From first-order logic to accessible categories

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(a) *positive-primitive* if it has the form $(\exists y)\psi(x, y)$ where $\psi(x, y)$ is a conjunction of atomic formulas,

(b) *positive-existential* if it is a disjunction of positive-primitive formulas,

(c) *basic* it it has the form

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Any basic theory is closed under directed colimits of its models.

Definition. (Makkai, Paré 1989). A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-directed colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

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2) **Met** (complete metric spaces and contractions), **Ban** (Banach spaces and linear contractions), or **Hilb** (Hilbert spaces and linear contractions) are $\aleph_1$-accessible. $\aleph_1$-presentable objects are separable spaces.

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Definition. (Shelah 1987) Let Σ be a signature, $\mathcal{K}$ a subcategory of the category Σ-structures and embeddings. Then $\mathcal{K}$ is an *abstract elementary class* if the following conditions are satisfied:

1. $\mathcal{K}$ is closed in $\mathbf{Str}(\Sigma)$ under directed colimits,
2. given homomorphisms $g : A \to B$ and $h : B \to C$ with $hg, h \in \mathcal{K}$ then $g \in \mathcal{K}$, and
3. there is a cardinal λ such that if $f : A \to B$ is a submodel embedding with $B \in \mathcal{K}$ then there is $h : A' \to B$ in $\mathcal{K}$ such that f factorizes through h and $|A'| \leq |A| + \lambda$.

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A subcategory $\mathcal{K}$ of a category $\mathcal{L}$ is

1. *iso-full* if every isomorphism $f : K \to K$ in $\mathcal{L}$ with K in $\mathcal{K}$ belongs to $\mathcal{K}$,
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Theorem 4. (Beke, Lieberman, JR 2012) A category is equivalent to an abstract elementary class iff it is an accessible category with directed colimits whose morphisms are monomorphisms and which admits an iso-full and coherent embedding into a finitely accessible category preserving directed colimits and monomorphisms.

It is easy to find an abstract elementary class $\mathcal{K} \subseteq \mathbf{Str}(\Sigma)$ which cannot be axiomatized by an $L_{\kappa\omega}$ theory in Σ. It is difficult to find an abstract elementary class $\mathcal{K}$ which is not equivalent to $\mathbf{Elem}(T)$ for any $L_{\kappa\omega}$-theory T, these categories are called (∞, ω)-*elementary*.

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The size of an infinite complete metric space is its density character, i.e., the smallest cardinality of a dense subset. The same for infinite dimensional Banach spaces. The size of an infinite dimensional Hilbert space is the cardinality of its orthonormal base.

Any infinite-dimensional Banach space has cardinality $\lambda^{\aleph_0}$ for some infinite cardinal λ. Thus there are no Hilbert spaces in cardinality λ of countable cofinality. But there are Hilbert spaces of any infinite size. Thus in **Hilb** sizes never start to coincide with cardinalities and there are arbitrarily large gaps of cardinalities but not in sizes.

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An accessible category is called *LS-accessible* if this cannot happen.

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This is an ultimate solution of the problem of Bankston (1982) – in 1984 Banaschewski and JR independently showed that this is not possible by any first-order theory whose models are closed under products.

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This leads to an understanding of (faithful) functors $\mathcal{K} \to \mathbf{Set}$ preserving directed colimits.

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Conversely, to any Grothendieck topos $\mathcal{T}$ we can assign the category $P\mathcal{T}$ of *points*, i.e., functors $\mathcal{T} \to \mathbf{Set}$ preserving colimits and finite limits. The category $P\mathcal{T}$ is (∞, ω)-elementary. Moreover, (∞, ω)-elementary categories are precisely categories of points of Grothendieck toposes.

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If $\mathbf{Acc}_\omega$ is the category of accessible categories with directed colimits and functors preserving directed colimits and $\mathbf{GTop}$ the category of Grothendieck toposes and functors preserving colimits and finite limits (geometric morphisms) then $S : \mathbf{Acc}_\omega \to \mathbf{GTop}$ is left adjoint to $P : \mathbf{GTop} \to \mathbf{Acc}_\omega$.

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Scott adjunction seems to be a strong tool for studying accessible categories with directed colimits.

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Shelah's Categoricity Conjecture claims that for every AEC $\mathcal{K}$ there is a cardinal κ such that $\mathcal{K}$ is either λ-categorical for all $\lambda \geq \kappa$ or $\mathcal{K}$ is not λ-categorical for any $\lambda \geq \kappa$.

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Recently, Vasey proved the conjecture for universal abstract elementary classes and, together with Shelah, for general abstract elementary classes assuming the existence of a proper class of strongly compact cardinals. Very recently, C. Espindola used Scott adjunction to transfer the problem to topos theory.