)

\[
| -\lambda & \gamma^{-1} & a_1 & b_1 & \cdots & a_{n-1} & b_{n-1} \\
-\gamma^{-1} & -\lambda & c_1 & d_1 & \cdots & c_{n-1} & d_{n-1} \\
-a_1 & -c_1 & -\lambda & v_1^{-1} & \cdots & 0 \\
-b_1 & -d_1 & v_1^{-1} & -\lambda & \cdots & \cdots \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
-a_{n-1} & -c_{n-1} & -\lambda & v_{n-1}^{-1} \\
-b_{n-1} & -d_{n-1} & 0 & v_{n-1}^{-1} & -\lambda
\]

are \( \pm i \mu_1^{-1}, \ldots, \pm i \mu_n^{-1} \) for some choice of the real numbers \( \gamma^{-1} \) and \( a_i, b_j, c_j, d_j, j \in \{1, \ldots, n-1\} \). We again take note that \( \iota = \omega^{-1} \sigma \) is a complex structure and we thus may consider \( (n-1) \)-dimensional complex subspaces in \( V \). This is equivalent to the special choice \( a_i = d_i \) and \( b_i = -c_i \) for \( i \in \{1, \ldots, n-1\} \). We observe that in this case the characteristic polynomial of the \( 2n \times 2n \) real matrix is equal to the characteristic polynomial of the \( n \times n \) complex matrix

\[
| -\lambda + i\gamma^{-1} & z_1 & \cdots & z_{n-1} & \bar{z}_1 \\
-\bar{z}_1 & -\lambda & i\gamma^{-1} & 0 \\
\vdots & \vdots & \ddots & \ddots \\
-\bar{z}_{n-1} & 0 & -\lambda & i\gamma^{-1}_{n-1}
\]

multiplied by its complex conjugate polynomial. Thus, we have again reduced the problem to the known Hermitian case exposed in Ref. 10 (or Ref. 9, Theorem III.1.9.), where the problem is formulated as finding a positive solution of a certain system of linear equations (or considering
eigenvalues of exterior powers of the operator $iF$). Finally, clearly
\[ \mu_1^{-1} \geq v_1^{-1} \geq \mu_2^{-1} \geq \cdots \geq \mu_{n-1}^{-1} \geq v_{n-1}^{-1} \geq \mu_n^{-1} \]
holds if and only if
\[ \mu_1 \leq v_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq v_{n-1} \leq \mu_n. \]

The general case of inequalities which arises by restricting to a $(2n - 2k)$-dimensional symplectic subspace $W$ is easily obtained by induction from Theorem 1 and reads
\[ \mu_j \leq v_j \leq \mu_{j+2k}, \quad j \in \{1, \ldots, n - 2k\}, \quad \mu_j \leq v_j, \quad j \in \{n - 2k + 1, \ldots, n - k\} \]
for the general case and
\[ \mu_j \leq v_j \leq \mu_{j+k}, \quad j \in \{1, \ldots, n - k\}, \quad \mu_j \leq v_j, \quad j \in \{n - k + 1, \ldots, n - 1\} \]
for the case $W^\omega = W^\omega$. The sufficiency proof for the generalization of (5) also holds by application of Theorem 2 and the reduction of the dimension of the problem in case any $\mu_j = v_k$.

In this section we have shown that the inequalities (3) constitute necessary conditions for symplectic eigenvalues $\mu, \nu$ of the whole $n$-mode system and its $(n-1)$-mode subsystem. We have also shown that the stronger inequalities (5) represent sufficient conditions for the symplectic eigenvalues. We conjecture that also the weaker inequalities (3) are sufficient, i.e., they represent both necessary and sufficient conditions for the Gaussian marginal problem as we have formulated it. Although we have not been able to prove this claim, we present evidence for it based on numerical simulations in Sec. V.

IV. INTERPRETATION OF THE INEQUALITIES

In Sec. III, we have performed a rather abstract derivation of the inequalities for the symplectic eigenvalues. To give them a physical interpretation, we consider the realization of the quantum system in terms of electromagnetic field.

The density operator of an $n$-mode state in which each mode is thermal and the modes are uncorrelated can be expressed as $\hat{\rho} = \bigotimes_{k=1}^n \hat{\rho}_k$, where $\hat{\rho}_k$ is a thermal state of the $k$th mode,
\[ \hat{\rho}_k = \frac{1}{N_k + 1} \sum_{\ell = 0}^\infty \left( \frac{N_k}{N_k + 1} \right)^\ell \left| \ell \right\rangle \left\langle \ell \right|, \]
Here $\left| \ell \right\rangle$ is the $\ell$-photon Fock state and $N_k$ is the mean photon number in the thermal state $\hat{\rho}_k$. The covariance in this case is given by $\text{diag}(2N_1 + 1, \ldots, 2N_n + 1, 2N_1 + 1, \ldots, 2N_n + 1)$.

Imagine we perform a general active (or just passive in the restricted case) symplectic transformation with such a system, which classically correlates and/or entangles the modes. After dropping one mode, we use active (or just passive in the restricted case) transformations to bring the system of the $n - 1$ remaining modes into an uncorrelated state in which its covariance has the form $\text{diag}(2N'_1 + 1, \ldots, 2N'_{n-1} + 1, 2N'_1 + 1, \ldots, 2N'_{n-1} + 1)$. The inequalities (3) and (5) then become inequalities for the average number of thermal excitations $N_1, \ldots, N_n$ and $N'_1, \ldots, N'_{n-1}$ in the original system and the subsystem. If we again assume the non-decreasing ordering of the numbers $N_i$ and $N'_i$, these inequalities can be written as
\[ N_1 \leq N'_1 \leq N_3, \quad N_2 \leq N'_2 \leq N_4, \quad \ldots, \quad N_{n-2} \leq N'_{n-2} \leq N_n, \quad N_{n-1} \leq N'_{n-1} \]
for active transformations and
\[ N_1 \leq N'_1 \leq N_2 \leq \cdots \leq N_{n-1} \leq N'_{n-1} \leq N_n \]
for passive transformations. In the latter case the inequalities represent complete system of constraints. We can also apply the other results in Sec. III, e.g., generalize the inequalities for local systems of $n - k$ modes.
V. NUMERICAL SIMULATION

In this section we would like to support our conjecture that the inequalities (3) are not just necessary but also sufficient conditions for the symplectic eigenvalues $\mu$ and $\nu$ by numerical simulations. In all cases we chose $n = 8$ and the global symplectic eigenvalues $\mu = (2, 5, 11, 25, 45, 77, 100, 144)$.

For the simulations we employ symplectic matrices, i.e., matrices $S$ fulfilling $SJS^t = J$. First, we generate a random symplectic matrix by using the singular decomposition in the symplectic group, the singular values for the noncompact factor being chosen in the range $\left(\frac{1}{2}, 5\right)$. Then we restrict $S \text{diag}(\mu, \mu) S^t$, where $\text{diag}(\mu, \mu)$ is the diagonal matrix containing the global symplectic eigenvalues, to a $(2n-2)$-dimensional symplectic subspace and compute the $(n-1)$ local symplectic eigenvalues.

![Diagram](image)

**FIG. 1.** Results of the simulations for $n = 8$ global modes and $n - k = 7$ local modes. The global symplectic eigenvalues were chosen as $\mu = (2, 5, 11, 25, 45, 77, 100, 144)$. In each case $N = 100,000$ subspaces were chosen randomly. (a) general symplectic transformations, (b) passive transformations, (c) “shear” symplectic transformations, (d) point transformations. Although in (a) it seems as though the allowed intervals for $\nu_2$, $\nu_4$ and $\nu_7$ are not filled completely, this is the artefact of having too few simulation runs and/or too small range of singular values. This is exemplified by comparing with case (d) and realizing that point transformations form a subset of symplectic transformations.
Fig. 1 shows the results of our simulations for several classes of symplectic transformations.

(a) general symplectic transformations,
(b) orthogonal symplectic transformations, i.e., passive transformations,
(c) “shear” symplectic transformations, i.e., transformations whose matrix $S$ is of the block form

\[
S = \begin{pmatrix}
1_n & B \\
0_n & 1_n
\end{pmatrix},
\]

where $B$ is an $n \times n$ symmetric matrix,
(d) point transformations, i.e., transformations whose matrix $S$ is of the block form

\[
S = \begin{pmatrix}
A & 0_n \\
0_n & (A^t)^{-1}
\end{pmatrix},
\]

where $A$ is an $n \times n$ invertible matrix. These transformations do not mix canonical positions with momenta.

For each case, the results depict the resulting local symplectic eigenvalues on $N = 100\,000$ randomly generated $(2n-2)$-dimensional symplectic subspaces. The simulations for the most general class of transformations (a), but also for the point transformations (d) show that each of the values $\nu_i$ used the whole interval allowed by the inequalities (3), which suggests they represent complete system of constraints.

VI. CONCLUSIONS

We have obtained inequalities bounding symplectic eigenvalues in the local system by means of the symplectic eigenvalues of the global system. The question of the sufficiency of the inequalities has been solved in the case of passive transformations, we have not found the proof for active transformations although the numerical evidence suggests that the inequalities are sufficient even in this case. Further research in this area could concern itself with relating the symplectic eigenvalues of both the local subsystems and the symplectic eigenvalues of the global system. This would relate the average photon numbers in the local systems of $n-k$ and $k$ modes with the average photon numbers in the global system of $n$ modes (assuming the modes are uncorrelated) in the case of the electromagnetic field.

We would like to point out that while deriving the inequalities (3) for the symplectic eigenvalues, we have not used the fact that the states are Gaussian. We can define symplectic eigenvalues for non-Gaussian states in terms of covariance as well (as long as covariance is defined). Therefore, the inequalities (3) are valid also for non-Gaussian states. In this case, however, they certainly do not constitute a sufficient set of inequalities.

9 R. Bhatia, Matrix Analysis, Graduate Texts in Mathematics (Springer, 1997).