Gödel’s Foundational Influence

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Prehistory of Formal Reasoning

- Aristotle
  * Deductive proofs as demonstration arguments in Euclid geometry
  * Logic in the form of syllogisms independent of mathematical/geometrical proofs

- Sufficient for more than two millenia

Logicism

- Kant – Arithmetic and Geometry as synthetic a priori issues akin to Metaphysics
- Dedekind - culmination of arithmetization of Arithmetic and Geometry
- Frege – Analyticity of arithmetic truths derived from their justification
  7+5=12 as analytical truth (contrary to Kant)
  sense and reference, semantics
  symbolic language for expressing everything explicitly & finite set of rules

Logicism

- Peano - Formalization of logical inference
  Axioms and Modus Ponens
- Russel - Principia Mathematica (1910-13)
  axioms as basic truths,
  deriving logical truths by MP and generalization

David Hilbert
Grundlagen der Geometrie (1899)

For foundational problems:

1. Formalization of mathematical theory
2. Proof of consistency of the the axioms
3. Independence and completeness of the axioms
4. The decision problem: is there a method answering any question in the theory<

David Hilbert

- By 1920’s, “Hilbert style” exiomatic approach dominates
- Propositional logic proved complete and decidable
- Predicate logic presented in Hilbert style by Ackermann by 1920

Hilbert Program ~1920

- Hilbert Program: expressing higher mathematics in terms of elementary Arithmetics;
  formalizing all Mathematics in axiomatic form together with a proof of completeness
  (finitistic methods, purely intuitive basis of concrete signs)
- P. Bernays, W. Ackermann, J. von Neumann, J. Herbrand

- Ackermann and von Neumann – proof of consistency of number theory, Ackermann thought near completion for analysis
- Hilbert claimed in 1928 in Bologna that the work is essentially completed

Kurt Gödel: completeness of First-Order Predicate Logic

- Completeness of First-Order Predicate Logic stated by Hilbert and Ackermann in 1928
- Kurt Gödel tackled this in his doctoral thesis in 1929
- Thm: Every logical expression is either satisfiable or refutable, aka Every valid logical expression is provable

- Presented September 1930 in Königsberg

Kurt Gödel: incompleteness of formal systems

- Also in September 1930 in Königsberg, presented as an “aside”
- The First Incompleteness Thm showing arithmetic nor refutable in Peano arithmetic
- The Seconf Incompleteness Thm showing that consistency of arithmetic cannot be proved in arithmetic itself, $Con(P) \equiv \neg Prov(\lceil 0 \rceil = \lceil 1 \rceil)$
- von Neumann interrupted his lectures on Hilbert proof theory in the Fall of 1930; seeing Hilbert program could not be achieved at all
  → Destruction of Hilbert Program (impossibility of proving consistence of a formal system inside of it)

Gödel and Gentzen

- Translation from Peano arithmetic to intuitionistic Heyting arithmetic, in parallel with Gentzen, 1932-33

Gentzen

- Gentzen thesis (1934-35) on analysis of mathematical proofs
- Natural Deduction (intro & elim rules)
- “Sequenzenkalkul”, Sequent Calculus
- Normalization and cut-elimination

Gödel and Gentzen

- Gentzen gave alternative proof of the Incompleteness Thm (written 1939, published 1943) by showing a formula unprovable in Peano arithmetic (thus also showing consistency of Peano arithmetic)
- Gödel’s proof of consistency using Dialectica interpretation

(Unclear mutual interaction in 1939.)

Computability and Undecidable problems

- Hilbert (1928): “Entscheidungsproblem”: Is there a general effective procedure deciding whether or not a given formula A of a calculus is provable?
- 1936: Alonzo Church proved on the basis of λ-calculus and Alan Turing on the basis of Turing machines several months later that

  the answer to the Entscheidungsproblem is negative

- Existence of *undecidable problems* in Informatics

Church, Turing, and Gödel

- Church using λ-calculus as a formal tool tried to formalize mathematics
- Learning about Gödel's result, claimed that it does not apply to this system
- Kleene - recursive functions
- Rosser - reductions
- Church-Turing thesis (Kleene, 1952) for computable functions / computability:
  - Turing machines
  - λ-definability
  - Gödel's general recursive functions (Princeton, 1934)

Curry-Howard correspondence aka “formulae-as-types”

- Computational semantics for intuitionistic logic
- Computations = normalization
- Intuitionistic logic not tied to any philosophy of Mathematics, but corresponds to program execution
- Girard’s Linear logic as a Sequent-Calculus-style system capable expressing parallel operations (via proof normalization)

Information

- Claude Shannon: the "fundamental problem of communication" as the ability of the receiver to reproduce a message sent by the sender via a communication channel
- 1948 paper "The Mathematical Theory of Communication"
- Information as a measure of difference
- Entropy as a measure of uncertainty of status prediction ($\log_2(n)$ bits for n values, entropy assuming the same probability of each)

Thermodynamics

- Second Law of the Thermodynamics (total entropy of an isolated system can never decrease over time)
- Rolf Landauer (1961): “any logically irreversible manipulation of information must be accompanied by a corresponding entropy increase of the information-processing apparatus or its environment”
- Only reversible operations can be performed without heat dissipation

Limits of computing

- Beckenstein bound: upper limit to the entropy, or information in a finite region of space (or the maximal amount of information required to perfectly describe a given physical system down to the quantum level)
- Human brain with a volume of 1260 cm^3 has Beckenstein bound of $2.6x10^{42}$ bits
- Margolus–Levitin theorem (1998): the processing rate výpočtů cannot be higher than 6×10^{33} operations / sec / joule (https://arxiv.org/pdf/quant-ph/9710043)
- Seth Lloyd (2010): The Universe could not perform more than 10^{120} operations on 10^{90} bits (https://arxiv.org/abs/quant-ph/0110141)