IOP Publishing Journal of Optics
J. Opt. **13** (2011) 115704 (5pp) doi:10.1088/2040-8978/13/11/115704

METATOYs and optical vortices

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Received 5 September 2011, accepted for publication 12 October 2011
Published 2 November 2011
Online at stacks.iop.org/JOpt/13/115704

Abstract
METATOYs are regular structures of optical elements that change the direction of light rays according to various generalized refraction laws. They may introduce discontinuities into transmitted wavefronts and create ray fields that do not have wave-optical analogies. We show that in such cases optical vortices are created during propagation and that this is in close relation to the properties of the rotations of superfluid helium. We also analyze the vortex density for a particular example of a METATOY.

Keywords: METATOYs, optical vortices

(Some figures may appear in colour only in the online journal)

1. Introduction

Vortices are remarkable structures that can be encountered in many areas of physics from fluid mechanics and optics to superconductors. The typical feature of vortices is that a certain characteristic quantity, e.g. fluid speed in the case of a fluid vortex, or the phase gradient in the case of an optical vortex, diverges as one approaches the vortex line. In some situations vortices provide the only possibility of how a certain process can happen. For example, the formation of vortices is the only way by which superfluid helium can rotate; it cannot rotate as a whole as usual liquids do, but vortices are formed upon rotation and their density grows with the speed of rotation.

In optics, vortices appear when wavefronts of different phases meet at a vortex line, where there is complete darkness. They can be most easily observed as speckle patterns when a laser beam is reflected from a rough surface [1]. Optical vortices have been widely explored and have found applications in optical micromanipulation [2], astronomy [3–6], quantum entanglement [7, 8], and free-space communication [5, 9, 6].

METATOYs, or metamaterials for rays, have been recently proposed [10] as regular structures of optical elements such as prisms [10], telescopes [11] etc that are tiny but at the same time larger than the wavelength of light for which they are designed. These structures are capable of refracting light in a way that can be very useful but difficult to achieve in other ways. For example, certain types of METATOYs refract light in a similar way to a layer of negatively refractive material [12] and it is hence possible to use them for imaging that is perfect from the point of view of geometrical optics [10, 11, 13]. METATOYs were proposed in the context of geometrical optics which turns out to be a very good approximation for describing METATOYs in most situations. On the other hand, the wave optics of METATOYs is largely unexplored. Some interesting effects can be expected here, for example diffraction at the interfaces of the optical elements forming the METATOY, or effects related to phase and wavefront jumps that naturally occur in METATOYs.

Moreover, recently it was found that METATOYs can, under certain conditions, create configurations of light rays that are impossible to achieve exactly for light waves [14]. In this sense METATOYs enable ray fields that do not have a wave-optical analog. On the other hand, clearly in this situation some electromagnetic wave will emerge from the METATOY, although the associated rays will be different from the ideal rays obtained from the geometrical optics description. In this paper, we focus on this problem and try to answer the question of what the resulting light wave beyond the METATOY looks like and what characteristics it has that cause the discrepancy between geometrical and wave optics descriptions. We will show that the key to the answer is an array of optical vortices that are formed in the outgoing wave. We will also point out the similarity between this behavior and the behavior of rotating superfluid helium.

The paper is organized as follows. In section 2 we define the ray field and discuss some of its properties. In section 3 we resume the example from [14] of a METATOY that produces light-ray fields without a wave-optical analog and in section 4 we analyze a light wave actually produced by this METATOY. In section 5 we show the analogy of light propagating beyond a METATOY with rotation of superfluid helium, and we conclude in section 6.

2. Phase and rays

Scalar-wave optics provides a very good description of light in many situations, in particular when the polarization is irrelevant and when the field is paraxial. Light of frequency ω can then be described by a complex scalar wave, $u(\mathbf{r})\exp(-\mathrm{i}\omega t)$; the modulus of $u(\mathbf{r})$ is the amplitude of the wave, and $\phi(\mathbf{r}) = \arg(u(\mathbf{r}))$, the argument of $u(\mathbf{r})$, is the phase of the wave relative to the global phase $-\omega t$. In the ray optics limit of scalar-wave optics, the light-ray field $\mathbf{d}(\mathbf{r})$ corresponding to a complex scalar wave $u(\mathbf{r})$ with phase $\phi(\mathbf{r})$ is given by the phase gradient,

$$\mathbf{d}(\mathbf{r}) = \nabla\phi(\mathbf{r}). \tag{1}$$

Wherever $\phi(\mathbf{r})$ has continuous partial second derivatives, which is not the case at phase discontinuities such as optical vortices [1], the curl of the gradient of ϕ has to be zero due to commutativity of partial derivatives, i.e. $\nabla\times\nabla\phi(\mathbf{r}) = 0$, and so the curl of the light-ray field $\mathbf{d}(\mathbf{r})$ has to vanish also [15, 16]:

$$\nabla\times\mathbf{d}(\mathbf{r}) = 0. \tag{2}$$

This wave-optical limitation on light-ray fields is discussed in more detail in [14]. Specifically, it is shown that light-ray fields with a non-zero z component of the curl correspond to complex scalar waves with a non-zero optical-vortex density (which was previously shown in [15]). This is similar to other situations where light waves that look like they have non-zero curl occur. For example, Andrew Hicks's distorting mirrors [16] adjust one of the transverse light-ray direction components such that the curl vanishes, and Carl Paterson's holograms that transform the intensity cross-section of a light beam [15, 17] collect the local vortex-charge density into vortices with charge $+1$ or -1.

In the following we will see that METATOYs [14] also create light waves that look like they have non-zero curl, and collect the local vortex-charge density into lines of discontinuous phase, which can alternatively be described as a series of optical vortices.

3. METATOYs and ray fields without wave-optical analog

It was shown recently [14] that under certain circumstances, METATOYs can produce light-ray fields that, in a certain sense, do not have a wave-optical analog. In particular, light rays emerging from a METATOY that are obtained from the geometrical optics description have such directions that it is impossible to find a distribution of phase $\phi(\mathbf{r})$ in space which would yield this ray field.

We will demonstrate this by the following example. Consider a spherical wave converging on the point $(0, 0, a)$ and incident on a METATOY that is placed in the $z = 0$ plane. The light-ray field in this plane is given by

$$\mathbf{d}_0(x, y, 0) = \frac{k}{\sqrt{a^2+x^2+y^2}}\begin{pmatrix}-x\\-y\\a\end{pmatrix}, \tag{3}$$

where $k = \omega/c$ is the wavenumber. Now suppose that the METATOY rotates the wave vector about the z axis clockwise by 90°. This can be achieved, for example, by attaching two parallel sheets of Dove prisms mutually rotated by 45° [18]. The light-ray field just behind the METATOY is then

$$\mathbf{d}(x, y, 0) = \frac{k}{\sqrt{a^2+x^2+y^2}}\begin{pmatrix}-y\\x\\a\end{pmatrix}. \tag{4}$$

Suppose that the phase ϕ exists of which $\mathbf{d}$ is a gradient, $\mathbf{d} = \nabla\phi$. The curl of $\mathbf{d}$ should then be zero due to the identity $\nabla\times\nabla\phi = 0$. However, from equation (4) clearly $\nabla\times\mathbf{d} \neq 0$, for example for $x, y \ll a$ one has approximately $|\nabla\times\mathbf{d}| \approx 2k/a$. This is a contradiction and hence the ray field (4) obtained by the METATOY does not have a wave-optical analog. In other words, there exists no phase distribution yielding the rays described by $\mathbf{d}$ as its gradient.

It is then natural to ask what happens to a light wave passing through the METATOY in the situation just described, or more generally when the ray field derived from the geometrical optics description of METATOY does not have a wave-optical analog. Even more importantly, what causes the discrepancy between the geometrical and wave optics descriptions? We will address these questions in the following section.

4. Wave field produced by a METATOY

Let us now look more closely at the ray field given by equation (4). As we have seen, the curl of $\mathbf{d}$ is non-zero. As the vector field $\mathbf{d}$ is smooth, it follows from Stokes' theorem that circulation of $\mathbf{d}$ along a closed loop ∂S is equal to the integral of the curl of $\mathbf{d}$ over the area S enclosed by this loop:

$$C_{\partial S}[\mathbf{d}] \equiv \oint_{\partial S}\mathbf{d}\cdot\mathrm{d}\mathbf{l} = \int_S(\nabla\times\mathbf{d})\cdot\mathrm{d}\mathbf{S}. \tag{5}$$

Placing the loop into the plane $z = 0$ immediately beyond the METATOY and using the approximate expression $\nabla\times\mathbf{d} \approx (0, 0, 2k/a)$ for $x, y \ll a$, we find that

$$C_{\partial S}[\mathbf{d}] = \frac{2k}{a}S. \tag{6}$$

The quantity $2k/a$, the curl of the vector field $\mathbf{d}$, can be interpreted as a density of circulation of $\mathbf{d}$.

Now if some vector field is a gradient of a scalar field (here the phase), its curl must vanish. Is the same true for the circulation? Not necessarily, if the vector field is discontinuous. For example, take the field $\mathbf{u} = (-y/[x^2 +$

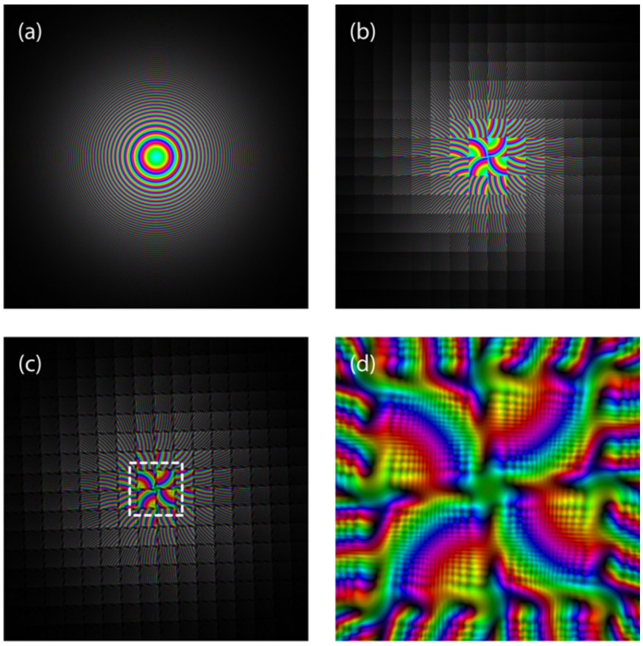

Figure 1. Simulated wave fields in front of and behind a METATOY that perform piecewise clockwise rotation of the wavefront through 90°. Intensity is shown by brightness, phase by color. A Gaussian-modulated contracting spherical wave (a Gaussian beam that has passed through a lens with $f > 0$) is shown immediately in front of (a) and behind (b) the METATOY, and after further propagation by 1 cm (c). (d) shows a magnified view of the center of (c), marked by the white dashed square. The beam shown in (a) is a Gaussian beam (waist size 3 mm) of wavelength $\lambda = 633$ nm and radius of curvature $a = 0.3$ m. In (b) the rotated square wavefront pieces can clearly be seen. In (c) and (d) vortices show up as intensity zeros. In (a)–(c), a square of side length 1 cm (the entire modeled area) is shown; in (d), a square of side length 1.67 mm is shown. Careful counting of the vortices reveals that their concentration is $\rho_{\rm sim} \approx 10.5$ mm^{-2} (0.001 vortex per pixel) while equation (7) gives the value $\rho = 10.54$ mm^{-2}, which is in very good agreement. In all cases, a physical area of size 1 cm × 1 cm was represented on a grid of 1024 pixels × 1024 pixels. Beam propagation was calculated using a standard Fourier-space method [19].

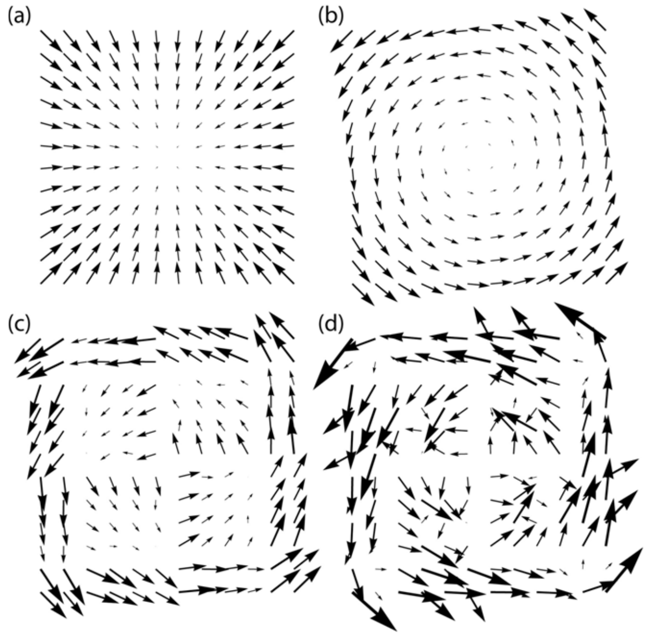

Figure 2. Transverse phase-gradient fields for the cross-sections shown in figure 1: (a) immediately in front of the METATOY; (b) as in (a), but rotated clockwise through 90°; (c) immediately behind the METATOY; and (d) 1 cm behind the METATOY. The local fluctuations of the field apparent in (d) are caused by the optical vortices, but the phase gradient shows that the pattern is rotating anti-clockwise. The phase-gradient vectors were calculated numerically from the field cross-sections shown in figure 1, but note that the fields shown in (a) and (b) are respectively described by the x and y components of equations (3) and (4). In all cases, the field across the central 1.67 mm × 1.67 mm is shown.

$y^2], x/[x^2 + y^2], 0)$, which is the gradient of the cylindrical coordinate φ and describes a perfect vortex of charge one. Its curl is zero everywhere with the exception of the z-axis where it is infinite[3] and its circulation along a curve that encircles the z-axis once is equal to 2π.

This suggests an interesting option for the light field beyond the METATOY. The circulation of the phase gradient can be made non-zero by phase singularities (vortices) into which the curl is concentrated. In the space between the singularities the curl of the phase gradient is zero, as it should be.

We can estimate the density of these vortices. For a loop that encircles once the vortex with charge one, the phase change is equal to 2π. The circulation of the phase gradient along a closed loop is therefore $2\pi n$ where n is the number of vortices enclosed. To match the required circulation density equal to $2k/a$ in our example, the 2D vortex density (number of vortices per area) has to be

$$\rho = \frac{k}{\pi a}. \tag{7}$$

We have performed numerical simulations of the electromagnetic wave beyond a specific, idealized, METATOY that realizes the situation just described, i.e. that rotates the local light-ray direction through 90° around the z direction. Our idealized METATOY splits the incident wave into square pieces and rotates these individually through 90°, an idealization of the effect of piecewise wavefront rotation that can be achieved with pairs of Dove-prism arrays that are rotated with respect to each other by 45° [18]. Figure 1 shows the simulated field in several planes $z =$ const.: immediately before and beyond the METATOY and after some propagation distance. It can be seen clearly that in the last case vortices are present. This shows that what we have anticipated indeed happens: the curl of the phase gradient is concentrated to vortex lines that flow in the positive z-direction. The non-zero curl is simply replaced by a spatial distribution of singularities and zero curl between them. Moreover, the density of vortices $\rho = 10.54$ mm^{-2} predicted by equation (7) for the example of figure 1 is in excellent agreement with the value $\rho_{\rm sim} \approx 10.5$ mm^{-2} obtained from our numerical simulation.

It is also interesting to compare the actual light-ray field obtained as a gradient of the actual phase obtained from

[3] The curl of $\mathbf{u}$ is given by the Dirac delta function, $\nabla \times \mathbf{u} = 2\pi\delta(x)\delta(y)$.

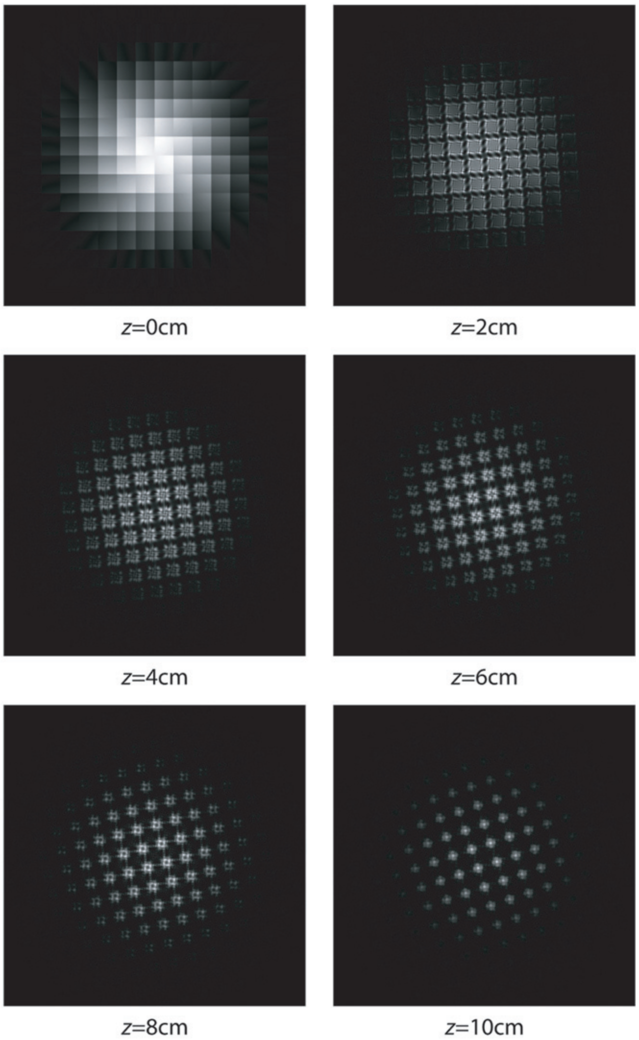

Figure 3. Intensity of the field shown in figure 1(b) after propagation through a distance z. The rotation of the bright parts with propagation distance and the vortices along the dislocation lines have become clearly visible. The vortex positions can be seen as dark spots in the frame for $z = 2$ cm.

numerical simulations with the ideal ray field obtained from the geometrical optics description. Figure 2 shows the phase gradients of both fields for the particular example discussed above.

5. Analogy with superfluid helium

It is well known [20] that superfluid helium rotates in a different way to an ordinary liquid. If the latter rotates as a whole, for instance when placing it into a uniformly rotating container and waiting sufficiently long, the velocity is

$$\mathbf{v} = \omega \times \mathbf{r} = (-\omega y, \omega x, 0) \tag{8}$$

(we are assuming rotation around the z-axis with angular velocity ω), and its curl is $\nabla \times \mathbf{v} = 2\omega$. This cannot happen for a superfluid since the quantum phase is directly related to the momentum of the particles, $\mathbf{p} = \hbar\nabla\phi$, and hence $\nabla \times \mathbf{p} = 0$. The problem is resolved in exactly the same way as in our example of a METATOY. The curl of momentum is concentrated into vortices, and outside their vortex lines the curl of $\mathbf{p}$ is zero as it should be. This way circulation of momentum is non-zero while its curl is zero almost everywhere. The only way a superfluid can rotate is to form vortices. And similarly, the only way light rays beyond the METATOY from our example can rotate is to form vortices of phase. The rotation of the beam cross-section, and the phase vortices, can clearly be seen in figure 3.

6. Conclusions

In conclusion, we have discussed wave effects related to METATOYs that, in terms of geometrical optics, produce light-ray fields that do not have wave-optical analogs. We have seen that in such a situation an array of optical vortices is created that concentrates the vorticity of the phase gradient into vortex lines. This is very similar to the situation when superfluid helium is brought to rotation: vortices are formed in the helium that concentrate the circulation of momentum vorticity into vortex lines. We have also calculated the concentration of the vortices in the field emerging from the METATOY and confirmed the results of the calculation by numerical simulations.

In return for all this wave-optical trouble, METATOYs appear to enable ray optics that are wave-optically forbidden; a visually convincing demonstration is described in [21]. This wave-optical prohibition is the same that was encountered in [16, 15, 17], but the ways in which it is overcome (or not) are different: in the mirrors discussed in [16], the mirror shape is adjusted such that the reflected light-ray field does not violate the wave-optical prohibition; in the optical holographic transformations discussed in [15, 17], the hologram is designed to introduce a fixed array of vortices into the beam; in METATOYs, vortices form on the lines where the constituent optical elements touch.

Acknowledgments

TT acknowledges support of the grants MSM0021622409 and MSM0021622419.

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