Atoms In Compactified Universes

Master's Thesis



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Abstract

It is the purpose of this thesis to investigate the stability and energy spectra of the non-relativistic hydrogen atom in four-dimensional spaces. The additional spatial dimension is considered to be either infinite or curled-up in a circle of radius R.

After a short historical introduction, we study the case of spaces with an infinite extra dimension. We solve the Schrödinger equation of the hydrogen atom and analyze the results. Considerable attention is devoted to discussion of the three qualitatively distinct solutions that appear. We argue that there is no stable hydrogen atom in this case.

The second part of this work deals with four-dimensional spaces, where one of the space-like dimensions is compactified, i.e. it has the topology of a circle at a very small radius. We solve the Schrödinger equation and explore the main task, namely, how an additional curled-up dimension affects the spectrum of hydrogen atoms. Finally, we argue that if the potential is sufficiently strong, the hydrogen atom is no longer stable.

Statutory declaration

With this I declare that I have developed and written the enclosed thesis completely by myself and have not used sources without declaration in the text.

Brno, May 25th 2007

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1 Introduction

Is it not there first proved that there are no more than three dimensions, since Three is everything, and everywhere?

Galileo Galilei (referring to Aristotle's 'On The Heavens'), Dialogue Concerning the Two Chief World Systems (1632) [8, 17]

The idea of extra spatial dimensions has a long and rich history. Its origin lies in the search for a unified description of the forces observed in nature. An early attempt traces back to the Finnish theoretical physicist Gunnar Nordström (1914), who proposed, in the context of his theory of gravity, a five-dimensional theory to simultaneously describe electromagnetic and gravitational forces [26]. This theory was the first known example of a metric theory of gravitation. However, as it turned out, it was not in agreement with observation and experiment.

After the invention of general relativity, a German mathematician and physicist Theodor Kaluza noticed that gravitational and electromagnetic interactions can be simultaneously described by a five-dimensional generalization of Einstein's theory [21]. Giving this idea a precise formulation in 1926, a Swedish theoretical physicist Oskar Klein further proposed that the extra dimensions can be curled-up in a circle of a very small radius [22]. Thus, he resolved the apparent contradiction to the fact that we observe only three space-like dimensions and one time-like. But also the Kaluza-Klein theory, lacking of experimental implications and suffering from a number of faults, did not unify the two fundamental forces of gravitation and electromagnetism, failed in its purpose and was essentially abandoned for a couple of decades.

During the mid-seventies, the emergence of string theory made the old idea of extra spatial dimensions come alive, since the existence of additional dimensions is one of its requirements. Because string theory is the most promising candidate for a consistent theory of quantum gravity, the study of higher-dimensional problems is of big importance.

One of the basic and most interesting questions concerning problems in higher-dimensional spaces is the stability of atoms. The first investigation is due to Ehrenfest [12]. In his paper from 1919, he studies among other things the stability of planetary orbits and the Bohr model of hydrogen atom in higher-dimensional spaces. Thereafter, many interesting investigations followed [11, 15, 10, 27, 19, 23, 31].

Overview of the thesis

In the present work, we will study the stability and energy spectra of the non-relativistic hydrogen atom in four-dimensional spaces. The first part of the thesis is devoted to the hydrogen atom in spaces with an infinite additional space-like dimension. Although there are some papers (especially [19]) dealing with the topic of stability of hydrogen atom in the general case of n-dimensional spaces, we will give a more detailed discussion of the four-dimensional case that we are interested in. We will solve the defining Schrödinger equation and discuss the results.

As it is found in this work that the hydrogen atom is not stable in infinite four-dimensions, it is important to mention (see note at end of section 2) that there are articles stating the existence of a stable hydrogen atom in higher dimensions (e.g. [10], [25], [6]). The physically important point is that the potential is considered to be proportional to $1/r$, irrespective of the the number of spatial dimensions. This is, however, not the potential that would correspond to the solution of Maxwell's equations in n-dimensional space. Hence, the immediate consequence of such an approach is the modification of Maxwell's equations in higher dimensions.

The second part of this thesis deals with four-dimensional spaces, where one of the space-like dimensions is compactified, i.e. it has the topology of a circle at a very small radius. We will start by explaining a simple method used for treatment of spaces with such topology. Then, we will solve the Schrödinger equation and explore the main task, namely, how an additional curled-up dimension affects the spectrum of hydrogen atoms. For this purpose, we use time-independent perturbation theory and calculate corrections to energy eigenvalues.

Finally, we will argue that if the potential is sufficiently strong, the hydrogen atom is no longer stable. To show this, we will employ a trial function and see that the continuous energy spectrum extends from zero to minus infinity.

The next section is devoted to discussion of the results. Lastly, some possibilities for further development are presented. The thesis also contains several appendices covering some support material of mainly mathematical character.

2 Infinite Extra Dimension

In this section, we take up the case of four-dimensional spaces, where the extra space-like dimension is considered to be extended. We will solve the Schrödinger equation for the hydrogen atom and discuss the results. It will be shown that we must distinguish three cases, differing in the nature of solutions.

It turns out that in the so-called repulsive and weak case there are no bound states, since there are no physically acceptable solutions to the Schrödinger equation if the energy is taken as negative. In the strong case on the other hand the energy spectrum is continuous and extends from zero to minus infinity. Thus, in the presence of an infinite extra dimension, the hydrogen atom is not stable.

Next, we present an alternative treatment of how to decide the question of stability of the studied system. Namely, we use a trial function to show that the continuous spectrum is unbounded from below. Lastly, we make some comments regarding articles, in which the hydrogen atom in N-dimensional space is defined by the potential proportional to $1/r$.

2.1 Solution of a four-dimensional hydrogen atom

Let us start with the Schrödinger equation for a closed system of two non-relativistic point masses interacting via a central force:

$$\left(-\frac{1}{\kappa}\nabla^2 + V(r)\right)\psi(\boldsymbol{r}) = E\psi(\boldsymbol{r}). \tag{2.1}$$

where $\kappa \equiv 2m/\hbar^2$. [1] Since the potential is centrally symmetric, hyperspherical coordinates are best adapted to the problem (figure 1):

$$\begin{aligned} x &= r\sin\eta\sin\theta\cos\phi, \\ y &= r\sin\eta\sin\theta\sin\phi, \\ z &= r\sin\eta\cos\theta, \\ w &= r\cos\eta. \end{aligned} \tag{2.2}$$

Thus, we are to find the expression for the Laplacian operator in hyperspherical coordinates. The result is (appendix A):

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{\mathcal{L}^2(3)}{r^2}, \tag{2.3}$$

[1] We shall use this notation throughout the rest of the work.

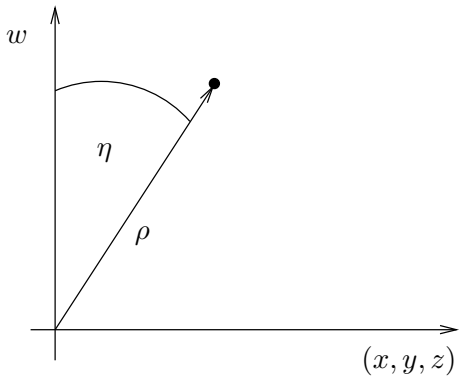


Figure 1: Hyperspherical coordinates in four-dimensional space.

where

$$\mathcal{L}^2(3) = -\frac{1}{\sin^2\eta}\left[\frac{\partial}{\partial\eta}\left(\sin^2\eta\frac{\partial}{\partial\eta}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{2.4}$$

is the square of the angular momentum operator on a 3-sphere (see appendix A).

The general solution of Poisson's equation yields its potential as a function of the radial distance r to the source. The potential energy in four-dimensions is thus given by (appendix B):

$$V(r) = -\frac{e^2}{r^2}, \tag{2.5}$$

where e^2 is the four-dimensional charge (with the unit energy $\times$ length^2). Hence, our Hamiltonian has the form

$$\hat{H} = \frac{1}{\kappa}\left(\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{\mathcal{L}^2(3)}{r^2}\right) - \frac{e^2}{r^2}, \tag{2.6}$$

and we can rewrite the Schrödinger equation (2.1) as:

$$\left[\frac{\partial^2}{\partial r^2} + \frac{3}{r}\frac{\partial}{\partial r} - \frac{\mathcal{L}^2(3)}{r^2} + \kappa\left(E + \frac{e^2}{r^2}\right)\right]\psi(\boldsymbol{r}) = 0. \tag{2.7}$$

As in the case of the three-dimensional hydrogen atom (and generally for any spherically symmetric potential), the Schrödinger equation can be separated into two following parts - the first one is an ordinary differential equation for the radial function, the other is a partial differential equation for the angular function. Thus, we make the Ansatz $\psi(\boldsymbol{r}) =$

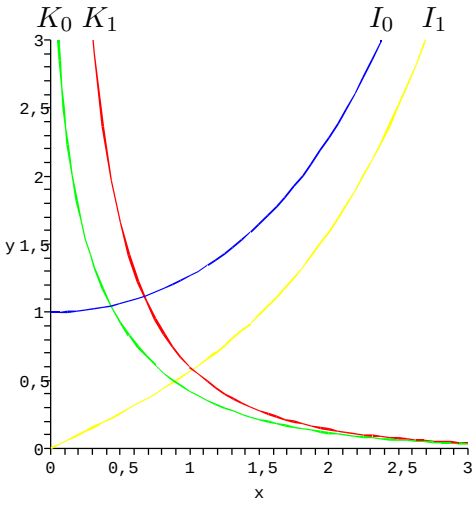


Figure 2: Modified Bessel functions of the first and second kind I_0, I_1 and K_0, K_1 respectively.

$R(\boldsymbol{r})Y^{(3)}(\eta,\theta,\phi)$. After the separation of the angular variables in (2.7), we obtain for the radial function $R(r)$ the equation

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{l(l+2)}{r^2}+\kappa\left(E+\frac{e^2}{r^2}\right)\right]R(r)=0, \tag{2.8}$$

where we have applied the fact that the eigenvalues of $\mathcal{L}^2(3)$ are given by [28]:

$$\mathcal{L}^2(3)Y_l^{(3)}(\eta,\theta,\phi)=l(l+2)\,Y_l^{(3)}(\eta,\theta,\phi), \qquad l=0,1,2,\ldots. \tag{2.9}$$

The functions $Y_l^{(3)}$ are the hyperspherical harmonics on a 3-sphere. For $E<0$, e.g. for bound states, we use the abbreviations

$$\alpha^2=-\kappa E, \quad \nu^2=l(l+2)+1-\lambda^2, \quad \text{with} \quad \lambda^2=\kappa e^2. \tag{2.10}$$

Setting $R(r)\equiv\chi(r)/r$, we replace equation (2.8) by an equivalent radial equation

$$r^2\frac{\mathrm{d}^2\chi(r)}{\mathrm{d}r^2}+r\frac{\mathrm{d}\chi(r)}{\mathrm{d}r}-(\alpha^2r^2+\nu^2)\chi(r)=0. \tag{2.11}$$

Equation (2.11) is the modified Bessel equation. The solutions are the modified (or sometimes called hyperbolic) Bessel functions of the first and second kind $I_\nu(\alpha r)$ and $K_\nu(\alpha r)$, respectively. The functions I_0, I_1, K_0 and K_1 are shown in figure 2.

Our boundary condition at infinity eliminates the solutions I_ν, since I_ν/r diverge exponentially for large values of r ([1], p. 377):

$$\frac{I_\nu(\alpha r)}{r} \sim \frac{1}{r}\frac{e^{\alpha r}}{\sqrt{2\pi\alpha r}}. \tag{2.12}$$

It remains for us to discuss the second independent solution to equation (2.11), the modified Bessel functions of the second kind K_ν. Three cases must be considered according to whether the parameter ν^2 belongs to the interval $\langle 1, \infty)$, $\langle 0, 1)$ or $(-\infty, 0)$. We shall call these cases repulsive, weak and strong, respectively. This nomenclature naturally corresponds to the character of the effective potential energy (from (2.8))

$$V_{eff} = \frac{1}{\kappa}\frac{l(l+2)}{r^2} - \frac{e^2}{r^2} = \frac{1}{\kappa}\frac{\nu^2 - 1}{r^2}. \tag{2.13}$$

2.1.1 Repulsive case

First, let us consider ν^2 taking values from the interval

$$\nu^2 \geq 1 \quad \text{i.e.} \quad \kappa e^2 - l(l+2) \leq 0. \tag{2.14}$$

In this case the Bessel functions of the second kind K_ν are too divergent at the origin and the corresponding radial functions $K_\nu(\alpha r)/r$ are not square integrable. Indeed, the lowest-order terms in the series expansion of $K_\nu(\alpha r)$ are given by ([7], p. 711):

$$K_\nu(\alpha r) = \quad 2^{\nu-1}(\nu-1)!(\alpha r)^{-\nu} + \ldots, \quad \nu > 0, \tag{2.15}$$

with $K_\nu = K_{-\nu}$. Thus, expanding the integrand of the normalization integral in a series, the lowest-order term is proportional to

$$r^3(r^\nu/r)^2 = r^{1-2\nu}. \tag{2.16}$$

In order for the integral to converge, the power of r must be greater than -1. But this condition is not met (recall $\nu^2 \geq 1$) and the wave function constructed with this solution cannot be accepted.

Let us recall that we search only for bound states. Considering, however, also unbounded states, the functions I_ν might correspond to some solution.

2.1.2 Weak case

Next, we restrict ν^2 to the interval

$$0 \leq \nu^2 < 1 \quad \text{i.e.} \quad 0 < \kappa e^2 - l(l+2) \leq 1. \tag{2.17}$$

With the exception of K_0, the lowest-order terms in the series expansion of $K_\nu(\alpha r)$ are again given by (2.15). For $\nu^2 = 0$ the expansion is as follows ([7], p. 711):

$$K_0(\alpha r) \quad = \quad -\ln(\alpha r) - \gamma + \ln 2 + \ldots. \tag{2.18}$$

with γ being a constant. [2] Looking at (2.16) we find out that the problem with convergence at the origin vanished, since the power of r in the normalization integral takes values from the interval $(-1, 3)$. For large r, the asymptotic expansion of $K_\nu(\alpha r)$ is given by ([7], p. 717):

$$K_\nu(\alpha r) \sim \sqrt{\frac{\pi}{2\alpha r}} e^{-\alpha r} \tag{2.19}$$

and the wave function constructed with K_ν is square integrable. Thus, it seems that it could be accepted as eigenfuction.

However, this is not the case. The reason is as follows: Because the Hamiltonian is a Hermitian operator, the equality

$$\langle \psi_1(r)|\hat{H}|\psi_2(r)\rangle = \langle \psi_2(r)|\hat{H}|\psi_1(r)\rangle^* \tag{2.20}$$

must hold for any states $\psi_1(r)$ and $\psi_2(r)$. Using integration by parts (the explicit form of Hamiltonian is given by (2.6)) we obtain:

$$\langle \psi_1(r)|\hat{H}|\psi_2(r)\rangle - \langle \psi_2(r)|\hat{H}|\psi_1(r)\rangle^* = \left[\left(-\psi_1^* \frac{\mathrm{d}\psi_2}{\mathrm{d}r} + \frac{\mathrm{d}\psi_1^*}{\mathrm{d}r}\psi_2\right) r^3\right]_0^\infty. \tag{2.21}$$

Thus, if $\hat{H}$ is Hermitian, the right-hand side of (2.21) must vanish. The boundary condition at infinity is satisfied (see (2.19)), but that at the origin is not. For, taking the wave functions constructed with K_ν:

$$\psi_1(r) \propto \frac{1}{r} K_\nu(\alpha_1 r), \quad \psi_2(r) \propto \frac{1}{r} K_\nu(\alpha_2 r), \tag{2.22}$$

[2] It is the Euler-Mascheroni constant defined by $\gamma = \lim_{n\to\infty}\left(\sum_{m=1}^{n} m^{-1} - \ln n\right) = 0.57721566\ldots$, [7], p. 312.

and using the series expansion of K_ν (equation (2.15)), we find the terms in the square bracket to be proportional to $r^{-2\nu}$. Accordingly, the condition ensuring that the boundary terms at zero vanish is $\nu < 0$. But this is outside the interval (2.17). Similarly, for $\nu = 0$ we take the expansion (2.18) and find the terms to be proportional to $\ln(\alpha r)(1 - \ln(\alpha r))$. Consequently, the Hamiltonian is not Hermitian and we must again reject this solution as unacceptable.

2.1.3 Strong case

Finally, let us choose ν^2 taking values from the interval

$$\nu^2 < 0 \quad (\text{or } \nu \in i\mathcal{R} - \{0\}) \quad \text{i.e.} \quad \kappa e^2 - l(l+2) > 1. \tag{2.23}$$

Now we will show that the wave functions constructed with $K_{i|\nu|}$ form an orthonormal set. We must, however, choose a suitable discrete subset of energy eigenvalues.

Consider two solutions $\chi_1(r)$, $\chi_2(r)$ of equation (2.11), corresponding to two values of energy E_1, E_2, represented by (see (2.10)):

$$\alpha_1 = \sqrt{-\kappa E_1}, \;\; \alpha_2 = \sqrt{-\kappa E_2}, \qquad E_1,\, E_2 < 0. \tag{2.24}$$

Let us take equation (2.11), corresponding to the solution χ_1, and multiply it by χ_2/r. Subtracting the same equation, corresponding to χ_2 and multiplied by χ_2/r, we obtain, by integrating this expression:

$$\begin{aligned}\int_0^\infty \left(r\chi_2 \frac{\mathrm{d}^2\chi_1}{\mathrm{d}r^2} - r\chi_1 \frac{\mathrm{d}^2\chi_2}{\mathrm{d}r^2} + \chi_2 \frac{\mathrm{d}\chi_1}{\mathrm{d}r} + \chi_1 \frac{\mathrm{d}\chi_2}{\mathrm{d}r} - r\alpha_1^2\chi_2\chi_1 + r\alpha_2^2\chi_1\chi_2 \right) \mathrm{d}r \\ = \int_0^\infty \left[\chi_2 \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\chi_1}{\mathrm{d}r} \right) - \chi_1 \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}\chi_2}{\mathrm{d}r} \right) \right] \mathrm{d}r - (\alpha_1^2 - \alpha_2^2) \int_0^\infty r\chi_1\chi_2 \mathrm{d}r \\ = \left[r\chi_2 \frac{\mathrm{d}\chi_1}{\mathrm{d}r} - r\chi_1 \frac{\mathrm{d}\chi_2}{\mathrm{d}r} \right]_0^\infty - (\alpha_1^2 - \alpha_2^2) \int_0^\infty r\chi_1\chi_2 \mathrm{d}r = 0. \end{aligned} \tag{2.25}$$

Using the fact that

$$\psi_1(r) \propto \frac{\chi_1(r)}{r} = \frac{1}{r} K_{i|\nu|}(\alpha_1 r), \quad \psi_2(r) \propto \frac{\chi_2(r)}{r} = \frac{1}{r} K_{i|\nu|}(\alpha_2 r), \tag{2.26}$$

and noting that there are no boundary terms at infinity (see (2.19)), the last line of (2.25)

gives us the scalar product:

$$\int_0^\infty r\chi_1\chi_2 \mathrm{d}r = \int_0^\infty r^3\psi_1(r)\psi_2(r)\mathrm{d}r = \langle\psi_1(r)|\psi_2(r)\rangle = \frac{1}{\alpha_1^2-\alpha_2^2}\lim_{r\to 0}\left[rK_{i|\nu|}(\alpha_2 r)\frac{\mathrm{d}K_{i|\nu|}(\alpha_1 r)}{\mathrm{d}r} - rK_{i|\nu|}(\alpha_1 r)\frac{\mathrm{d}K_{i|\nu|}(\alpha_2 r)}{\mathrm{d}r}\right]_0^\infty. \quad (2.27)$$

The lowest-order term in the series expansion of $K_{i|\nu|}$ is, according to [13]:

$$K_{i|\nu|}(\alpha r) \approx -\frac{1}{|\nu|^2}\sqrt{\frac{|\nu|\pi}{\sinh|\nu|\pi}}\sin\left(|\nu|\ln\left(\frac{\alpha r}{2}\right) - \arg\left[\Gamma(i|\nu|+1)\right]\right). \quad (2.28)$$

Inserting this into (2.27) produces

$$\langle\psi_1(r)|\psi_2(r)\rangle = \frac{\pi}{|\nu|^2\sinh(|\nu|\pi)}\frac{\sin\left(|\nu|\ln\frac{\alpha_1}{\alpha_2}\right)}{\alpha_1^2-\alpha_2^2} \quad (2.29)$$

Fixing some certain value $|E_0|$ and forming the following discrete subset of energy eigenvalues

$$E = -|E_0|e^{2\pi n/|\nu|}, \quad E_0 \in \mathbb{R},\ n \in \mathbb{Z}, \quad (2.30)$$

we have from (2.24):

$$\alpha_1 = \sqrt{\kappa|E_0|}e^{\pi n_1/|\nu|},\ \ \alpha_2 = \sqrt{\kappa|E_0|}e^{\pi n_2/|\nu|},\ \ n_1, n_2 \in \mathbb{Z}, \quad (2.31)$$

Inserting α_1 and α_2 into (2.27) yields an orthogonality relation:

$$\langle\psi_1|\psi_2\rangle = \begin{cases} \dfrac{\pi}{2\kappa|\nu||E_0|\sinh(|\nu|\pi)} & \text{for } n_1 = n_2 \\ 0 & \text{for } n_1 \neq n_2 \end{cases}, \quad (2.32)$$

where we have used the l'Hôpital's rule for $n_1 = n_2$. The eigenstates are thus given by

$$\psi_\nu(r) = \mathcal{N}(\nu)\frac{K_\nu(\alpha r)}{r}, \quad (2.33)$$

where $\alpha = \sqrt{\kappa|E_0|}e^{\pi n/|\nu|}$ and with

$$\mathcal{N}(\nu) = \frac{\pi}{2\kappa|\nu||E_0|\sinh(|\nu|\pi)} \quad (2.34)$$

being a normalization factor.

Note that we can obtain the above orthogonality relation easily using ([18], p. 686):

$$\int_0^\infty r K_\nu(\alpha_1 r) K_\nu(\alpha_2 r)\,\mathrm{d}r = \frac{\pi(\alpha_1\alpha_2)^{-\nu}(\alpha_1^{2\nu}-\alpha_2^{2\nu})}{2\sin(\nu\pi)(\alpha_1^2-\alpha_2^2)}, \quad |\mathrm{Re}\ \nu|<1, \quad \mathrm{Re}\ (\alpha_1+\alpha_2)>0, \tag{2.35}$$

Clearly, if $0 \leq \nu^2 < 1$ (the weak case), then (2.35) does not lead to any orthogonal relation.

Again, we can check for Hermiticity of $\hat{H}$. By inserting (2.28) into the the right-hand side of equation (2.21), together with (2.26), we obtain

$$\left(-\psi_1^*\frac{\mathrm{d}\psi_2}{\mathrm{d}r}+\frac{\mathrm{d}\psi_1^*}{\mathrm{d}r}\psi_2\right) r^3 \approx \frac{\pi}{|\nu|^2\sinh(|\nu|\pi)}\sin\left(|\nu|\ln\frac{\alpha_1}{\alpha_2}\right) \tag{2.36}$$

and the boundary terms in zero vanish by inserting (2.31).

However, the energy eigenvalues, corresponding to the wave functions constructed with K_ν, can be arbitrarily large. Therefore, the continuous spectrum (recall $|E_0| \in \mathbb{R}_+$) is unbounded from below and the atom is not stable.

It is also worth noting that the absence of a lower bound in the energy spectrum can also be deduced from the following property of the K_ν functions. Because the argument of the sine in equation (2.28) grows without limit as r approaches the origin, this oscillating function has an infinite number of zeros. Since the wave function of the ground state can have no zeros, the ground state corresponds to the energy $E = -\infty$, i.e. the spectrum is unbounded from below and the particle falls on the nucleus (cf. [24], p. 116).

2.2 Trial function

In the following we will argue that the energy of our system can take arbitrarily large negative values if $\nu^2 < 0$. More precisely, we will show that

$$\lim_{\alpha\to\infty}\langle E\rangle = -\infty, \tag{2.37}$$

where

$$\langle E\rangle = \frac{\langle\psi_{trial}|\hat{H}|\psi_{trial}\rangle}{\langle\psi_{trial}|\psi_{trial}\rangle} \tag{2.38}$$

is the mean value of energy for a system described by the wave function

$$\psi_{trial}(r) = r^{\mu} e^{-\alpha r}. \tag{2.39}$$

Note that by inserting our trial function (2.39) into equation (2.21) we can easily verify that the Hamiltonian is Hermitian. Normalization gives us

$$\langle \psi_{trial} | \psi_{trial} \rangle = \int_0^{\infty} r^3 r^{2\mu} e^{-2\alpha r} \mathrm{d}r = \frac{\Gamma(2\mu+4)}{(2\alpha)^{2\nu+4}}, \tag{2.40}$$

where we have used the integral

$$\int_0^{\infty} x^{\nu} e^{-ax} \mathrm{d}x = \frac{\Gamma(\nu+1)}{a^{\nu+1}}, \quad a > 0, \quad \nu > -1. \tag{2.41}$$

By inserting the Hamiltonian (2.6) into the numerator of (2.38), we obtain:

$$\langle \psi_{trial} | \hat{H} | \psi_{trial} \rangle = -\frac{1}{\kappa} \langle \psi_{trial} | \nabla^2 | \psi_{trial} \rangle - e^2 \langle \psi_{trial} | \frac{1}{r^2} | \psi_{trial} \rangle. \tag{2.42}$$

Thus, it remains for us to evaluate

$$\langle \psi_{trial} | \frac{1}{r^2} | \psi_{trial} \rangle = \int_0^{\infty} r^3 r^{-2} r^{2\mu} e^{-2\alpha r} \mathrm{d}r = \frac{\Gamma(2\mu+2)}{(2\alpha)^{2\nu+2}} \tag{2.43}$$

and

$$\begin{aligned} \langle \psi_{trial} | \nabla^2 | \psi_{trial} \rangle &= \int_0^{\infty} r^3 r^{2\mu} e^{-2\alpha r} \left(\frac{\mu(\mu+2)}{r^2} - \alpha \frac{2\mu+3}{r} + \alpha^2 \right) \mathrm{d}r \\ &= \mu(\mu+2) \frac{\Gamma(2\mu+2)}{(2\alpha)^{2\nu+2}} - \alpha(2\mu+3) \frac{\Gamma(2\mu+3)}{(2\alpha)^{2\nu+3}} + \alpha^2 \frac{\Gamma(2\mu+4)}{(2\alpha)^{2\nu+4}}. \end{aligned} \tag{2.44}$$

The most restrictive condition imposed on μ is given by equations (2.43) and (2.44). Namely, we must have

$$\mu > -1 \tag{2.45}$$

in order to get convergent integrals. Inserting (2.40), (2.43) and (2.44) into (2.38) produces:

$$\langle E \rangle = \frac{\alpha^2}{\kappa} \frac{\mu + 3 - 2\kappa e^2}{(\mu+1)(\mu+3)} = \frac{\alpha^2}{\kappa} \frac{\mu + 1 + 2\nu^2}{(\mu+1)(\mu+3)}. \tag{2.46}$$

Now, for any $\nu^2 < 0$ we can always choose μ so that the numerator of (2.46) is negative. Consequently, the energy can take arbitrarily large negative values as $\alpha \to \infty$. For this to hold, the condition imposed on μ is:

$$\mu < -1 - 2\nu^2. \tag{2.47}$$

On the other hand, if $\nu^2 \geq 0$, this cannot be achieved for any permissible value of μ.

2.3 Note

If one assumes the potential to be $\sim 1/r$, independent of the spatial dimension N, then it is possible to have stable atoms in higher infinite dimensions (see e.g. [10, 25, 6, 2, 33]). However, this is not the potential which would correspond to the solution of Maxwell's equations in N-dimensional space (and therefore this approach does not lead to a Gaussian law for charges - appendix B). As a consequence, Maxwell's equations have to be modified in higher dimensions.

The solution of the corresponding Schrödinger equation involves a procedure which is similar to that for the standard three-dimensional hydrogen atom. The radial part of the wave function is [25]:

$$R_{nl}(\rho) = \mathcal{N}(n,l)e^{-\rho/2}\rho^l L_{n-l-1}^{(2l+N-2)}(\rho)\,, \tag{2.48}$$

where

$$\rho = \frac{r}{r_0[n+(N-3)/2]}, \quad \text{with} \quad r_0 = \hbar^2/2me^2 \tag{2.49}$$

The functions $L_n^{(\alpha)}(t)$ are the generalized Laguerre polynomials, defined as

$$L_n^{(\alpha)}(t) \equiv \sum_{j=0}^{n} \binom{n+\alpha}{n-j} \frac{(-t)^j}{j!} = \frac{(-1)^\alpha L_{n+\alpha}^{\alpha}(t)}{(n+\alpha)!}, \tag{2.50}$$

with α being a non-negative integer. The allowed values of the quantum numbers n, l are:

$$n = 1, 2, 3, \ldots, \tag{2.51}$$

$$l = 0, 1, \ldots, n-1. \tag{2.52}$$

The energy eigenvalues E_n are given by

$$E_n = -\frac{\mathcal{E}_0}{[n + (N-3)/2]^2}, \tag{2.53}$$

with the ground state energy $\mathcal{E}_0 = me^4/2\hbar^2$.

For papers dealing with some other quantum-mechanical models in higher dimensional spaces see, for example, [3, 4, 5] and references therein.

3 Compactified Extra Dimension

Now that we know how the hydrogen atom behaves in infinite spaces, we can investigate what happens if one of the dimensions is compactified, i.e. it has the topology of a circle at a very small radius (figure 3). For this purpose, we use what is known as the method of images, which will be explained in the following paragraph. In section 3.2 we will argue that if the potential is weak, a lower bound is present in the energy spectrum. For small compactification radii, we can treat the compactified dimension as a perturbation. We will solve the Schrödinger equation and give energy eigenvalues in second order perturbation theory.

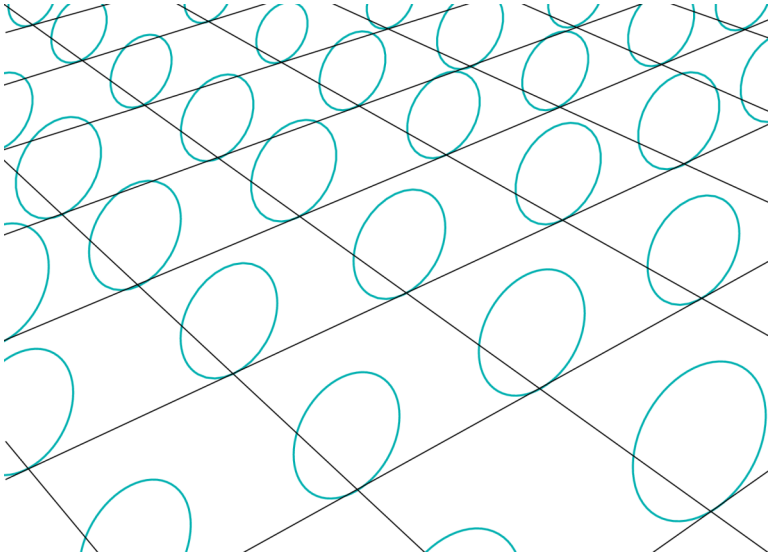


Figure 3: An additional curled-up dimension is represented by a circle located at each point of the three-dimensional space.

Finally, we will use a trial-function to show that if the attractive potential is sufficiently strong, then the energy spectrum is continuous and extends from zero to minus infinity. Thus, as in the strong infinite case, the hydrogen atom is not stable.

3.1 Method of images

Method of images is the best way to tackle the calculations in problems concerning extra compactified dimensions. What one in principle does, is to unroll the curled-up dimension to get an infinite space repeating itself with a period of $2\pi R$, where R is the compactification radius. To calculate the force that one particle 'feels' from some other particle,

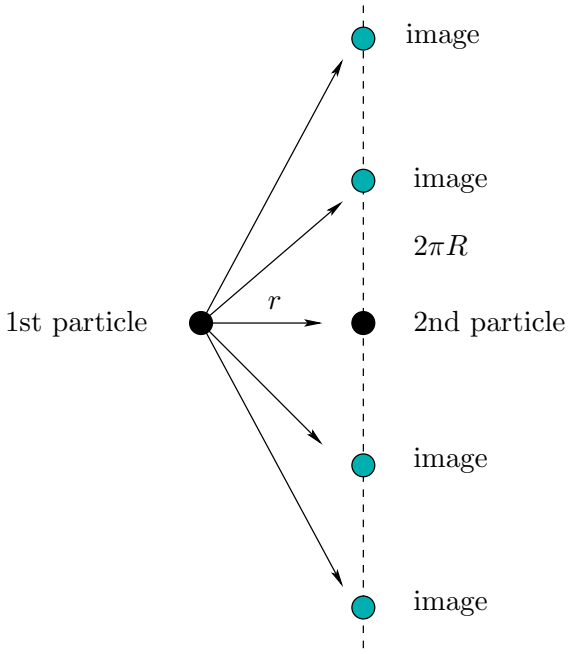


Figure 4: To calculate the force between two particles, the method of images makes it easier. The basic idea is to unroll the curled-up dimension to get an infinite space that repeats itself with a period of $2\pi R$. (Redrawn according to [29].)

we just have to sum up the expressions for the force it 'feels' from all the 'images' of the other particle (figure 4).

3.2 Lower bound in the energy spectrum

In this paragraph we shall argue that there is a lower bound present in the energy spectrum, i.e. there can be no states with energy lower than a certain value E_B. We will base our proof on the result of paragraph 2.1.2. There, we have found that if the parameter ν^2, corresponding to the strength of the potential, takes values from the interval $0 \leq \nu^2 < 1$ (the weak case), then the system has no negative energy levels.

First, suppose that we are given two systems described by the Hamiltonians $\hat{H}_1 = T + V_1$, $\hat{H}_2 = T + V_2$. Let the mean values of energy in some state $|\psi\rangle$ be $\langle\psi|\hat{H}_1|\psi\rangle = E_1$ and $\langle\psi|\hat{H}_2|\psi\rangle = E_2$, respectively. If $V_1 > V_2$, then

$$E_1 - E_2 = \langle\psi|\hat{H}_1|\psi\rangle - \langle\psi|\hat{H}_2|\psi\rangle = \langle\psi|V_1|\psi\rangle - \langle\psi|V_2|\psi\rangle > 0, \tag{3.1}$$

since

$$\langle\psi|V_1|\psi\rangle - \langle\psi|V_2|\psi\rangle = \langle\psi|V_1 - V_2|\psi\rangle = \int \psi^*(V_1 - V_2)\psi = \int |\psi|^2(V_1 - V_2) > 0. \quad (3.2)$$

Thus, we have $E_1 > E_2$ and we can apply this result to show that there exists a lower bound in the energy spectrum.

Let us consider an artificial system, which is periodic in the w-direction. For any value of $n \in \mathbb{Z}$, the potential V_{art} of this system is for w from the interval $|w - 2\pi Rn| \leq \pi R$ given by

$$V_{art}(r, w) = -\frac{e^2}{r^2 + (w - 2\pi Rn)^2}, \quad (3.3)$$

where the coordinate r stands, unlike section 2, only for $(x^2 + y^2 + z^2)^{1/2}$, i.e. it is the ordinary distance in our three-dimensional space. In other words, we "cut out" a part of the shape of the potential, corresponding to the hydrogen atom in four-dimensional infinite space (equation (2.5)), and make its periodic copies, putting them successively one behind another in the w-direction (figure 5). The cutout has the width of $2\pi R$ and is symmetric with respect to the plane perpendicular to the w-axis and running through the origin. This potential, of course, does not correspond to the real potential (equation (3.14)) of the compactified space (the dot-dashed curve in the figure), which results from including the potential

$$V_{anti}(r, w) = -\sum_{n\neq 0} \frac{e^2}{r^2 + (w - 2\pi Rn)^2}. \quad (3.4)$$

of the other images of the nucleus. In the following, we will refer to the inclusion of the potential $V_{anti}(r, w)$ to the "single-nucleus potential" as *antiscreening*.

Now we come to the crucial point. As we know that there are no states with negative energy in the weak infinite case, we make a reasonable assumption that our artificial system, resulting from copying the parts of the original one, does not have negative energy levels either. The main idea of the remaining proof, schematically illustrated in figure 5, is as follows: The antiscreening at the origin is $-e^2/12R^2$ (see (3.12)) and increases towards the boundary of the individual regions, where it reaches its maximum

value [3]

$$E_M = -\frac{e^2}{R^2}\left(\frac{1}{4} - \frac{1}{\pi^2}\right). \tag{3.5}$$

To calculate this value, we take the limit $r \to 0, w \to \pm\pi R$ in equation (3.14) (next paragraph). Consequently, if we shift down the shape of the potential energy corresponding to the artificial system about the value of E_M, the resulting potential will lie below the real potential (in the figure the dot-dashed line). Thus, according to (3.2), the real system can have no energy levels below E_M. Using similar arguments it can be proved that the

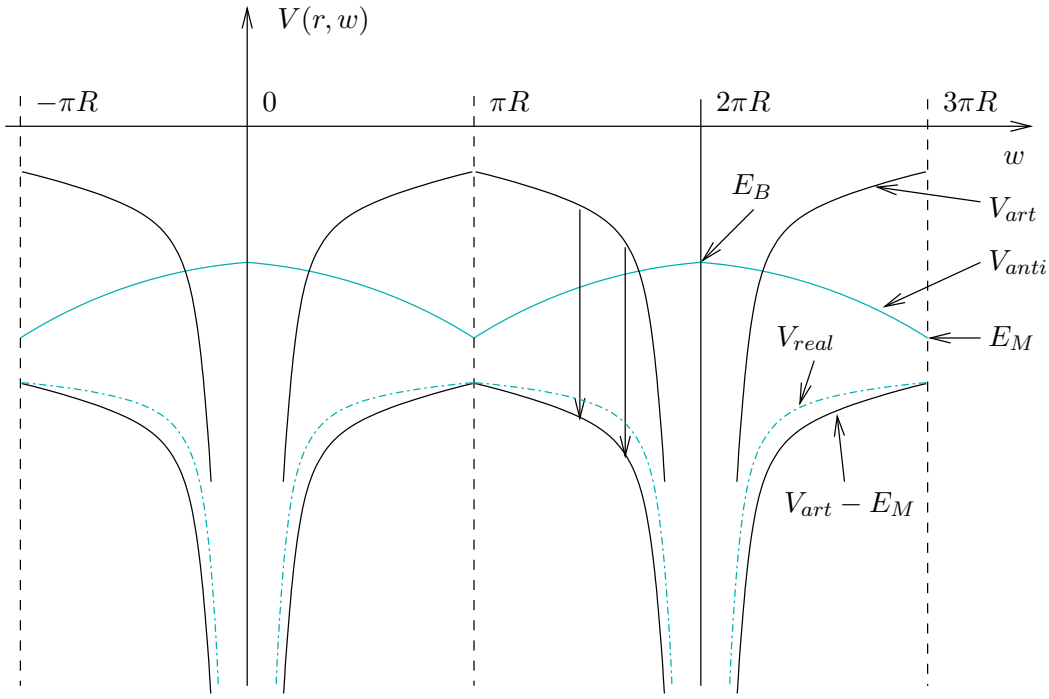


Figure 5: Shape of the potential energy, corresponding to the artificial system (upper solid line) and the real system (dot-dashed line) V_{art}, V_{real}, respectively. The borders between individual cutouts of the infinite 4-dim potential are indicated by vertical dashed lines. The bottom solid curve represents the artificial potential V_{art} shifted about the value of E_M, which corresponds to the maximum antiscreening. Because the maximum antiscreening is at the boundaries, the artificial potential V_{art} matches the real potential V_{real} at the points $w = (2n+1)\pi R$, where $n \in \mathbb{Z}$. However, as we depart further from these planes, the difference between these two potentials becomes larger and reaches its maximum value $E_B - E_M$ at the center (points $w = 2\pi nR$, $n \in \mathbb{Z}$).

[3] As seen from (3.4), the maximum value lies on the w-axis. For, as we depart from the w-axis, all terms in (3.4) are decreasing and the antiscreening becomes smaller.

lower bound lies actually higher and corresponds to the level of antiscreening at the origin (see (3.12)). We conclude that

$$E_B = -e^2/12R^2. \tag{3.6}$$

In the following sections we shall calculate, by means of perturbation theory, the energy levels of the hydrogen atom (in the weak case) and we will see that the energy of the ground state, i.e. the lowest level, does not violate this bound.

3.3 Solution of the radial equation

We consider two charged particles in a space with the ordinary three extended spatial dimensions and one extra compactified dimension with radius R. Let us again start with the Schrödinger equation:

$$\left(-\frac{1}{\kappa}\nabla^2 + V(r,w)\right)\Psi(\boldsymbol{r},w) = E\,\Psi(\boldsymbol{r},w). \tag{3.7}$$

Since we now have a system with "cylindrical" symmetry, it is appropriate to introduce the following coordinate system:

$$\begin{aligned} x &= r\sin\theta\cos\phi, \\ y &= r\sin\theta\sin\phi, \\ z &= r\cos\theta, \\ w &= w, \end{aligned} \tag{3.8}$$

where w is the coordinate along the compactified spatial dimension (figure 6). The Laplacian operator is now given by (see appendix A)

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{l}^2}{r^2} + \frac{\partial^2}{\partial w^2}\,, \tag{3.9}$$

where

$$\hat{l}^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{3.10}$$

is the three-dimensional angular momentum operator.

Potential energy consist of two terms. The first one is due to the interaction of the

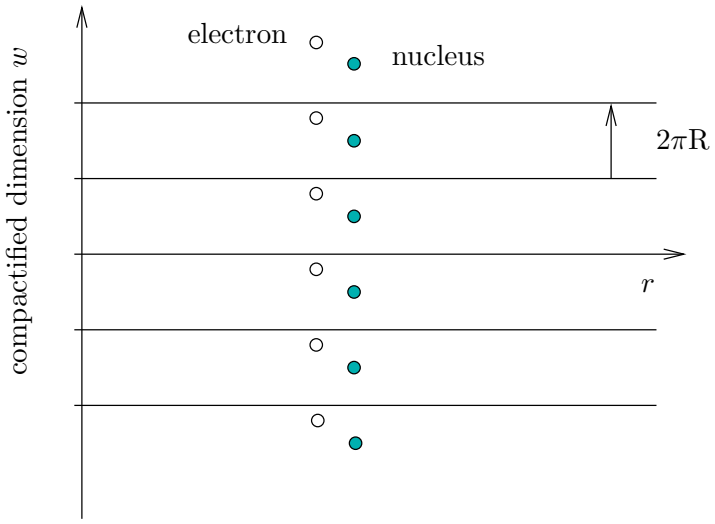


Figure 6: Coordinate system: the protons are $2\pi R$ apart on the w-axis and r is the distance to the w-axis.

electron with the nucleus and its images:

$$V(r,w) = -\sum_{n=-\infty}^{\infty} \frac{e^2}{r^2 + (w - 2\pi Rn)^2}\,. \tag{3.11}$$

The second part, which is merely a constant factor shifting the level of the potential energy, is the interaction energy between electron and its images

$$V' = \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{e^2}{(2\pi Rn)^2} = \frac{2e^2}{(2\pi R)^2} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{2e^2}{(2\pi R)^2}\zeta(2) = \frac{e^2}{12R^2}, \tag{3.12}$$

where $\zeta(2) = \pi^2/6$ is the Riemann zeta function. We have, of course, omitted the term with $n = 0$ representing self interaction. Thus, our Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \frac{1}{\kappa}\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{l}^2}{r^2} + \frac{\partial^2}{\partial w^2}\right) - \sum_{n=-\infty}^{\infty} \frac{e^2}{r^2 + (w - 2\pi Rn)^2}. \tag{3.13}$$

Expression (3.11) can be summed up by using the calculus of residues (see appendix C). The result is:

$$V(r,w) = -\frac{e^2}{2rR}\,\frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)}. \tag{3.14}$$

Before continuing further, let us have a look at the limiting cases. For $r \ll R$ and $w \ll R$ the lowest-order term is

$$V(r,w) \approx -\frac{e^2}{r^2+w^2}, \tag{3.15}$$

and we can see that the behaviour of the potential around the origin is the same as in the uncompactified case. This is also the reason why the result of the trial function treatment (paragraph 3.7) is the same as in the infinite case. On the other hand, if $r \gg R$, we get

$$V(r,w) \approx -\frac{e^2}{2rR}, \tag{3.16}$$

which means that the usual three-dimensional behaviour is restored and (3.14) turns into the common $1/r$ potential, but with a factor given by the volume R of the extra dimension. The behaviour of the field is shown schematically in Figure 7.

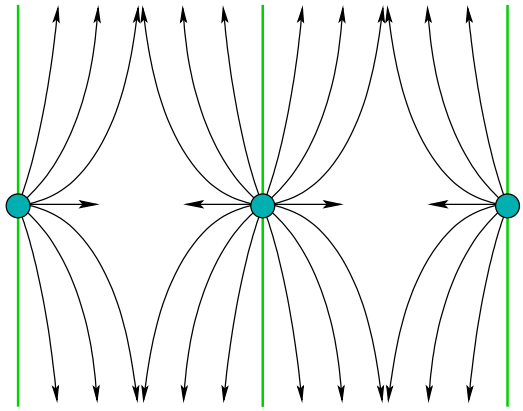


Figure 7: This picture schematically shows how electromagnetic force behaves in $\mathbb{R}^1 \times S^1$. At longer distances the field becomes parallel and we get the usual three-dimensional behaviour. It is obvious that the compactification radius has to be small. Otherwise, we would observe deviations in the behaviour of electromagnetic force at short distances.

Because the potential is periodic in the w-direction, it can be expanded in a Fourier series:

$$V(r,w) = \sum_n v_n(r)\, e^{inw/R}. \tag{3.17}$$

The expansion coefficients are given by (appendix D):

$$v_n(r) = -\frac{e^2}{2rR}\, e^{-|n|r/R} \tag{3.18}$$

and we have

$$V(r,w) = -\frac{e^2}{2rR} \sum_{n=-\infty}^{\infty} e^{-|n|r/R}\, e^{inw/R}. \tag{3.19}$$

Similarly, the wave function must be a periodic function with a period of $2\pi R$. This implies that it can be written as

$$\Psi(\boldsymbol{r}, w) = \sum_n \psi_n(\boldsymbol{r})\, e^{inw/R}\,. \tag{3.20}$$

Inserting (3.17) and (3.20) into the Schrödinger equation (3.7), we obtain:

$$\nabla^2 \left(\sum_n \psi_n(\boldsymbol{r})\, e^{inw/R} \right) + \kappa \left(E - \sum_m v_m(r)\, e^{imw/R} \right) \sum_n \psi_n(\boldsymbol{r})\, e^{inw/R} = 0. \tag{3.21}$$

Furthermore, using the explicit form of Laplacian (3.9) and letting the w-part act on the wave function, equation (3.21) becomes

$$\sum_n e^{inw/R} \left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{\hat{l}^2}{r^2} - \frac{n^2}{R^2} + \kappa \left(E - \sum_m v_m(r)\, e^{imw/R} \right) \right] \psi_n(\boldsymbol{r}) = 0. \tag{3.22}$$

Making the Ansatz $\psi_n(\boldsymbol{r}) = R_n(r) Y_n(\theta, \phi)$, we get

$$\sum_n e^{inw/R} \left[Y_n(\theta,\phi) \left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{n^2}{R^2} + \kappa E - \kappa \sum_m v_m(r)\, e^{imw/R} \right) R_n(r) + \frac{R_n(r)}{r^2}\, \hat{l}^2\, Y_n(\theta,\phi) \right] = 0 \tag{3.23}$$

For spherically symmetric solutions, we can separate the angular variables. We obtain the following radial equation:

$$\sum_n e^{inw/R} \left[\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{n^2}{R^2} + \kappa E - \kappa \sum_m v_m(r)\, e^{imw/R} \right] R_n(r) = 0. \tag{3.24}$$

Multiplying (3.24) by $\exp(-ikw/R)$ and integrating over the range $0 \le w \le 2\pi R$, to-

gether with

$$\int_0^{2\pi R} e^{i(n-k)w/R}\mathrm{d}w = 2\pi R\,\delta_{n,k} \quad \text{and} \quad \int_0^{2\pi R} e^{i(n-k+m)w/R}\mathrm{d}w = 2\pi R\,\delta_{n-k,m} \tag{3.25}$$

yields

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r} - \frac{k^2}{R^2} + \kappa E\right) R_k(r) - \kappa \sum_n v_{k-n}(r) R_n(r) = 0. \tag{3.26}$$

This is a set of coupled differential equations. Each Fourier component of the radial wave function is determined by an infinite set of Fourier components of the potential. To find the energy eigenvalues, we apply perturbation theory. We split the Hamiltonian into two time-independent parts

$$\hat{H} = \hat{H}_0 + \hat{H}_p, \tag{3.27}$$

where the unperturbed system is considered to be described by the Hamiltonian

$$\hat{H}_0 = \frac{1}{\kappa}\nabla^2 + v_0(r), \tag{3.28}$$

i.e. we take into account only the zero-th component of the Fourier expansion of the potential. This system can be solved exactly. The remaining terms of the Fourier series are regarded as a perturbation:

$$\hat{H}_p = -\frac{e^2}{2rR} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} e^{-|n|r/R} e^{-inw/R}. \tag{3.29}$$

3.4 Discrete spectrum of the unperturbed Hamiltonian

Considering only the unperturbed system, equation (3.26) takes the following simple form:

$$\frac{\mathrm{d}^2 R_k^{(0)}(r)}{\mathrm{d}r^2} + \frac{\mathrm{d}R_k^{(0)}(r)}{\mathrm{d}r} + \left(\kappa E - \frac{k^2}{R^2} + \frac{\kappa e^2}{2rR}\right) R_k^{(0)}(r) = 0. \tag{3.30}$$

This is nothing but the radial part of the three-dimensional Schrödinger equation for the hydrogen atom - apart from a constant factor $-k^2/R^2$. Through the use of dimensionless abbreviations

$$\rho = \alpha r \qquad \text{with} \qquad \alpha^2 = 4\left(-\kappa E + \frac{k^2}{R^2}\right), \quad E < 0, \tag{3.31}$$

we obtain

$$\frac{\mathrm{d}^2\chi_k^{(0)}(\rho)}{\mathrm{d}\rho^2} + \frac{2}{\rho}\frac{\mathrm{d}\chi_k^{(0)}(\rho)}{\mathrm{d}\rho} + \left(-\frac{1}{4} + \frac{\sigma}{\rho}\right)\chi_k^{(0)}(\rho) = 0\,, \tag{3.32}$$

where we have used the notation

$$\sigma = \frac{1}{\alpha}\frac{\kappa e^2}{2R} \quad \text{and} \quad \chi_k^{(0)}(\rho) = R_k^{(0)}(\rho/\alpha). \tag{3.33}$$

A comparison of equation (3.32) with the associated Laguerre differential equation yields the following solution:

$$\chi_k^{(0)}(\rho) = e^{-\rho/2}\, L^1_{\sigma-1}(\rho)\,. \tag{3.34}$$

We must now restrict the parameter σ by requiring it to be a positive integer $\mathfrak{n}$:[4]

$$\sigma \equiv \mathfrak{n} = \frac{1}{\alpha}\frac{\kappa e^2}{2R}. \tag{3.35}$$

This is necessary, since, as we know from the standard solution of the hydrogen atom, the Laguerre function of non-integral $\mathfrak{n}$ would diverge as $\rho^{\mathfrak{n}}e^{\rho}$. This restriction on σ, imposed by our boundary condition, leads to the quantization of energy. From the previous equation it follows that

$$\alpha_{\mathfrak{n}} = \frac{\kappa e^2}{2R}\frac{1}{\mathfrak{n}}. \tag{3.36}$$

Substituting (3.36) into (3.31) gives us the spectrum of energy eigenvalues:

$$E^{(0)}_{\mathfrak{n},k} = -\frac{\kappa e^4}{16R^2}\frac{1}{\mathfrak{n}^2} + \frac{1}{\kappa}\frac{k^2}{R^2}, \quad \mathfrak{n} \in \mathbb{N}, k \in \mathbb{Z}. \tag{3.37}$$

Turning back to the radial function $R_k^{(0)}(r)$, the solution to (3.30) is given by

$$R_k^{(0)}(r) = e^{-\alpha_{\mathfrak{n}} r/2}\, L^1_{\mathfrak{n}-1}(\alpha_{\mathfrak{n}} r)\,. \tag{3.38}$$

Thus, the spherically symmetric solutions corresponding to the unperturbed Hamiltonian are given by

$$\psi^{(0)}_{\mathfrak{n},k}(\boldsymbol{r}, w) = \mathcal{N}(\mathfrak{n}, k)\, L^1_{\mathfrak{n}-1}(\alpha_{\mathfrak{n}} r)\, e^{-\alpha_{\mathfrak{n}} r/2}\, e^{ikw/R}, \tag{3.39}$$

[4]The quantum number $\mathfrak{n}$ will be typeset with another font in order to avoid confusion with n, used for labeling the Fourier components.

with $\mathcal{N}(\mathfrak{n}, k)$ being a normalization factor:

$$\mathcal{N}(\mathfrak{n}, k) = \frac{\alpha_{\mathfrak{n}}^{3/2}}{\sqrt{8\pi^2 R}} \frac{1}{\mathfrak{n}}. \tag{3.40}$$

Before we turn our attention to perturbation theory, we check whether the expected energy bound $E_B = -e^2/12R^2$ (paragraph 3.2) is satisfied. Taking the ratio of the energy of the lowest state, the ground state $E_{1,0}^{(0)}$, to the bound E_B (equation (3.6)) yields

$$\frac{E_{1,0}^{(0)}}{E_B} = \frac{3}{4}\kappa e^2 = \frac{3}{4}(1 - \nu^2). \tag{3.41}$$

Since $0 \leq \nu^2 < 1$ in the weak case (section 2), the ground state energy lies within three-fourths of the bound E_B, depending on the strength of the potential.

3.5 Perturbation theory

In this section we shall use the time-independent perturbation theory to calculate the corrections to the energy eigenvalues.

3.5.1 First-order correction

To determine the first-order correction to the eigenvalue $E_{\mathfrak{n},k}^{(0)}$, we need to calculate

$$E_{\mathfrak{n},k}^{(1)} = \langle \psi_{\mathfrak{n},k}^{(0)} | \hat{H}_p | \psi_{\mathfrak{n},k}^{(0)} \rangle. \tag{3.42}$$

The insertion of (3.29) and (3.39) yields the energy to first order perturbation:

$$E_{\mathfrak{n},k}^{(1)} = -\frac{4\pi e^2}{2R} \mathcal{N}^2(\mathfrak{n}, k) \int_0^\infty r \left[L_{\mathfrak{n}-1}^1 (\alpha_{\mathfrak{n}}) \right]^2 \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} e^{-(|n|/R + \alpha_{\mathfrak{n}})r} \mathrm{d}r \int_{-\pi R}^{\pi R} e^{-inw/R} \, \mathrm{d}w, \tag{3.43}$$

where the integration over θ, ϕ has been carried out. However, inspection shows that the integral over w produces zero (since $n \neq 0$) and the first-order correction always vanishes. Thus, we have to use second order perturbation theory.

3.5.2 Second-order correction

In the second approximation, perturbation theory gives:

$$E^{(2)}_{\mathfrak{n},k} = \sum_{(\mathfrak{n}',k')\neq(\mathfrak{n},k)} \frac{\left|\langle\psi^{(0)}_{\mathfrak{n}',k'}|\hat{H}_p|\psi^{(0)}_{\mathfrak{n},k}\rangle\right|^2}{E^{(0)}_{\mathfrak{n},k} - E^{(0)}_{\mathfrak{n}',k'}}. \tag{3.44}$$

Using (3.29) and (3.39), we find for the matrix elements in (3.44):

$$\begin{aligned}\langle\psi^{(0)}_{\mathfrak{n}',k'}|\hat{H}_p|\psi^{(0)}_{\mathfrak{n},k}\rangle &= -(2\pi e)^2\mathcal{N}(\mathfrak{n})\mathcal{N}(\mathfrak{n}') \\ &\times \int_0^\infty r e^{-\left(\frac{|k-k'|}{R}+\frac{\alpha_\mathfrak{n}+\alpha_{\mathfrak{n}'}}{2}\right)r} L^1_{\mathfrak{n}-1}(\alpha_\mathfrak{n} r)\, L^1_{\mathfrak{n}'-1}(\alpha_{\mathfrak{n}'} r)\,\mathrm{d}r, \quad k\neq k', \end{aligned}\tag{3.45}$$

where we have used the fact that

$$\begin{aligned}&\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{|n|r/R}\int_{-\pi R}^{\pi R} e^{i(n+k-k')w/R}\mathrm{d}w \\ &\qquad = 2\pi R\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} e^{|n|r/R}\delta(n+k-k') = 2\pi R\, e^{|k-k'|r/R}, \quad k\neq k'. \end{aligned}\tag{3.46}$$

Evaluating the integral in (3.45), we obtain ([18], p. 858):

$$\begin{aligned}\langle\psi^{(0)}_{\mathfrak{n}',k'}|\hat{H}_p|\psi^{(0)}_{\mathfrak{n},k}\rangle &= -(2\pi e)^2\mathcal{N}(\mathfrak{n})\mathcal{N}(\mathfrak{n}')\frac{\mathfrak{n}+\mathfrak{n}'}{(\mathfrak{n}-1)!(\mathfrak{n}'-1)!}\frac{(b-\alpha_\mathfrak{n})^{\mathfrak{n}-1}(b-\alpha_{\mathfrak{n}'})^{\mathfrak{n}'-1}}{b^{\mathfrak{n}+\mathfrak{n}'}} \\ &\times F\left[-\mathfrak{n}+1,-\mathfrak{n}'+1,-\mathfrak{n}-\mathfrak{n}'+1;\frac{b(b-\alpha_\mathfrak{n}-\alpha_{\mathfrak{n}'})}{(b-\alpha_\mathfrak{n})(b-\alpha_{\mathfrak{n}'})}\right], \quad k\neq k', \end{aligned}\tag{3.47}$$

where $b \equiv \frac{|k-k'|}{R} + \frac{\alpha_1+\alpha_{\mathfrak{n}'}}{2}$. Obviously, it is not easy to evaluate the sum in (3.44). For the ground state $(\mathfrak{n},k) = (1,0)$, however, (3.44) reduces to

$$E^{(2)}_{1,0} = \sum_{(\mathfrak{n}',k')\neq(1,0)} \frac{\left|\langle\psi^{(0)}_{\mathfrak{n}',k'}|\hat{H}_p|\psi^{(0)}_{1,0}\rangle\right|^2}{E^{(0)}_{1,0} - E^{(0)}_{\mathfrak{n}',k'}}. \tag{3.48}$$

We note that all terms in (3.48) are negative, since $E^{(0)}_{\mathfrak{n},k}$ corresponds to the lowest value of energy. Thus, the second-order correction to the energy of the ground state is always

negative. Thus, equation (3.45) has a simpler form

$$\langle\psi^{(0)}_{\mathfrak{n}',k'}|\hat{H}_p|\psi^{(0)}_{1,0}\rangle = -2(2\pi e)^2\mathcal{N}(1)\mathcal{N}(\mathfrak{n}')\int_0^\infty re^{-\left(\frac{k'}{R}+\frac{\alpha_1+\alpha_{\mathfrak{n}'}}{2}\right)r}L^1_{\mathfrak{n}'-1}\left(\alpha_{\mathfrak{n}'}r\right)\mathrm{d}r, \tag{3.49}$$

where $k' \in \mathbb{N}$. Expressing the Laguerre polynomials in terms of confluent hypergeometric functions ([7], p. 857)

$$L^1_{\mathfrak{n}'-1}\left(\alpha_{\mathfrak{n}'}r\right) = \mathfrak{n}'\,F(-\mathfrak{n}'+1,2;\alpha_{\mathfrak{n}'}r), \tag{3.50}$$

enables us to make use of the following integral ([24], p. 666):

$$\int_0^\infty e^{-bx}x^{\gamma+p-1}\,F(a,\gamma;kx)\mathrm{d}x = (-1)^p\,\Gamma(\gamma)\frac{\mathrm{d}^p}{\mathrm{d}b^p}[b^{a-\gamma}(b-c)^{-a}], \tag{3.51}$$

with $b \equiv k'/R + (\alpha_1+\alpha_{\mathfrak{n}'})/2$, $a \equiv -\mathfrak{n}'+1$, $\gamma \equiv 2$, $p \equiv 0$ and $c \equiv \alpha_{\mathfrak{n}'}$. Thus, (3.49) can be expressed in terms of elementary functions:

$$\begin{aligned}\langle\psi^{(0)}_{\mathfrak{n}',k'}|\hat{H}_p|\psi^{(0)}_{1,0}\rangle &= -2(2\pi e)^2\mathcal{N}(1)\mathcal{N}(\mathfrak{n}')\mathfrak{n}'b^{-\mathfrak{n}'-1}(b-\alpha_{\mathfrak{n}'})^{\mathfrak{n}'-1}\\ &= -2(2\pi e)^2\mathcal{N}(1)\mathcal{N}(\mathfrak{n}')\mathfrak{n}'R^2\left[\lambda^2\left(1+\frac{1}{n'}\right)+k'\right]^{-\mathfrak{n}'-1}\left[\lambda^2\left(1-\frac{1}{n'}\right)+k'\right]^{\mathfrak{n}'-1}\end{aligned} \tag{3.52}$$

where $\lambda^2 \equiv \kappa e^2$. From (3.40) it follows that

$$\mathcal{N}(1)\mathcal{N}(\mathfrak{n}') = \frac{\lambda^6}{8\pi^2R^4}\frac{1}{\mathfrak{n}'^{5/2}}. \tag{3.53}$$

Next, from (3.37) we have

$$E^{(0)}_{1,0} - E^{(0)}_{\mathfrak{n}',k'} = -\frac{1}{\kappa R^2}\left[\frac{\lambda^4}{16}\left(1-\frac{1}{\mathfrak{n}'^2}\right)+k'^2\right], \tag{3.54}$$

and thus, inserting (3.52), (3.53) and (3.54) into (3.48), we get finally

$$\begin{aligned}E^{(2)}_{1,0} &= \sum_{\mathfrak{n}'>1,k'>0}\frac{\left|\langle\psi^{(0)}_{\mathfrak{n}',k'}|\hat{H}_p|\psi^{(0)}_{1,0}\rangle\right|^2}{E^{(0)}_{1,0}-E^{(0)}_{\mathfrak{n}',k'}}\\ &= E^{(0)}_{1,0}\,\frac{4\lambda^{12}}{\pi^2}\sum_{\mathfrak{n}'>1,k'>0}\frac{1}{\mathfrak{n}'^3\left[\frac{\lambda^4}{16}\left(1-\frac{1}{\mathfrak{n}'^2}\right)+k'^2\right]}\frac{\left[\lambda^2\left(1-\frac{1}{n'}\right)+k'\right]^{\mathfrak{n}'-1}}{\left[\lambda^2\left(1+\frac{1}{n'}\right)+k'\right]^{\mathfrak{n}'+1}}.\end{aligned} \tag{3.55}$$

We now take the limit of small $\lambda^2 = \kappa e^2$, corresponding to a weak potential. Another way of interpreting this limit is the following. By equating the expression (3.36) for energy eigenvalues of the unperturbed Hamiltonian to the energy of the three-dimensional hydrogen atom

$$E_n = -\frac{m e_{3d}^4}{2\hbar^2}\frac{1}{n^2}, \tag{3.56}$$

we find that the four-dimensional charge is related to the three-dimensional by $e^2 = 2Re_{3D}^2$ and consequently:

$$\lambda^2 = 2R\kappa e_{3D}^2. \tag{3.57}$$

From the last equation we may develop the physical interpretation that small λ^2 corresponds to a limit of small compactification radii R. Taking this limit, (3.55) reduces to

$$E_{1,0}^{(2)} \approx E_{1,0}^{(0)}\ \frac{4\lambda^{12}}{\pi^2}\sum_{\mathfrak{n}'>1}\frac{1}{\mathfrak{n}'^3}\sum_{k'>0}\frac{1}{k'^4} = E_{1,0}^{(0)}\frac{4\lambda^{12}}{\pi^2}\left[\zeta(3)-1\right]\zeta(4), \tag{3.58}$$

where $\zeta(3)$ and $\zeta(4)$ are the Riemann zeta functions. [5] Since $\zeta(3) \approx 1.20$ and $\zeta(4) = \pi^4/90$, we have

$$E_{1,0}^{(2)} \approx 0.18\lambda^{12}E_{1,0}^{(0)}. \tag{3.59}$$

Taking again the ratio of the energy of the lowest state, the ground state $E_{1,0}$, to the bound E_B yields (cf. (3.41)):

$$\frac{E_{1,0}^{(0)} + E_{1,0}^{(2)}}{E_B} \approx 0.75 + 0.18\lambda^{12}. \tag{3.60}$$

Because $\lambda^2 = 1 - \nu^2$ lies within $\langle 0, 1)$ in the weak case, the ground state energy to second order also does not violate the energy bound $E_B = -e^2/12R^2$ (paragraph 3.2).[6]

[5] $\zeta(3)$ is also called the Apéry's constant.

[6] Actually, we take the limit of small λ^2. Consequently, it makes no sense to think of $E_{1,0}^{(0)} + E_{1,0}^{(2)}/E_B \approx 0.75 + 0.18 = 0.93$ as an upper limit.

3.6 Green's function method

To calculate the higher-order corrections, the Green's function method is very useful. Here we limit ourselves to show how to use this method in our problem.

Let us return to equation (3.26) and rewrite it as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}-\frac{k^2}{R^2}+\kappa E+\frac{\kappa e^2}{2rR}\right)R_k(\boldsymbol{r})=\kappa\sum_{n\neq k}v_{k-n}(r)R_n(r), \tag{3.61}$$

where we have inserted $v_0(r)=-e^2/2rR$. Assuming the coefficients v_{k-n} to be small, we can write

$$R_k^{(1)}(r)=\kappa\sum_{n\neq k}\int g_l(r,r';b)v_{k-n}(r')R_k^{(0)}(r)r'^2\mathrm{d}r'. \tag{3.62}$$

Thus, it remains for us to find the corresponding Green's function. The defining equation for the radial part $g_l(r,r';b)$ of the Green's function is (cf. [30])

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2}+\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{2}{ar}-\frac{1}{b^2a^2}\right)g_l(r,r';b)=-\kappa\frac{\delta(r-r')}{rr'}, \tag{3.63}$$

where we have written

$$\frac{1}{b^2a^2}=-\kappa E+\frac{k^2}{R^2},\quad\text{with}\quad a=\frac{4R}{\kappa e^2}. \tag{3.64}$$

The imposed boundary conditions are

$$\lim_{r\to r'}rg_l(r,r';b)=0\quad\text{and}\quad rg_l(r,r';b)\in L^2(\mathbb{R}). \tag{3.65}$$

It is possible to solve the differential equation (3.63) explicitly. The solution (for a more general equation with $l\neq 0$) is given by [30]:

$$\begin{aligned}g_l(r,r';b)=&\frac{2m}{\hbar^2}\left(\frac{2}{ab}\right)^{2l+1}(rr')^l e^{-(r+r')/ab}\\&\times\sum_{m=0}^{\infty}\frac{m!}{(2l+1+m)!(l+1+m-b)}L_m^{2l+1}\left(\frac{2r}{ab}\right)L_m^{2l+1}\left(\frac{2r'}{ab}\right),\end{aligned} \tag{3.66}$$

where the functions L_m^{2l+1} are the associated Laguerre polynomials. Since we are now interested only in the spherically symmetric solutions ($l=0$), the radial Green's func-

tion (3.66) reads

$$g_l(r,r';b) \;=\; \frac{2m}{\hbar^2}\frac{2}{ab}e^{-(r+r')/ab} \times \sum_{m=0}^{\infty} \frac{m!}{(1+m)!(1+m-b)} L_m^1\left(\frac{2r}{ab}\right) L_m^1\left(\frac{2r'}{ab}\right). \quad (3.67)$$

It is worth noting (cf. [30], p. 98) that the poles $\mathfrak{n} \equiv b = 1+m$ of the Green's function correspond to the energy eigenvalues we have found. Indeed, (3.64) gives:

$$E_{\mathfrak{n},k}^{(0)} = -\frac{1}{\kappa a^2 \mathfrak{n}^2} + \frac{1}{\kappa}\frac{k^2}{R^2}, \quad (3.68)$$

which is the discrete energy spectrum (3.37) of the unperturbed Hamiltonian we have calculated. Now that we know the Green's function, we can insert (3.67) into the expression (3.62) and get the first-order correction to the unperturbed radial functions $R_k^{(0)}(r)$. The results will be presented elsewhere.

3.7 Trial function

In this paragraph we shall show, with the aid of a reasonable trial function, that the energy of a system described by the Hamiltonian (3.13) can take negative eigenvalues, which are arbitrarily large in absolute value, i.e. the continuous spectrum is unbounded from below. Namely, we want to show that

$$\lim_{\alpha\to\infty} \langle E\rangle = -\infty, \quad (3.69)$$

where

$$\langle E\rangle = \frac{\langle \psi_{trial}|\hat{H}|\psi_{trial}\rangle}{\langle \psi_{trial}|\psi_{trial}\rangle} \quad (3.70)$$

is the mean value of energy belonging to a system described by the trial function

$$\psi_{trial}(r,w) = \sum_{n=-\infty}^{\infty} r^p\,|w-2\pi Rn|^q\, e^{-\alpha\left[r^2+(w-2\pi Rn)^2\right]}, \quad (3.71)$$

which is real, always positive and, of course, periodic in the w-direction, as required. Here $p > -1/2$ and $q > 1/2$ are real numbers. As will be seen later, this conditions follows from the requirement that the integrals in (3.70) be convergent.
Let us split the trial function $\psi_{trial}(r,w)$ into two parts $\psi_1(r,w)$ and $\psi_2(r,w)$, where

$$\psi_1(r,w) \equiv r^p|w|^q e^{-\alpha\left(r^2+w^2\right)} \quad (3.72)$$

and

$$\psi_2^q(r,w) \equiv r^p e^{-\alpha r^2} \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |w - 2\pi Rn|^q \, e^{-\alpha(w-2\pi Rn)^2}. \tag{3.73}$$

The superscript q in (3.73) corresponds to the power of $|w-2\pi Rn|$. [7] In the following, we shall see that for $\alpha(\pi R)^2 \gg 1$ the chief part in the expressions present in (3.70) is played by terms containing only $\psi_1(r,w)$. The remaining terms will be shown to be exponentially small. Hence, we can write:

$$\langle E\rangle = \frac{\langle\psi_{trial}|\hat{H}|\psi_{trial}\rangle}{\langle\psi_{trial}|\psi_{trial}\rangle} = \frac{\langle\psi_1|\hat{H}|\psi_1\rangle + R(\alpha)}{\langle\psi_1|\psi_1\rangle + R(\alpha)}, \tag{3.74}$$

where in the limit $\alpha \to \infty$ the remainder $R(\alpha)$ vanishes.

First of all, we note that the sum present in (3.73) has, for $w \in \langle -\pi R, \pi R\rangle$, an upper bound given by

$$0 \leq \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |w - 2\pi Rn|^q \, e^{-\alpha(w-2\pi Rn)^2} \leq Ce^{-\alpha(2\pi R)^2}, \tag{3.75}$$

where C is a finite positive number. With zero, as an obvious lower bound, it is clear that this series must converge. Indeed, for $w \in \langle -\pi R, \pi R\rangle$, we may write

$$\begin{aligned}
&\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} |w - 2\pi Rn|^q \, e^{-\alpha(w-2\pi Rn)^2} \\
&\quad\leq \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} (\pi R)^q (2|n|+1)^q e^{-\alpha(\pi R)^2(2|n|-1)^2} = 2\sum_{n>0} (\pi R)^q (2n+1)^q e^{-\alpha(\pi R)^2(2n-1)^2} \\
&\quad= e^{-\alpha(2\pi R)^2} 2\sum_{n>0} (\pi R)^q (2n+1)^q e^{-\alpha(\pi R)^2\left[(2n+1)^2-4(2n+1)\right]},
\end{aligned} \tag{3.76}$$

where the exponent has been rewritten as $(2n-1)^2 = (2n+1-2)^2 = (2n+1)^2 - 4(2n+1) + 4$. The last sum in (3.76) is a convergent series, which can be easily verified. For instance, by using the l'Hôpital's rule for indeterminate forms, the Cauchy ratio test gives us

$$\lim_{n\to\infty} = \frac{a_{n+1}}{a_n} = 0, \tag{3.77}$$

[7] Later, we will make use of this notation.

which implies convergence. Thus, we conclude that for w taking values from $\langle -\pi R, \pi R\rangle$ it holds:

$$0 \leq \psi_2^q(r,w) \leq \phi(r), \tag{3.78}$$

where

$$\phi(r) \equiv Ce^{-\alpha(2\pi R)^2} r^p e^{-\alpha r^2}, \tag{3.79}$$

with C being a finite positive number.

3.7.1 Normalization

Using this result, we now proceed in the spirit of paragraph 2.2. First, we shall calculate the normalization (recall that ψ_1 and ψ_2 are real functions):

$$\langle \psi_{trial}|\psi_{trial}\rangle = \langle \psi_1|\psi_1\rangle + 2\langle \psi_1|\psi_2\rangle + \langle \psi_2|\psi_2\rangle. \tag{3.80}$$

Using (3.72), the first term is given by (we drop the pre-factors resulting from the integration over the angular variables):

$$\begin{aligned}\langle \psi_1|\psi_1\rangle = \int_0^\infty r^{2p+2}e^{-2\alpha r^2}\mathrm{d}r \int_{-\pi R}^{\pi R} w^{2q}e^{-2\alpha w^2}\mathrm{d}w \\ = 2\int_0^\infty r^{2p+2}e^{-2\alpha r^2}\mathrm{d}r \int_0^\infty w^{2q}e^{-2\alpha w^2}\mathrm{d}w - R(\alpha),\end{aligned} \tag{3.81}$$

where we extended the ranges of integration over z to infinity, for the integrand is essentially zero when z departs appreciably from the origin. More precisely, the remainder

$$R(\alpha) = 2\int_{\pi R}^\infty w^{2q}e^{-2\alpha w^2}\mathrm{d}w, \tag{3.82}$$

resulting from changing the ranges of integration, is an exponentially small quantity (see appendix E) and thus, in the limit $\alpha \to \infty$, does not contribute. Making use of the formula [8]

$$\int_0^\infty x^\nu e^{-ax^2} = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{2a^{\frac{\nu+1}{2}}}\mathrm{d}x, \quad a>0, \quad \nu > -1\,, \tag{3.83}$$

[8] This formula can be easily verified substituting $x^2 = t$ and applying the definition of the gamma function.

integral (3.81) becomes

$$\langle\psi_1|\psi_1\rangle = 2\frac{\Gamma\left(\frac{2p+3}{2}\right)}{2(2\alpha)^{\frac{2p+3}{2}}}\frac{\Gamma\left(\frac{2q+1}{2}\right)}{2(2\alpha)^{\frac{2q+1}{2}}} - R(\alpha). \tag{3.84}$$

Hence, it remains for us to show that the remaining terms in (3.80) are exponentially small quantities. Employing (3.78), we get the inequality

$$\begin{aligned} 0 \leq \langle\psi_1|\psi_2\rangle \leq \langle\psi_1|\phi\rangle &= Ce^{-\alpha(2\pi R)^2}\int_0^\infty r^{2p+2}e^{-2\alpha r^2}\mathrm{d}r\int_{-\pi R}^{\pi R}|w|^q e^{-\alpha w^2}\mathrm{d}w \\ &\leq 2Ce^{-\alpha(2\pi R)^2}\int_0^\infty r^{2p+2}e^{-2\alpha r^2}\mathrm{d}r\int_0^\infty |w|^q e^{-\alpha w^2}\mathrm{d}w - R(\alpha) \\ &= Ce^{-\alpha(2\pi R)^2}\frac{\Gamma\left(\frac{2p+3}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2(2\alpha)^{p+q/2+2}} - R(\alpha), \end{aligned} \tag{3.85}$$

where we again used the formula (3.83). Finally, the last term in (3.80) is

$$\begin{aligned} &0 \leq \langle\psi_2|\psi_2\rangle \leq \langle\phi|\phi\rangle \\ &= C^2e^{-2\alpha(2\pi R)^2}\int_0^\infty r^{2p+2}e^{-2\alpha r^2}\mathrm{d}r\int_{-\pi R}^{\pi R}\mathrm{d}w = 2\pi R\,C^2e^{-2\alpha(2\pi R)^2}\frac{\Gamma\left(\frac{2p+3}{2}\right)}{2(2\alpha)^{\frac{2p+1}{2}}}. \end{aligned} \tag{3.86}$$

Summing up our results, we have

$$\langle\psi_{trial}|\psi_{trial}\rangle = \frac{\Gamma(p+3/2)\Gamma(q+1/2)}{2(2\alpha)^{p+q+2}} + rem, \tag{3.87}$$

where "rem" stands for all exponentially small quantities.

3.7.2 Mean value of the kinetic energy

Having determined the normalization, we now proceed to calculate the next term appearing in (3.70), corresponding to kinetic energy:

$$\langle\psi_{trial}|\nabla^2|\psi_{trial}\rangle = \langle\psi_1|\nabla^2|\psi_1\rangle + \langle\psi_1|\nabla^2|\psi_2\rangle + \langle\psi_2|\nabla^2|\psi_1\rangle + \langle\psi_2|\nabla^2|\psi_2\rangle. \tag{3.88}$$

Using (3.72), (3.73) and (3.9) we find for the Laplacian of ψ_1 and ψ_2:

$$\nabla^2\psi_1(r,w) \;=\; \left[\frac{p(p+1)}{r^2} + \frac{q(q-1)}{w^2} - 4\alpha(p+q+2) + 4\alpha^2(r^2+w^2)\right]\psi_1(r,w) \tag{3.89}$$

and

$$\nabla^2\psi_2(r,w) = r^p e^{-\alpha r^2}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\Bigg[\frac{p(p+1)}{r^2}+\frac{q(q-1)}{(w-2\pi Rn)^2} - 4\alpha(p+q+2)+4\alpha^2r^2+4\alpha^2(w-2\pi Rn)^2\Bigg]|w-2\pi Rn|^q\, e^{-\alpha(w-2\pi Rn)^2}$$
$$= \left(\frac{p(p+1)}{r^2}-4\alpha(p+q+2)+4\alpha^2r^2\right)\psi_2(r,w)+q(q-1)\psi_2^{q-2}(r,w)+4\alpha^2\psi_2^{q+2}(r,w), \tag{3.90}$$

where we have used the notation introduced in (3.73). With (3.72) and (3.89), the first summand in (3.88) becomes

$$\langle\psi_1|\nabla^2|\psi_1\rangle = \int_0^{\infty}\int_{-\pi R}^{\pi R}\Bigg[\frac{p(p+1)}{r^2}+\frac{q(q-1)}{w^2} -4\alpha(p+q+2)+4\alpha^2(r^2+w^2)\Big]\, r^{2p+2}w^{2q}e^{-2\alpha(r^2+w^2)}\mathrm{d}w\,\mathrm{d}r. \tag{3.91}$$

Evaluation of this integral is much the same as that of $\langle\psi_1|\psi_1\rangle$. Again, we extend the ranges of integration over w to infinity, since the remainder $R(\alpha)$, resulting from changing the ranges of integration, is an exponentially small quantity and thus in the limit $\alpha\to\infty$ does not contribute (again, see appendix E). Integration by employing (3.83) gives finally:

$$\langle\psi_1|\nabla^2|\psi_1\rangle = -\frac{\Gamma\left(\frac{2p+1}{2}\right)\Gamma\left(\frac{2q-1}{2}\right)}{2^3(2\alpha)^{p+q+1}}\left[(2p+\frac{3}{2})(2q-1)+(2q-\frac{1}{2})(2p+1)\right]+rem. \tag{3.92}$$

As in section 3.7.1, it remains for us to show that the terms $\langle\psi_1|\nabla^2|\psi_2\rangle$, $\langle\psi_2|\nabla^2|\psi_1\rangle$ and $\langle\psi_2|\nabla^2|\psi_2\rangle$ are exponentially small quantities. Using (3.90), the first term is given by:

$$\langle\psi_1|\nabla^2|\psi_2\rangle = \int_0^{\infty}\int_{-\pi R}^{\pi R} r^2\psi_1(r,w)\Bigg[\left(\frac{p(p+1)}{r^2}-4\alpha(p+q+2)+4\alpha^2r^2\right)\psi_2(r,w) + q(q-1)\psi_2^{q-2}(r,w)+4\alpha^2\psi_2^{q+2}(r,w)\Bigg]\mathrm{d}w\,\mathrm{d}r. \tag{3.93}$$

Inserting (3.72), (3.73), together with using the result (3.78), we find out that the result of integration consists of terms similar to that appearing in (3.85). The same argument holds for $\langle\psi_2|\nabla^2|\psi_1\rangle$ and $\langle\psi_2|\nabla^2|\psi_2\rangle$.

3.7.3 Mean value of the potential energy

Let us move forward to calculate the last term appearing in (3.70) and corresponding to potential energy. Rewriting it as

$$\langle\psi_{trial}|\sum_{n=-\infty}^{\infty}\frac{1}{r^2+(w-2\pi Rn)^2}|\psi_{trial}\rangle = \langle\psi_{trial}|\frac{1}{r^2+w^2}|\psi_{trial}\rangle + \langle\psi_{trial}|\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\frac{1}{r^2+(w-2\pi Rn)^2}|\psi_{trial}\rangle, \quad (3.94)$$

the first term is given by

$$\langle\psi_{trial}|\frac{1}{r^2+w^2}|\psi_{trial}\rangle = \langle\psi_1|\frac{1}{r^2+w^2}|\psi_1\rangle + 2\langle\psi_1|\frac{1}{r^2+w^2}|\psi_2\rangle + \langle\psi_2|\frac{1}{r^2+w^2}|\psi_2\rangle. \quad (3.95)$$

Again, the main contribution is due to the first summand:

$$\langle\psi_1|\frac{1}{r^2+w^2}|\psi_1\rangle = \int_0^{\infty}\int_{-\pi R}^{\pi R}\frac{r^{2p+2}}{r^2+w^2}w^{2q}e^{-2\alpha(r^2+w^2)}\mathrm{d}w\mathrm{d}r. \quad (3.96)$$

Extending the limits of the w-integration to infinity, we carry out the integration by transforming to polar coordinates ρ, θ:

$$\begin{aligned}\langle\psi_1|\frac{1}{r^2+w^2}|\psi_1\rangle &= 2\int_0^{\infty}\int_0^{\infty}\frac{r^{2p+2}}{r^2+w^2}w^{2q}e^{-2\alpha(r^2+w^2)}\mathrm{d}r\mathrm{d}w - R(\alpha)\\ &= 2\int_0^{\infty}\rho^{2p+2q+1}e^{-2\alpha\rho^2}\mathrm{d}\rho\int_0^{\pi/2}\cos^{2p+2}\theta\sin^{2q}\theta\mathrm{d}\theta - R(\alpha). \quad (3.97)\end{aligned}$$

The remainder $R(\alpha)$ satisfies the following inequality

$$\begin{aligned}R(\alpha) &= 2\int_0^{\infty}\int_{\pi R}^{\infty}\frac{r^{2p+2}}{r^2+w^2}w^{2q}e^{-2\alpha(r^2+w^2)}\mathrm{d}r\mathrm{d}w \quad (3.98)\\ &\leq 2\int_0^{\infty}r^{2p}e^{-2\alpha r^2}\mathrm{d}r\int_{\pi R}^{\infty}w^{2q}e^{-2\alpha w^2}\mathrm{d}w, \quad (3.99)\end{aligned}$$

since $1/(r^2+w^2)\leq 1/r^2$. The integral over r is a finite number, the other is an exponentially small quantity (appendix E).
Thus, employing (3.83) and the fact that

$$2\int_0^{\pi/2}\cos^{2p+2}\theta\sin^{2q}\theta\mathrm{d}\,\theta \equiv B(p+3/2,q+1/2) = \frac{\Gamma(p+3/2)\Gamma(q+1/2)}{\Gamma(p+q+2)}. \quad (3.100)$$

is the Beta function gives for (3.97):

$$\langle\psi_1|\frac{1}{r^2+w^2}|\psi_1\rangle = \frac{\Gamma(p+3/2)\Gamma(q+1/2)}{\Gamma(p+q+2)}\frac{\Gamma(p+q+1)}{2(2\alpha)^{p+q+1}} - R(\alpha)$$
$$= \frac{1}{2(2\alpha)^{p+q+1}}\frac{\Gamma(p+3/2)\Gamma(q+1/2)}{p+q+1} - R(\alpha). \quad (3.101)$$

As before, (3.101) is the only surviving term in (3.95) as $\alpha \to \infty$. Transforming to polar coordinates, we can follow the reasoning of paragraph 3.7.1.

As for the second term in (3.94), we will make use of the fact that for $w \in \langle -\pi R, \pi R\rangle$ the sum

$$\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{r^2+(w-2\pi Rn)^2} \quad (3.102)$$

has an upper limit C_M given by

$$C_M = \frac{1}{R^2}\left(\frac{1}{4} - \frac{1}{\pi^2}\right) \quad (3.103)$$

and corresponding to the maximum level of antiscreening (see paragraph 3.2). Consequently,

$$0 \leq \langle\psi_{trial}| \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} \frac{1}{r^2+(w-2\pi Rn)^2}|\psi_{trial}\rangle \leq C_M\langle\psi_{trial}|\psi_{trial}\rangle. \quad (3.104)$$

Divided by the normalization (3.80), we get a finite contribution to $\langle E\rangle$, which is not essential in our discussion (recall that we want to show (3.69)). In the following, we will drop this factor.

3.7.4 Mean value of energy

Now we are ready to summarize our results. Inserting (3.87), (3.92) and (3.101) into (3.70), we obtain:

$$\langle E\rangle = \frac{2\alpha}{\kappa}\left(\frac{2p+3/2}{2p+1} + \frac{2q-1/2}{2q-1} - \frac{1-\nu^2}{p+q+1}\right), \quad (3.105)$$

where all exponentially decreasing terms (for large α) have been dropped and we have written, as usual, $\kappa e^2 = 1-\nu^2$. By inspection of (3.105) we find that for $\nu^2 < -1(1+\sqrt{2})$

the energy can take arbitrarily large negative values as $\alpha \to \infty$. On the other hand, if $0 \leq \nu^2 < 1$ (the weak case), the right-hand side of (3.105) is always positive and this is no longer possible. Although we have not showed that $\lim_{\alpha\to\infty}\langle E\rangle = -\infty$ for all $\nu^2 < 0$, we expect that this can be achieved through some other trial function. The reason for this belief results from the behaviour of potential near the origin - it is the same as in the infinite case (see (3.15)).

Thus, the fact that hydrogen atom is not stable in the strong case also applies to spaces with compactified extra dimension.

4 Summary and Outlook

In this work, we have studied the stability and spectra of the hydrogen atom in spaces with an extra dimension.

In the first part, we have investigated the case of an extra spatial dimension which is infinite. Separating the angular part, we have solved the corresponding Schrödinger equation. The radial equation was identified with the modified Bessel equation with a parameter ν^2, taking all real values. The asymptotic behaviour enabled us to immediately eliminate one of the solutions, the modified Bessel functions of the first kind. To discuss the second independent solution, the modified Bessel functions of the second kind, we have distinguished three cases, according to the value of ν^2 - repulsive, weak and strong.

In the repulsive case, the modified Bessel functions of the second kind are not square integrable and must therefore be rejected. We should mention that the modified Bessel functions of the first kind might actually correspond to some solution, since we have a repulsive potential. Nevertheless, we consider here only bound states. In the weak case, the Hermiticity of Hamiltonian is violated. Thus, we must conclude that there are no bound states in the repulsive and weak case. These results are in agreement with the treatment given in paper [19].

The treatment of the strong case is somewhat peculiar and has not yet been clarified. Namely, we have found that by fixing some (arbitrary) value of energy, it is possible to construct a sequence (discrete subset) of energy eigenvalues such that the corresponding wave functions form a set of orthonormal states. To find some interpretation of this result, one possibility might be a calculation of the probability current to show that the particle falls to the center. Such a result would correspond to the treatment given in paragraph 2.2, where we have argued, using a trial function, that there is a continuous energy spectrum extending to minus infinity. This conclusion is also presented in [19].

This issue, leading to a possibly different picture of the strong case, remains to be investigated. In any case, we can say that there is no stable hydrogen atom in four-dimensional spaces, where an infinite additional dimension is present.

After the discussion of infinite spaces, we have approached the study of spaces with an extra compactified dimension. We have argued that, as a result of compactification, a lower bound is present in the energy spectrum of the weak case. For small compactification radii, we can treat the additional dimension as a perturbation. We have solved the Schrödinger equation and found the energy eigenvalues to second order in perturbation theory. Energy eigenstates for the unperturbed Hamiltonian were also given.

In the strong case, we have shown that the result of the trial function treatment (paragraph 3.7) is the same as in the infinite case. This was to be anticipated, since the behaviour of the potential around the origin is the same as in the uncompactified case. Thus, the fact that the hydrogen atom is not stable in the strong case also applies to spaces with compactified extra dimension.

To summarize our results, we can say that the main goal has been achieved. Namely, we have found out how the energy spectrum is influenced in the presence of a curled-up dimension. It should be mentioned, however, that we have used the limit of small compactification radii for our calculations. The next step would be to give exact energy eigenvalues for the whole range of the weak case and to calculate, in particular, the maximum compactification radius that would correspond to the transition between the weak and the strong case. Thus, the most important issue to address is a more detailed discussion of the weak case, which was not yet fully understood.

A Differential Operators in Curvilinear Coordinates

In this appendix, we shall find a general expression for gradient and Laplace operator in N-dimensional, orthogonal curvilinear coordinates.

A.1 General expressions for gradient and Laplacian

In an arbitrary system of orthogonal curvilinear coordinates $u_1, u_2, \ldots u_N$, the element of length is

$$\mathrm{d}l^2 = \sum_{i=1}^{N} g_{ii}\mathrm{d}u_i^2 = h_1^2\mathrm{d}u_1^2 + h_2^2\mathrm{d}u_2^2 + \ldots + h_N^2\mathrm{d}u_N^2, \tag{A.1}$$

where the components of the metric $g_{ii} = h_i^2$ are functions of the coordinates. The volume element is given by

$$\mathrm{d}V = \prod_{i=1}^{N} \sqrt{g_{ii}}\mathrm{d}u_i = h_1 h_2 \ldots h_N \mathrm{d}u_1 \mathrm{d}u_2 \ldots \mathrm{d}u_N. \tag{A.2}$$

As we shall see below, the general expression for the gradient of a scalar function f has the form

$$\nabla = \left(\frac{1}{h_1}\frac{\partial f}{\partial u_1}, \frac{1}{h_2}\frac{\partial f}{\partial u_2}, \ldots, \frac{1}{h_N}\frac{\partial f}{\partial u_N} \right), \tag{A.3}$$

and for Laplacian

$$\nabla^2 = \frac{1}{h_1 h_2 \ldots h_N} \sum_{cycl} \frac{\partial}{\partial u^i} \left(\frac{h_1 h_2 \ldots h_N}{h_i^2} \frac{\partial f}{\partial u^i} \right). \tag{A.4}$$

Here, the summation is over cyclic interchanges of the suffixes 1,2...,N. An elegant way to obtain (A.3) and (A.4) is through the use of differential forms. We take advantage of the Hodge operator $*$, well-known from differential geometry. In terms of this operator, the Laplacian is given by

$$\nabla^2 = * \, \mathrm{d} * \mathrm{d}, \tag{A.5}$$

where d denotes differentiation, as usual.

Let us apply this definition. First, the differential is

$$\mathrm{d}f = \sum_i \frac{\partial f}{\partial u^i}\mathrm{d}u^i = \sum_i \frac{1}{h_i}\frac{\partial f}{\partial u^i}\Theta^i, \tag{A.6}$$

where $\Theta^i = h_i \mathrm{d}u^i$ are the coframe vectors. Now, since the components df in the coframe represent the gradient, we already have the expression (A.3). The application of the Hodge operator gives

$$*\,\mathrm{d}f = \sum_i (-1)^{i+1}\frac{1}{h_i}\frac{\partial f}{\partial u^i}\Theta^1 \wedge \ldots \wedge \Theta^{i-1} \wedge \Theta^{i+1} \wedge \ldots \wedge \Theta^N. \tag{A.7}$$

Rewriting this expression in terms of frame vectors

$$*\,\mathrm{d}f \;=\; \sum_i (-1)^{i+1}\frac{h_1 \ldots h_{i-1}h_{i+1}\ldots h_N}{h_i}\frac{\partial f}{\partial u^i}\mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^{i-1} \wedge \mathrm{d}u^{i+1} \wedge \ldots \wedge \mathrm{d}u^N, \tag{A.8}$$

enables us to apply the exterior derivate:

$$\mathrm{d} * \mathrm{d}f = \sum_i \frac{\partial}{\partial u^i}\left(\frac{h_1 \ldots h_{i-1}h_{i+1}\ldots h_N}{h_i}\frac{\partial f}{\partial u^i}\right)\mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^N. \tag{A.9}$$

Again, using the basis of coframe vectors:

$$\mathrm{d} * \mathrm{d}f = \frac{1}{h_1 h_2 \ldots h_N}\sum_i \frac{\partial}{\partial u^i}\left(\frac{h_1 \ldots h_{i-1}h_{i+1}\ldots h_N}{h_i}\frac{\partial f}{\partial u^i}\right)\mathrm{d}\Theta^1 \wedge \ldots \wedge \mathrm{d}\Theta^N. \tag{A.10}$$

According to (A.5), the last step gives

$$*\mathrm{d} * \mathrm{d}f = \frac{1}{h_1 h_2 \ldots h_N}\sum_i \frac{\partial}{\partial u^i}\left(\frac{h_1 \ldots h_{i-1}h_{i+1}\ldots h_N}{h_i}\frac{\partial f}{\partial u^i}\right), \tag{A.11}$$

which is the desired expression (A.4). In addition, we note that this is a special case of the more general Laplace-Beltrami operator (e.g. [7], p. 161):

$$\nabla^2 f = |\mathrm{det} g_{ij}|^{-1/2}\,\frac{\partial}{\partial x^k}\left(g^{lk}\,|\mathrm{det} g_{ij}|^{1/2}\,\frac{\partial f}{\partial x^l}\right), \tag{A.12}$$

where g^{ij} are the components of the inverse metric tensor. This operator is an extension of the Laplace operator to functions defined on Riemannian and pseudo-Riemannian manifolds. Since we have a diagonal metric, inserting $g_{ij} = h_i^2\delta_{ij}$ into (A.12) gives again (A.4).

A.2 The Laplacian in particular coordinate systems

Here, we express the Laplace operator in the two particular coordinate systems used in the text. First, we consider the hyperspherical coordinate system:

$$\begin{aligned} x_1 &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-2}\sin\theta_{N-1},\\ x_2 &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-2}\cos\theta_{N-1},\\ x_3 &= r\sin\theta_1\sin\theta_2\ldots\cos\theta_{N-2},\\ &\vdots\\ x_{N-1} &= r\sin\theta_1\cos\theta_2,\\ x_N &= r\cos\theta_1, \end{aligned} \tag{A.13}$$

where $x_1, x_2 \ldots x_N$ are Cartesian coordinates, r is the hyperradius and $\phi_1, \phi_2 \ldots \phi_{N-1}$ are the hyperspherical angles, with $0 \leq \theta_j \leq \pi$ for $j = 1, \ldots, N-2$, and $0 \leq \theta_{N-1} \leq \pi$. Thus, the corresponding coefficients h_i are given by

$$\begin{aligned} h_r &= 1\\ h_{\theta_1} &= r\\ h_{\theta_2} &= r\sin\theta_1\\ h_{\theta_3} &= r\sin\theta_1\sin\theta_2\\ &\vdots\\ h_{\theta_{N-2}} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-3}\\ h_{\theta_{N-1}} &= r\sin\theta_1\sin\theta_2\ldots\sin\theta_{N-3}\sin\theta_{N-2}. \end{aligned} \tag{A.14}$$

Inserting (A.14) into (A.4) produces:

$$\nabla^2 = \frac{1}{r^{N-1}}\frac{\partial}{\partial r}\left(r^{N-1}\frac{\partial}{\partial r}\right) - \frac{\mathcal{L}^2(N-1)}{r^2}, \tag{A.15}$$

with

$$\mathcal{L}^2(N-1) = -\sum_{cycl}\frac{1}{\sin^2\theta_1\sin^2\theta_2\ldots\sin^2\theta_{i-1}}\frac{1}{\sin^{N-1-i}\theta_i}\left(\frac{\partial}{\partial\theta_i}\sin^{N-1-i}\theta_i\frac{\partial}{\partial\theta_i}\right), \tag{A.16}$$

where $\mathcal{L}^2(N-1)$ is a partial differential operator on the unit sphere S^{N-1}. The summation is over cyclic interchanges of the suffixes $1, 2 \ldots, N-1$. Alternatively, we can

rewrite (A.16) as a recursive relation [16]:

$$\begin{aligned}
\mathcal{L}^2(N-1) &= -\frac{1}{\sin^{N-2}\theta_1}\left(\sin^{N-2}\theta_1\frac{\partial}{\partial\theta_1}\right)+\frac{1}{\sin^2\theta_1}\mathcal{L}^2(N-2)\\
\mathcal{L}^2(N-2) &= -\frac{1}{\sin^{N-3}\theta_2}\left(\sin^{N-3}\theta_2\frac{\partial}{\partial\theta_2}\right)+\frac{1}{\sin^2\theta_2}\mathcal{L}^2(N-3)\\
&\vdots\\
\mathcal{L}^2(2) &= -\frac{1}{\sin\theta_{N-2}}\frac{\partial}{\partial\theta_{N-2}}\left(\sin\theta_{N-2}\frac{\partial}{\partial\theta_{N-2}}\right)+\frac{1}{\sin^2\theta_{N-2}}\mathcal{L}^2(1)\\
\mathcal{L}^2(1) &= -\frac{\partial^2}{\partial\theta_{N-1}^2}.
\end{aligned} \tag{A.17}$$

For the special case of 4-dimensional space with coordinates denoted by r,η,θ,ϕ, we have from (A.13) and (A.14):

$$\begin{aligned}
x &= r\sin\eta\sin\theta\cos\phi,\\
y &= r\sin\eta\sin\theta\sin\phi,\\
z &= r\sin\eta\cos\theta,\\
w &= r\cos\eta,
\end{aligned} \tag{A.18}$$

and

$$h_r=1,\quad h_\theta=r,\quad h_\phi=r\sin\eta,\quad h_w=r\sin\eta\sin\theta\,. \tag{A.19}$$

Again, inserting into (A.17) yields:

$$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{3}{r}\frac{\partial}{\partial r}-\frac{\mathcal{L}^2(3)}{r^2}\,, \tag{A.20}$$

where

$$\mathcal{L}^2(3)=-\frac{1}{\sin^2\eta}\left[\frac{\partial}{\partial\eta}\left(\sin^2\eta\frac{\partial}{\partial\eta}\right)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] \tag{A.21}$$

is the square of the angular momentum operator on a 3-sphere.

In section 3, we introduced another system of coordinates:

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \\ w &= w \end{aligned} \tag{A.22}$$

with

$$h_r = 1, \quad h_\theta = r, \quad h_\phi = r\sin\theta, \quad h_w = 1\,. \tag{A.23}$$

thus, the corresponding Laplace operator is:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{\hat{l}^2}{r^2} + \frac{\partial^2}{\partial w^2}, \tag{A.24}$$

where

$$\hat{l}^2 = -\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]. \tag{A.25}$$

B Potential of a Point Charge in Higher Dimensions

In this appendix we shall investigate the behaviour of electromagnetic force in an infinite N-dimensional space. [9] We assume that Gauss' law for electrostatics is still valid in higher dimensions. This means that we demand the flux of the field intensity produced by a point charge, through a surface enclosing this charge, to be independent of the spatial dimension. The surface integral is then equal to the enclosed charge times some constant. Potential of a charge distribution $\rho(\boldsymbol{r})$ is the solution of Poisson's equation

$$\Delta V = -\rho(\boldsymbol{r}). \tag{B.1}$$

We have $\rho(\boldsymbol{r}) = e\,\delta(\boldsymbol{r})$, i.e. we consider a point charge e at the origin of our coordinate system. Because the field is spherically symmetric, we use hyperspherical coordinates (A.13). The Laplacian operator is given by (A.15), the delta function $\delta(\boldsymbol{r})$ is in this coordinate system expressed as:

$$\delta(\boldsymbol{r}) = \frac{1}{J(r, \phi_{N-1}, \phi_{N-2} \dots \phi_1)} \delta(r)\delta(\phi_{N-1})\delta(\phi_{N-2}) \dots \delta(\phi_1), \tag{B.2}$$

where $J \equiv J(r, \phi_{N-1}, \phi_{N-2} \dots \phi_1) = h_\rho h_{\theta_1} h_{\theta_2} \dots h_{\theta_{N-1}}$ is the Jacobian (see (A.14)) of the transformation. Hence, the Poisson's equation (B.1) takes the following form:

$$\frac{1}{r^{N-1}} \frac{\mathrm{d}}{\mathrm{d}r} \left(r^{N-1} \frac{\mathrm{d}V(r)}{\mathrm{d}r} \right) = -\frac{e}{J} \delta(r)\delta(\phi_{N-1})\delta(\phi_{N-2}) \dots \delta(\phi_1). \tag{B.3}$$

Integrating over the whole space yields the desired expression for potential:

$$V(r) = \begin{cases} \dfrac{1}{S_N} \dfrac{1}{N-2} \dfrac{e}{r^{N-2}} & \text{for } N \neq 2 \\ -\dfrac{e}{S_N} \ln r & \text{for } N = 2 \end{cases}, \tag{B.4}$$

where $S_N = 2\pi^{N/2}/\Gamma(N/2)$ is the surface area of a unit sphere in N spatial dimensions. In the special case of $N = 4$, we have:

$$V(r) = \frac{1}{8\pi^2} \frac{e}{r^2}. \tag{B.5}$$

Throughout this text, we drop the pre-factor $1/8\pi^2$.

[9] For discussion of the behaviour of gravitational force in models with extra spatial dimensions see, for instance, [20, 9, 14].

C Potential with Compactified Extra Dimension

In section 3 we need to sum up the following series, giving the potential with a compactified extra dimension:

$$V(r,w) = -\sum_{n=-\infty}^{\infty} \frac{e^2}{r^2 + (w - 2\pi R n)^2}. \tag{C.1}$$

Here we shall show that the result is:

$$V(r,w) = -\frac{e^2}{2rR} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)}. \tag{C.2}$$

To prove this assertion, we use the residue theorem. Let us start with the function

$$f(z) = \frac{\cot \pi z}{(a-z)^2 + b^2}, \qquad a, b \in \mathbb{R}. \tag{C.3}$$

Its numerator has two singularities at

$$z_I = a + ib, \quad z_{II} = a - ib, \tag{C.4}$$

The corresponding residues are:

$$\operatorname{res} \mathrm{f}(\mathrm{z_I}) = \lim_{z \to z_I} (z - z_I) \frac{\cot \pi z}{(a-z)^2 + b^2} = \lim_{z \to z_I} \frac{\cot \pi z}{z - z_{II}} = \frac{\cot \pi (a + ib)}{2ib} \tag{C.5}$$

and

$$\operatorname{res} \mathrm{f}(\mathrm{z_{II}}) = \lim_{z \to z_{II}} (z - z_{II}) \frac{\cot \pi z}{(a-z)^2 + b^2} = \lim_{z \to z_{II}} \frac{\cot \pi z}{z - z_I} = -\frac{\cot \pi (a - ib)}{2ib}. \tag{C.6}$$

The denominator has its singularities at

$$z_n = n, \quad n \in \mathbb{Z}, \tag{C.7}$$

with the corresponding residues:

$$\begin{aligned} \operatorname{res} \mathrm{f}(\mathrm{z_n}) &= \lim_{z \to z_n} (z - z_n) \frac{\cot \pi z}{(a-z)^2 + b^2} = \lim_{z \to z_n} \frac{\cos \pi z}{\sin \pi z} \frac{z - z_n}{(a-z)^2 + b^2} \\ &= \lim_{z \to z_n} \frac{\cos \pi z - \pi z \sin \pi z}{[(a-z)^2 + b^2] \pi \cos \pi z + 2(z-a) \sin \pi z} = \frac{1}{\pi [(a-n)^2 + b^2]}. \end{aligned} \tag{C.8}$$

By the residue theorem, the sum of all residues must be zero. Hence,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a-n)^2+b^2} = \frac{i\pi \cot \pi(a+ib)}{2b} - \frac{i\pi \cot \pi(a-ib)}{2b}\,. \tag{C.9}$$

Using hyperbolic functions, this expression can be brought into the form:

$$\sum_{n=-\infty}^{\infty} \frac{1}{(a-n)^2+b^2} = \frac{\pi}{b}\frac{\sinh(2\pi b)}{\cosh(2\pi b)-\cos(2\pi a)}. \tag{C.10}$$

We are almost at the end. Rewriting (C.1) as

$$V(r,w) = -\sum_{n=-\infty}^{\infty} \frac{e^2}{r^2+(w-2\pi Rn)^2} = -\frac{e^2}{(2\pi R)^2}\sum_{n=-\infty}^{\infty} \frac{1}{\frac{r}{2\pi R}+(\frac{w}{2\pi R}-n)^2} \tag{C.11}$$

and comparing (C.10) with (C.11) we find that $a = w/2\pi R$, $b = r/2\pi R$. Thus,

$$V(r,w) = -\frac{e^2}{2rR}\frac{\sinh(r/R)}{\cosh(r/R)-\cos(w/R)}, \tag{C.12}$$

which is already the result (C.2).

D Fourier Expansion of the Potential

In this appendix we shall be concerned with finding the coefficients of the Fourier expansion of the function:

$$f(\theta) = \frac{1}{a - \cos\theta} = \sum_{n=-\infty}^{\infty} \alpha_n e^{in\theta} \quad (a > 1). \tag{D.1}$$

As $f(\theta)$ is an even real function, the coefficients α_n must satisfy the condition

$$\alpha_n = \alpha_{-n}. \tag{D.2}$$

The Fourier coefficient α_n is given by:

$$\alpha_n = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-in\theta}}{a - \cos\theta} \mathrm{d}\theta. \tag{D.3}$$

For $n \geq 0$, let

$$z = e^{-i\theta} \quad \text{and} \quad \mathrm{d}z = -ie^{i\theta}\mathrm{d}\theta. \tag{D.4}$$

Then $\cos z = (z + 1/z)/2$ and (D.3) becomes

$$\alpha_n = \frac{i}{2\pi} \int_{\circlearrowleft} \frac{z^n}{a - (z + 1/z)/2} \frac{\mathrm{d}z}{z}, \tag{D.5}$$

where we integrate over the unit circle (clockwise). Rearranging (D.5), we obtain

$$\alpha_n = -\frac{i}{\pi} \int_{\circlearrowleft} \frac{z^n}{z^2 - 2az + 1} \mathrm{d}z = \frac{i}{\pi} \int_{\circlearrowleft} \frac{z^n}{(z - z_1)(z - z_2)} \mathrm{d}z, \tag{D.6}$$

where we have changed the orientation of the integration path. The integrand has two poles of order one:

$$z_1 = a + \sqrt{a^2 - 1}, \qquad z_2 = a - \sqrt{a^2 - 1}. \tag{D.7}$$

Only the pole at z_2 is located inside the unit circle. Then by the residue theorem

$$\alpha_n = 2\pi i \operatorname{res} f(z_2), \tag{D.8}$$

where

$$\operatorname{res} f(z_2) = \frac{i}{\pi} \lim_{z \to z_2} \frac{z^n}{(z - z_1)} = \frac{i}{\pi} \frac{z_2^n}{(z_2 - z_1)} = -\frac{i}{\pi} \frac{(a - \sqrt{a^2 - 1})^n}{2\sqrt{a^2 - 1}} \,. \tag{D.9}$$

The Fourier coefficient α_n is thus

$$\alpha_n = \frac{(a - \sqrt{a^2 - 1})^n}{\sqrt{a^2 - 1}}. \tag{D.10}$$

Employing (D.2), the final result is:

$$\frac{1}{a - \cos\theta} = \sum_{n=-\infty}^{\infty} \frac{(a - \sqrt{a^2 - 1})^{|n|}}{\sqrt{a^2 - 1}} \, e^{in\theta} \,. \tag{D.11}$$

Using this relation for the potential (3.14), we obtain

$$\begin{aligned} \frac{\sinh(r/R)}{\cosh(r/R) - \cos(w/R)} = \sinh(r/R) \sum_{n=-\infty}^{\infty} \frac{[\cosh(r/R) - \sinh(r/R)]^{|n|}}{\sinh(r/R)} \, e^{inw/R} \\ = \sum_{n=-\infty}^{\infty} e^{-|n|r/R} \, e^{inw/R} \,. \end{aligned} \tag{D.12}$$

E Incomplete Gamma Function

In the following, we shall estimate the value of the integral

$$I(p,b,c) \equiv \int_c^\infty y^p e^{-by^2} \mathrm{d}y \tag{E.1}$$

for large values of c or, more precisely, for $bc^2 \gg 1$. Here b, c are positive constants, $p > -1$. Substituting $by^2 = t$ in (E.1), we obtain

$$\int_c^\infty y^p e^{-by^2} \mathrm{d}y = \frac{1}{2b^{p+1/2}} \int_{bc^2}^\infty t^{\frac{p-1}{2}} e^{-t} \mathrm{d}t \,. \tag{E.2}$$

But this is the incomplete gamma function

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} \mathrm{d}t \tag{E.3}$$

of arguments $a = p + 1/2$ and $x = bc^2$. The asymptotic expansion of incomplete gamma function is as follows ([7], p. 661):

$$\Gamma(a,x) = x^{a-1} e^{-x} \sum_{n=0}^{\infty} \frac{(a-1)!}{(a-1-n)!} \frac{1}{x^n} = x^{a-1} e^{-x} \sum_{n=0}^{\infty} (-1)^n \frac{(n-a)!}{(-a)!} \frac{1}{x^n} \,. \tag{E.4}$$

Thus, since we have $bc^2 \gg 1$, the asymptotic expansion of (E.1) is given by

$$\int_c^\infty y^p e^{-by^2} \mathrm{d}y = \frac{c^{2p-1}}{2b} e^{-bc^2} \sum_{n=0}^{\infty} \frac{(p-1/2)!}{(p-1/2-n)!} \frac{1}{(bc^2)^n} \,. \tag{E.5}$$

We can see that for $bc^2 \gg 1$, expression (E.1) is an exponentially small quantity.

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