Plasma conductivity and diffusion

Pavel Souček

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Langevin equation

Before we wander into the realm of plasma conductivity and diffusion it is convenient to introduce a simple form of the equation of motion for cold and weakly ionized plasma.

- in a weakly ionized plasma the number density of the charged particles is much smaller than that of neutral particles $\rightarrow$ charged-neutral interactions are dominant
- only electric and magnetic force is taken into account (gravitational field and forces caused by pressure gradients are neglected)

Therefore the macroscopic equation of motion for electrons under the action of Lorentz force and collisional forces can be written as:

$$m_e \frac{D\mathbf{u}_e}{Dt} = -e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) + \left(\mathbf{F}_{coll}\right)_e , \tag{1}$$

where $\mathbf{u}_e(\mathbf{r}, t)$ is the average electron velocity and $\left(\mathbf{F}_{coll}\right)_e$ is the collision term.

Langevin equation II

The macroscopic collision term $(\mathbf{F}_{coll})_e$ can be expressed as the product of the average electron momentum with an effective constant collision frequency ν_c for momentum transfer between the electrons and the heavy (neutral) particles,

$$(\mathbf{F}_{coll})_e = -\nu_c m_e \mathbf{u}_e \tag{2}$$

In this expression we are neglecting the average motion of the neutral particles, as they are much more massive than the electrons (i.e. velocities of individual particles are random and their average is 0).
Using this expression for the collision term, we obtain the Langevin equation:

$$m_e \frac{D\mathbf{u}_e}{Dt} = -e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nu_c m_e \mathbf{u}_e \tag{3}$$

Collision term

What is the physical meaning of the collision term?
In the absence of electric and magnetic fields the Langevin equation (3) reduces to

$$\frac{D\mathbf{u}_e}{Dt} = -\nu_c \mathbf{u}_e \tag{4}$$

whose solution is

$$\mathbf{u}_e(t) = \mathbf{u}_e(0)\exp(-\nu_c t) \tag{5}$$

Thus, the electron-neutral collisions decrease the average electron velocity exponentially, at a rate governed by the collision frequency.

Langevin equation III

An equation similar to the Langevin equation for electrons can be written also for ions

$$m_i \frac{D\mathbf{u}_i}{Dt} = Ze\left(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}\right) + \left(\mathbf{F}_{coll}\right)_i \tag{6}$$

where $\mathbf{u}_i$ is the average ion velocity and Ze the ion charge. In many cases of interest, as in high-frequency phenomena, we can neglect the ion motion and assume $\mathbf{u}_i = 0$, since ion mass is typically about 10^3 or 10^4 times grater than the electron mass. The type of plasma, in which only the electron motion is important, is usually called a Lorentz gas.

Linearization of the Langevin equation

In the form presented (3) the Langevin equation contains nonlinear terms

- the total derivative contains term $(\mathbf{u}_e \cdot \nabla)\mathbf{u}_e$, which is called the inertial term in fluid mechanics. The omission of this term is justified when the average velocity and its space derivatives are small, or when $\mathbf{u}_e$ is normal to its gradient (in traverse waves)
- in the term $\mathbf{u}_e \times \mathbf{B}$ we can separate the magnetic flux density $\mathbf{B}(\mathbf{r}, t)$ into two parts

$$\mathbf{B}(\mathbf{r}, t) = \mathbf{B}_0 + \mathbf{B}'(\mathbf{r}, t) \tag{7}$$

where $\mathbf{B}_0$ is constant and $\mathbf{B}'(\mathbf{r}, t)$ is the variable component, so that

$$q\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}\right) = q\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0 + \mathbf{u}_e \times \mathbf{B}'\right) \tag{8}$$

Linearization of the Langevin equation II

in cases where

$$\left|\mathbf{u}_e \times \mathbf{B}'\right| \ll |E| \tag{9}$$

the term $\mathbf{u}_e \times \mathbf{B}'$ can be neglected, so that the Langevin equation becomes

$$m_e \frac{\partial \mathbf{u}_e}{\partial t} = -e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0\right) - \nu_c m_e \mathbf{u}_e \tag{10}$$

Plane waves

A case of practical interest is the one where variables $\mathbf{E}, \mathbf{B}'$ and $\mathbf{u}_e$ vary harmonically in time and space. Any complex and physically realizable wave motion can be synthesized in terms of superposition of plane waves.
Let us therefore consider solution in the form of

$$\mathbf{E}, \mathbf{B}', \mathbf{u}_e \propto \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)] \tag{11}$$

where ω denotes the wave angular frequency and $\mathbf{k}$ is the wave propagation vector (normal to the wave front)
After some mathematics we arrive to

$$|\mathbf{u}_e \times (\mathbf{k} \times \mathbf{E})/\omega| \ll |\mathbf{E}| \tag{12}$$

Hence the nonlinear term can be neglected if

$$|u_e| \ll |\omega/k| \tag{13}$$

Plane waves II

The term (ω/k) represents the phase velocity of the plane wave. Since this term is usually of the order of the speed of light, whereas the magnitude of the mean velocity of electrons u_e is much less, the nonlinear term can be generally neglected. However, in cases of resonance ω/k is very small, whereas u_e becomes large. Under these conditions the nonlinear terms are important and a nonlinear analysis must be used.

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DC case

Isotropic Plasma In the absence of magnetic field the Langevin equation becomes

$$-e\mathbf{E} - m_e \nu_c \mathbf{u}_e = 0 \tag{14}$$

The electric current density is

$$\mathbf{J} = -e n_e \mathbf{u}_e \tag{15}$$

Which can be combined to

$$\mathbf{J} = \frac{n_e e^2}{m_e \nu_c} \mathbf{E} \tag{16}$$

Do you see the Ohm's law?

$$\sigma_0 = \frac{n_e e^2}{m_e \nu_c} \tag{17}$$

The electron mobility is defined as the ratio of the mean velocity to the applied field. In our case

$$M_e = \frac{u_e}{E} = -\frac{e}{m_e \nu_c} = -\frac{\sigma_0}{n_e e} \tag{18}$$

Anisotropic Magnetoplasma

In the presence of magnetic field the plasma becomes spatially anisotropic. The Langevin equation can be written as

$$-e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) - m_e \nu_c \mathbf{u}_e \tag{19}$$

As for the current density

$$\mathbf{J} = \sigma_0(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0) \tag{20}$$

This is the simplified Ohm's law.
At this point we can consider a useful result which arises when the collisions are negligible. When $\nu_c \to 0$ the DC conductivity becomes very large ($\sigma_0 \to \infty$) so that from the previous equation

$$\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0 = 0 \tag{21}$$

In this case, taking the cross product of (21) with $\mathbf{B}_0$ and noting that

$$(\mathbf{u}_e \times \mathbf{B}_0) \times \mathbf{B}_0 = -\mathbf{u}_{e\perp} B_0^2 \tag{22}$$

we obtain

$$\mathbf{u}_{e\perp} = \frac{\mathbf{E} \times \mathbf{B}_0}{B_0^2} \tag{23}$$

Anisotropic Magnetoplasma II

This result shows us that in the absence of collisions, the electrons have a *drift velocity* $\mathbf{u}_{e\perp}$ perpendicular to both the electric and the magnetic fields. This result is independent on particle mass or charge, so the same can be said for ions. Thus there is no electric current ($\mathbf{J} = 0$) associated with their motion. When collisional effects are not negligible, the motions of the ions suffers a larger retardation than that of electrons as a result of collisions. In this case, there is an electric current given by (assuming $n_e = n_i$)

$$\mathbf{J}_\perp = en_e(\mathbf{u}_{i\perp} - \mathbf{u}_{e\perp}) \tag{24}$$

which is perpendicular to both $\mathbf{E}$ and $\mathbf{B}_0$, known as the *Hall current*. Since $\mathbf{u}_{e\perp} > \mathbf{u}_{i\perp}$, this current is in the direction opposite to the drift velocity of both types of particles.

Anisotropic Magnetoplasma III

If we now return to the simplified Ohm's law (20) and define a *DC conductivity tensor S* by the equation

$$\mathbf{J} = S \cdot \mathbf{E} \tag{25}$$

considering Cartesian coordinates with the z axis paralel to the magnetic field we reach after some maths to

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \sigma_0 \begin{pmatrix} \frac{\nu_c^2}{(\nu_c^2+\Omega_{ce}^2)} & -\frac{\nu_c \Omega_{ce}}{(\nu_c^2+\Omega_{ce}^2)} & 0 \\ \frac{\nu_c \Omega_{ce}}{(\nu_c^2+\Omega_{ce}^2)} & \frac{\nu_c^2}{(\nu_c^2+\Omega_{ce}^2)} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{26}$$

where Ω_{ce} denotes the electron cyclotron frequency.
The DC conductivity tensor is therefore given by

$$S = \begin{pmatrix} \sigma_\perp & -\sigma_H & 0 \\ \sigma_H & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix} \tag{27}$$

Anisotropic Magnetoplasma IV

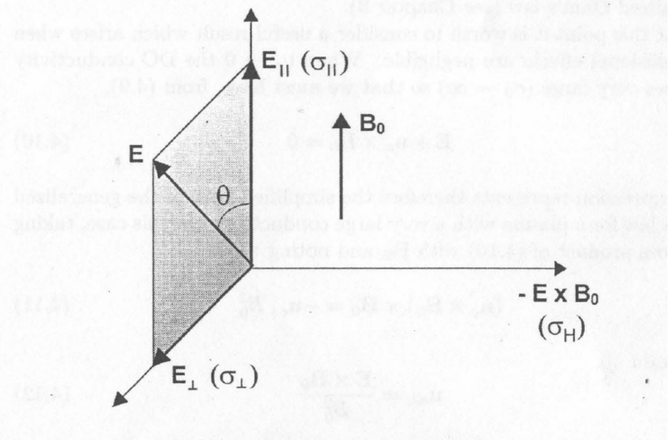

Fig. *Relative orientation of the vector fields* $\mathbf{E}_{\parallel}$, $\mathbf{E}_{\perp}$ *and of* $-\mathbf{E} \times \mathbf{B}_0$. *The conductivities* $\sigma_{\parallel}$, $\sigma_{\perp}$ *and* σ_H *govern the magnitude of the electric currents flowing along these directions, respectively.*

Anisotropic Magnetoplasma V

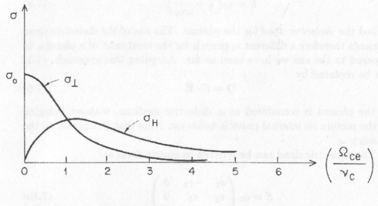

Fig. *Dependence of the Hall conductivity σ_H and of the perpendicular conductivity $\sigma_\perp$ on the ratio of the cyclotron frequency Ω_{ce} to the collision frequency ν_c.*

- when Ω_{ce}/ν_c is relatively large almost all current is produced along the magnetic field lines
- σ_0 increases as ν_c decreases and is independent on the magnitude of **B** and therefore of Ω_{ce}. Therefore in a rarefied plasma in a relatively strong magnetic filed, the electric current flows essentially along the magnetic field lines

AC conductivity

Let us consider the case when the electric field $\mathbf{E}(\mathbf{r},t)$ and the mean electron velocity $\mathbf{u}_e(\mathbf{r},t)$ vary harmonically in time as $\exp(-i\omega t)$. We have previously seen that for time disturbances $\partial/\partial t$ is replaced by $-i\omega$. Therefore the linearized Langevin equation becomes

$$-i\omega m_e \mathbf{u}_e = -e\left(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}_0\right) - \nu_c m_e \mathbf{u}_e \tag{28}$$

This equation is very similar to equation for anisotropic magnetoplasma, except for the change in collision frequency ν_c to $(\nu_c - i\omega)$. The solution is therefore also a similar tensor with

$$\sigma_\perp = \frac{\omega^2}{(\omega^2 - \Omega_{ce}^2)}\sigma_0 \tag{29}$$

$$\sigma_H = \frac{i\omega\Omega_{ce}}{(\omega^2 - \Omega_{ce}^2)}\sigma_0 \tag{30}$$

$$\sigma_0 = i\frac{n_e e^2}{m_e \omega} \tag{31}$$

A complex conductivity means that there is a difference between the current density and the applied field.

Conductivity with ion motion

The evaluation of the conductivity tensor, when the contribution due to the motion of the ions is included, can be performed in a straightforward way. The linearized Langevin equation is solved for all the particles and the total conductivity tensor is simply

$$S = \sum_{\alpha} S_{\alpha} \tag{32}$$

Plasma as a dielectric medium

Plasma can also be treated as a dielectric medium characterized by a dielectric tensor, in which the internal particle behavior is not considered. Instead of the Langevin equation let us consider the Maxwell equation

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right) \tag{33}$$

and incorporate the conductivity tensor S defined as

$$\mathbf{J} = S \cdot \mathbf{E} \tag{34}$$

After some calculations we can rewrite the Maxwell equation as

$$\nabla \times \mathbf{B} = -i\omega\mu_0\epsilon \cdot \mathbf{E} \tag{35}$$

where

$$\epsilon = \epsilon_0 \left(1 + \frac{iS}{\omega\epsilon_0} \right) \tag{36}$$

is called the dielectric tensor.

Plasma as a dielectric medium

The dielectric tensor can be written in matrix form as

$$S = \begin{pmatrix} \epsilon_1 & -\epsilon_2 & 0 \\ \epsilon_2 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_3 \end{pmatrix} \tag{37}$$

where the following notation was introduced

$$\epsilon_1 = 1 + \frac{i}{\omega \epsilon_0} \sigma_\perp \tag{38}$$

$$\epsilon_2 = \frac{i}{\omega \epsilon_0} \sigma_H \tag{39}$$

$$\epsilon_3 = 1 + \frac{i}{\omega \epsilon_0} \sigma_0 \tag{40}$$

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Free electron diffusion

The diffusion of particles results from the presence of a pressure gradient, which results in a force which tends to smooth out any inhomogenities.

To deduce the expression for the electron diffusion coefficient for a warm weakly ionized plasma we will use the momentum transport equation. Let us consider, that the inhomogenities are very small, so that they can be considered first order quantities. This means that the mean velocity of electrons $\mathbf{u}_e$ is also a first order quantity, and considering that the velocity distribution is approximately isotropic, we can replace the pressure tensor by a scalar pressure p_e.

For a slightly nonuniform electron number density, we can write

$$n_e(\mathbf{r}, t) = n_0 + n_e'(\mathbf{r}, t) \quad (41)$$

$$p_e(\mathbf{r}, t) = n_e(\mathbf{r}, t)kT_e = (n_0 + n_e')kT_e \quad (42)$$

For the momentum transport equation we can write

$$n_e m_e \left[\frac{\partial \mathbf{u}_e}{\partial t} + (\mathbf{u}_e \cdot \nabla)\mathbf{u}_e \right] = -\nabla p_e - n_e m_e \nu_c \mathbf{u}_e \quad (43)$$

Free electron diffusion II

After linearization, performing a divergence and substituting form continuity equation we obtain

$$\frac{\partial n_e'}{\partial t} = D_e \nabla^2 n_e' - \frac{1}{\nu_c} \frac{\partial^2 n_e'}{\partial t^2} \tag{44}$$

where we have defined the *electron free diffusion coefficient*

$$D_e = \frac{kT_e}{m_e \nu_c} \tag{45}$$

When the rate of change in the number density is slow compared to the collision frequency, the last term in (44) can be neglected and the *diffusion equation* becomes

$$\frac{\partial n_e'}{\partial t} = D_e \nabla^2 n_e' \tag{46}$$

Lastly we can write the linearized electron flux ($\mathbf{\Gamma}_e = n_0 \mathbf{u}_e$) as

$$\mathbf{\Gamma}_e = -D_e \nabla n_e' \tag{47}$$

Electron diffusion in a magnetic field

As in previous (way back) slide dealing with conductivity in anisotropic magnetoplasma a magnetic force term is added into the equation. This inclusion into the momentum transfer equation leads to very similar equation as in this case and so leads to a very similar equation. Instead of

$$\mathbf{J} = S \cdot \mathbf{E} \tag{48}$$

we come to an analogous expression

$$\mathbf{\Gamma}_e = -D \cdot \nabla n'_e \tag{49}$$

where D is the *tensor coefficient for free diffusion* given in the matrix form by

$$D = \begin{pmatrix} D_\perp & D_H & 0 \\ -D_H & D_\perp & 0 \\ 0 & 0 & D_\parallel \end{pmatrix} \tag{50}$$

Electron diffusion in a magnetic field II

where the following notation is used

$$D_\perp = \frac{\nu_c^2}{(\nu_c^2 + \Omega_{ce}^2)} D_e \quad (51)$$

$$D_H = \frac{\nu_c \Omega_{ce}}{(\nu_c^2 + \Omega_{ce}^2)} D_e \quad (52)$$

$$D_\parallel \equiv D_e = \frac{kT_e}{m_e \nu_c} \quad (53)$$

A diffusion equation for n^prime_e in a constant and uniform magnetic field can be derived from the continuity equation in the form

$$\frac{\partial n'_e}{\partial t} + \nabla \cdot \mathbf{\Gamma}_e = 0 \quad (54)$$

by substituting $\mathbf{\Gamma}_e$ from (49) and calculating in the Cartesian coordinates to

$$\frac{\partial n'_e}{\partial t} = D_\perp \left(\frac{\partial^2 n'_e}{\partial x^2} + \frac{\partial^2 n'_e}{\partial y^2} \right) + D_e \frac{\partial^2 n'_e}{\partial z^2} \quad (55)$$

Electron diffusion in a magnetic field III

Since $D_bot < D_e$ and since D_bot decreases with increasing values of Ω_{ce}/ν_c (similarly to $\sigma_\perp$) the diffusion of particles in a direction perpendicular to **B** is always less than that in the direction parallel to **B**. For values of Ω_{ce} much larger than ν_c, the diffusion of particles across the magnetic field lines is freatly reduced. Note that for $\Omega_{ce} \gg \nu_c$ we have approximately $D_\perp \propto 1/B^2$ and $D_H \propto 1/B$.

Ambipolar diffusion

We have seen that the steady state momentum equation, in the absence of electromagnetic forces and when the temperature is constant, gives the following diffusion equation for the electrons

$$\boldsymbol{\Gamma}_e = -D_e \nabla'_e \tag{56}$$

where the *electron free diffusion coefficient* is given by

$$D_e = \frac{kT_e}{m_e \nu_{ce}} \tag{57}$$

The subscript e has been added to ν_c to indicate that the effective collision frequency ν_{ce} refers to electron-neutral collisions. If we consider equations for ions, we get

$$\boldsymbol{\Gamma}_i = -D_i \nabla'_i \tag{58}$$

$$D_i = \frac{kT_i}{m_i \nu_{ci}} \tag{59}$$

Ambipolar diffusion II

Since the diffusion coefficient is inversely proportional to particle mass, the electrons tend to diffuse faster leaving an excess of positive charge behind them. This gives rise to a space charge electric field, which points in the same direction as the particle diffusion, and which accelerates the ions and slows the electrons.

For most problems of plasma diffusion, however, the space charge electric field cannot be neglected. According to Maxwell equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{e(n_i - n_e)}{\epsilon_0} \tag{60}$$

it is clear, that an electric field is present whenever the densities differ. To estimate the importance of the space charge electric field let us use dimensional analysis. Let L represent a characteristic length over which the number density changes significantly. Thus, from the Maxwell equation (60) we may write

$$E \sim \frac{enL}{\epsilon_0} \tag{61}$$

Ambipolar diffusion III

so that the electric force per unit mass f_E is in the order

$$f_E = \frac{eE}{m} \sim \frac{e^2 nL}{m\epsilon_0} \quad (62)$$

The diffusion force per unit mass f_D obtained from (56) is in the order

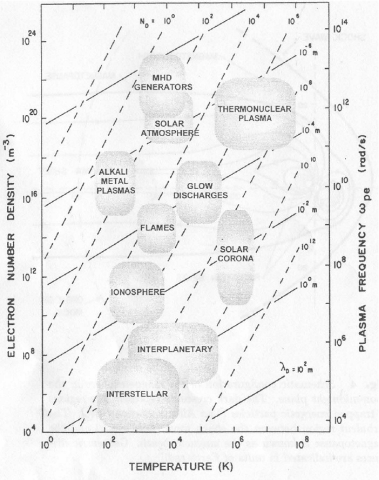
$$f_D = \frac{kT}{mn_0}|\nabla n| \sim \frac{kTn}{mn_0 L} \quad (63)$$

Therefore, the space charge can be neglected only if $f_E \ll f_D$, or equivalently if

$$L^2 \ll \frac{\epsilon_0 kT}{n_0 e^2} = \lambda_D^2 \quad (64)$$

where λ_D is the Debye length.

Ambipolar diffusion IV



- the condition $L^2 \ll \lambda_D^2$ is rarely satisfied and so cannot be neglected
- the combined diffusion of electrons and ions, forced by a space charge **E** field. is known as *ambipolar diffusion* two kinds of particles tend to reach a diffusion rate which is intermediate in value to their free diffusion rates

Ambipolar diffusion V

The ambipolar diffusion coefficient and equation can be calculated similarly to free diffusion counterparts and setting $n'_e = n'_i$ we obtain

$$\frac{\partial n'}{\partial t} = D_a \nabla^2 n' \tag{65}$$

and

$$D_a = \frac{k(T_e + T_i)}{m_e \nu_{ce} + m_e \nu_{ci}} \tag{66}$$

This situation ($n'_e = n'_i$) is called the *perfect ambipolar diffusion*. A less restrictive simplifying approximation would be

$$n'_i = C n'_e \tag{67}$$

where C is constant. In this case we obtain

$$\frac{\partial n'_e}{\partial t} = D_a \nabla^2 n'_e \tag{68}$$

and

$$D_a = \frac{k(T_e + CT_i)}{m_e \nu_{ce} + C m_e \nu_{ci}} \tag{69}$$

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Good to know

- Langevin equation
- what's the collision term good for
- Ohm's law
- often tensors
- dependence on plasma and cyclotron frequency
- diffusion $D_\perp \sim 1/B^2$
- ambipolar diffusion

Thank you for attention