Gödel in 20/20: Hindsight, Foresight, and Logical Blindness

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Part 1: Logical Blindness

“When such a procedure is followed, the question **at once** arises whether the initially postulated system of axioms and principles of inference is complete, that is, whether it actually suffices for the derivation of *every* logico-mathematical proposition, or whether, perhaps, it is conceivable that there are true [*wahre*] propositions . . . that cannot be derived in the system under consideration.”

Gödel 1930

Part 1: Logical Blindness

"Frege, however, never saw completeness as a problem, and indeed almost fifty years elapsed between the publication of *Frege 1879* and that of *Hilbert and Ackermann 1928*, where the question of the completeness of quantification theory was raised explicitly for the first time. Why? Because neither in the tradition in logic that stemmed from Frege through Russell and Whitehead, that is, logicism, nor in the tradition that stemmed from Boole through Peirce and Schröder, that is, algebra of logic, could the question of the completeness of a formal system arise."

Dreben and Van Heijenoort, "Introductory note to *Gödel 1929*, *Gödel 1930*, and *Gödel 1930a*"

Part 1: Logical Blindness

"As for Skolem, what he could justly claim, but apparently does not claim, is that, in his *1923* paper, he implicitly proved: 'Either A is provable or $\neg A$ is satisfiable' ('provable' taken in an informal sense). However, since he did not clearly formulate this result (nor, apparently, had he made it clear to himself), it seems to have remained completely unknown, as follows from the fact that Hilbert and Ackermann in 1928 do not mention it in connection with their completeness problem."

Gödel in a 1964 letter to Van Heijenoort

Part 1: Logical Blindness

"The completeness theorem, mathematically, is indeed an almost trivial consequence of *Skolem 1923*. However, the fact is that, at that time, nobody (including Skolem himself) drew this conclusion (neither from *Skolem 1923* nor, as I did, from similar considerations of his own.)"

Gödel in a 1967 letter to Hao Wang

Part 1: Logical Blindness

Herbrand even went so far as to say in *1930* that it is tempting to infer from his results a certain statement which combined with the inferences he does draw establishes the semantic completeness of quantification theory, but that the statement is too idealistic to make real, concrete sense.

Part 1: Logical Blindness

“The blindness of logicians is indeed surprising.”

Gödel in a letter to Hao Wang:

Part 1: Logical Blindness

"If comment is a measure of interest, then the completeness of quantification theory held absolutely no interest for Skolem. There is not one reference to completeness in the fifty-one papers on logic, dating from 1913 through 1963, collected in *Skolem 1970*."

Dreben and Van Heijenoort, "Introductory note to *Gödel 1929*, *Gödel 1930*, and *Gödel 1930a*"

Part 1: Logical Blindness

Not quite as statistically impressive, but perhaps equally of interest, is Gerhard Gentzen's attitude.

Only once in the ten papers compiled in *Szabo 1970* did Gentzen mention the completeness of quantification theory.

(The reference is in the *1936* paper "Die Widerspruchsfreiheit der reinen Zahlentheorie.")

Part 1: Logical Blindness

In his *1934–35* dissertation "Untersuchungen über das logische Schliessen" Gentzen developed the systems of natural deduction and sequent calculus for quantification theory and proved their deductive equivalence as well as their equivalence with "a calculus modeled on the formalism of Hilbert."

Because the latter system was known already to be sound and complete with respect to the standard quantificational semantics, Gentzen could have immediately inferred the same properties for his classical calculi NK and LK.

But he neither referenced the completeness theorem nor posed the question.

Part 1: Logical Blindness

Not even in his expository *1938* paper “Die gegenwärtige Lage in der mathematischen Grundlagenforschung,” did Gentzen mention the completeness of quantification theory.

This, despite Gentzen’s claim in a section called “Exact foundational research in mathematics: axiomatics, metalogic, metamathematics: The theorems of Gödel and Skolem” that his purpose was to “discuss some of the more recent findings and, in particular, some of the especially important earlier results obtained in the exact foundational research in mathematics.”

Part 1: Logical Blindness

"A main task of metamathematics is the development of the *consistency proofs* required for the realization of *Hilbert's* programme. Other major problems are: The *decision problem*, i.e., the problem of finding a procedure for a given theory which enables us to decide of every conceivable assertion whether it is true or false; further, the question of *completeness*, i.e., the question of whether a specific system of axioms and forms of inference for a specific theory is *complete*, in other words, whether the truth or falsity of every conceivable assertion of that theory can be proved by means of these forms of inference."

Gentzen 1938

Part 1: Logical Blindness

Against this backdrop Gentzen then reviewed:

- Gödel's second incompleteness theorem
- Gentzen's own arithmetical consistency proof using transfinite induction
- Church's theorem on the undecidability of quantification theory as well as Gödel's preliminary work in this direction
- Gödel's first incompleteness theorem
- Ackermann's proof of the consistency of "general set theory" relative to the consistency of elementary number theory
- the Löwenheim-Skolem theorem
- Skolem's proof of the existence of nonstandard models of first-order arithmetical systems

Part 1: Logical Blindness

Obviously the completeness of quantification theory is a fundamental metalogical result quite difficult to omit from such a discussion, but Gentzen never mentioned it.

Part 1: Logical Blindness

To sum up, there are features of the thought of figures like Herbrand, Skolem, and Gentzen that, variously, prevented them from recognizing semantic completeness as a phenomenon or dissuaded them from acknowledging the relevance of the completeness theorem after it was proved.

One might suppose that those ways of thinking would be uninteresting to a modern logician if they could be recovered today. One might despair of the possibility of recovering them anyway due to the ossification of the point of view that Gödel introduced—the point of view that inclines us to think that these "deviant" logicians were missing something instead of being on to something.

But I am more optimistic.

Part 1: Logical Blindness

Gödel attributed the “blindness” of his predecessors to their lacking the appropriate attitude “toward metamathematics and toward non-finitary reasoning.”

Others have suggested that the principal obstacle in other writers’ way to the completeness theorem was a failure to appreciate the value of restricting one’s attention to first-order quantification. “When Frege passes from first-order logic to a higher-order logic,” van Heijenoort writes, “there is hardly a ripple.”

Moore emphasizes instead the “failure” on the part of Gödel’s contemporaries “to distinguish clearly between syntax and semantics.”

Part 1: Logical Blindness

Each of these accounts points to a substantial aspect of Gödel's characteristic approach to the study of logic. But I want to make a stronger suggestion that "logical blindness" is reciprocating.

Gödel never considered that others' logical vision might be, rather than defective, simply different—that their inability to see their way to the completeness theorem derived from their focus being held elsewhere.

Part 2: Reading Aristotle (Prior Analytics i23)

It is clear from what has been said that the deductions in these figures are completed by means of the universal deductions in the first figure and are reduced to them.

That every deduction without qualification can be so treated will be clear presently when it has been proved that every deduction is formed through one or another of these figures.

...

But if this is true, every demonstration and every deduction must be formed be means of the three figures mentioned above.

But when this has been shown it is clear that every deduction is completed by means of the first figure and is reducible to the universal deductions in this figure.

Part 2: Reading Aristotle

What is Aristotle showing here, if not completeness?

Part 2: Reading Aristotle

"This account raises a question. When John Corcoran and I wrote about Aristotle's logic at the beginning of the seventies, and when Jonathan Lear did the same at the end of the decade, we were all three alert to the possibility of a completeness proof in the *Prior Analytics*. Why did we all decide against it . . . ?"

Smiley, "Aristotle's completeness proof"

Part 2: Reading Aristotle

"The natural inference from the [passages extracted from i23] is that the intervening material represents a completeness proof. Corcoran made this very point, but he was deterred from following it up by two objections. One was that the text did not fit his picture of a completeness proof. The other was that Aristotle was not 'clear enough about his own semantics to understand the problem' of completeness. Corcoran therefore fell back on seeing the chapter, not as finishing off a completeness proof for Aristotle's chosen rules of inference, but as supplying a proof of the equivalence between them and a second set of rules . . . ."

Smiley, "Aristotle's completeness proof"

Part 2: Reading Aristotle

"My own preoccupations at the time led me to skip over i23 in favor of i25, but this was a double blunder. [...]."

Smiley, "Aristotle's completeness proof"

Part 2: Reading Aristotle

“Lear made the point that ‘it would be anachronistic to attribute to Aristotle the ability to raise the question of completeness’ because, unlike a modern logician, ‘Aristotle had a unified notion of logical consequence—not the bifurcated notion of semantics and syntactic consequence.’”

Smiley, “Aristotle’s completeness proof”

Part 2: Reading Aristotle

"As to one's picture of a completeness proof, it is quite true that Aristotle's proof is unlike anything one would expect."

"…but in emphasizing the difference between Aristotle's project and the modern one there is a danger of overlooking their similarity; a similarity that seems to me to be more significant than their difference."

Smiley, "Aristotle's completeness proof"

Part 2: Reading Aristotle

"To [Lear's] objection that Aristotle was not 'conscious of the distinction between syntactic and semantic consequence—and therefore of the need to prove completeness,' I would rejoin that Aristotle was conscious of the distinction between what follows and what can be shown to follow—and therefore of the need to prove completeness."

Smiley, "Aristotle's completeness proof"

Part 3: The Chasm

It is often said that for most of history, the question of logical completeness did not arise . . .

. . . and one sometimes hears that the reason for this is that the question could not meaningfully be posed.

This is not quite true, we know.

Part 3: The Chasm

Aristotle asked whether every categorical statement that follows from a finite set of premises, in the sense that no substitution of categorical terms could simultaneously make those premises all true while falsifying the candidate conclusion, could be formally deduced from those premises with a predesignated stock of inference rules.

Part 3: The Chasm

But there is something else wrong with the idea that the ability to meaningfully pose a question, with precise criteria for what would count as an answer, about logical completeness is an obstacle to being able even to entertain the idea that one's logical system is fully adequate.

Part 3: The Chasm

In the description of a “symbolic calculus” with which he began his treatise on *Trigonometry and Double Algebra*, **Augustus de Morgan** listed three ways in which a formal system, even one whose “given rules of operation be necessary consequences of the given meanings as applied to the given symbols,” could nevertheless be “imperfect.” The last sort of imperfection he considered is that the system “may be incomplete in its rules of operation.”

Part 3: The Chasm

He explained: “This incompleteness may amount either to an absolute privation of results, or only to the imposition of more trouble than, with completeness, would be required. Every rule the want of which would be a privation of results, may be called *primary*: all which might be dispensed with, except for the trouble that the want of them would give, may be treated merely as consequences of the primary rules, and called *secondary*.”

Part 3: The Chasm

Some years later in an paper called “On the algebra of logic,” **C. S. Peirce** boldly asserted, “I purpose to develop an algebra adequate to the treatment of all problems of deductive logic,” but issued this caveat: “I shall not be able to perfect the algebra sufficiently to give facile methods of reaching logical conclusions; I can only give a method by which any legitimate conclusion may be reached and any fallacious one avoided.”

Part 3: The Chasm

It is entirely mysterious why Peirce felt entitled to claim that his logical system is complete. No argument of any sort to this effect appears in his paper. De Morgan made no such boast.

But the two logicians shared an ability to speak about the importance of logical completeness without being able to say anything about what it would be like to justify the claim that a logical system is complete.

Part 3: The Chasm

There is a tension in Gödel's claim that anyone working with a logical system immediately wonders about its completeness.

Clearly it is not the case that the completeness question immediately struck everyone as a mathematical problem deserving attention.

But the feeling that one one wants not to leave anything out of their system, perhaps even the feeling that one hasn't, doesn't depend on that acknowledgement.

Part 3: The Chasm

The first person after Aristotle to raise the question of logical completeness as an open problem appears to have been **Bernard Bolzano**.

He did not manage to solve the problem and was acutely aware of this.

But in the research program he embarked on he went very far in providing the raw materials that would be used later to recast the completeness question in ways amenable to mathematical solution.

Part 3: The Chasm

In particular, Bolzano distinguished two separate realms of logical investigation. And he described how one would establish the adequacy of the central notions in one of those realms by establishing a correspondence between the two sides of the divide.

Part 3: The Chasm

The work most remembered and highly regarded by modern logicians, because of its striking resemblance to twentieth century set-theoretical definitions of consequence, concerns the *Ableitbarkeit* (“derivability”) relation. In his 1837 masterpiece, *Wissenschaftslehre*, Bolzano in fact defines a network of concepts—validity, compatibility, equivalence, and derivability—in terms of one another in a way very similar to contemporary presentations. Here is his definition of the last of these:

Part 3: The Chasm

"Let us then first consider the case that there is a relation among the compatible propositions A, B, C, D, . . . M, N, O, . . . such that all the ideas that make a certain section of these propositions true, namely A, B, C, D, . . . , when substituted for i, j, . . . also have the property of making some other section of them, namely M, N, O, . . . true. The special relationship between propositions A, B, C, D, . . . on the one side and propositions M, N, O, . . . on the other which we conceive of in this way will already be very much worthy of attention because it puts us in the position, in so far as we once know it to be present, to be able to obtain immediately from the known truth of A, B, C, D, . . . the truth of M, N, O, . . . as well. . . ."

Part 3: The Chasm

"…Consequently I give the relationship which subsists between propositions A, B, C, D, …on the one hand and propositions M, N, O, …on the other the title, a relationship of *derivability* [*Ableitbarkeit*]. And I say that propositions M, N, O, …would be derivable from propositions A, B, C, D, …with respect to the variables i, j, …, if every set of ideas which makes A, B, C, D, …all true when substituted for i, j, …also makes M, N, O, …all true." (§155)

Part 3: The Chasm

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For one thing, Bolzano requires all the propositions involved in the *Ableitbarkeit* relationship to be "compatible" with one another. The result is analogous to a stipulation, absent from modern logical theory, that formulas be jointly-satisfiable in order to stand in a relationship of logical consequence with one another. One result of this unfamiliar requirement is that, for Bolzano, nothing at all is derivable from a self-contradictory proposition, whereas in modern logical theory all formulas are consequences of an unsatisfiable one.

Part 3: The Chasm

Although this notion of derivability prefigures modern definitions of logical consequence in many ways, there are several evident disparities between Bolzano's concept and our own.

It is also noteworthy that Bolzano attends to propositions, not formulas, and to their reinterpretations over a fixed domain of ideas. This is a more conservative approach to modality than the modern one, wherein not only may the extensions of predicate symbols and constant symbols vary, but so too may the underlying set of objects.

Part 3: The Chasm

Although this notion of derivability prefigures modern definitions of logical consequence in many ways, there are several evident disparities between Bolzano's concept and our own.

Furthermore, the individuation of "ideas" with respect to which one may vary one's interpretation is made imprecise by the focus on "propositions" and their constituents in place of the modern focus on formulas and the symbols they contain.

Part 3: The Chasm

Bolzano's stance on these matters, though it appears peculiar from a modern point of view, was not whimsical. He maintained his position consistently over many years.

However, the modern notion is not conceptually distant from Bolzano's on these points and can meaningfully be seen as a refinement or adjustment of his definition.

Part 3: The Chasm

Nevertheless, a strong contrast must be drawn between Bolzano's *Ableitbarkeit* relation and the modern notion of logical consequence, if not in terms of their technical details, in terms of the sort of relationship their authors took themselves to be defining.

Part 3: The Chasm

The logical consequences of a formula, on the modern view of things, are solely determined by the existence and details of certain set-theoretical structures, quite independently of our access to them or ability to draw inferences based on them.

Bolzano's *Ableitbarkeit* relation, by contrast, is procedural: There is nothing "out there" over and above particular deductions that we might perform that determines any special relationship between the propositions that bear this relation to one another.

Part 3: The Chasm

In Bolzano's preferred terminology, the relationship corresponds to no "objective dependence" of propositions on one another (*Bolzano 1810*, II.§12). It is merely the case that *we* are able "to obtain immediately from the known truth of A, B, C, D, ... the truth of M, N, O, ... as well."

Rather than explain the phenomenon of inferring correctly in terms of a metaphysical relationship between propositions that our inferences might track, Bolzano treated right reasoning as primitive, the variation of ideas in propositions as part of the inferential process.

Part 3: The Chasm

Of course, Bolzano was not claiming that propositions only stand in the *Ableitbarkeit* relation with one another after someone has in fact carried out a logical deduction.

It is an objective and eternal fact, for Bolzano, whether or not such a relationship attains.

So what is his point in saying that the relationship is only subjective, that a truth obtained in this way is a "mere conclusion [*bloßer Schlußsatz*]" and not a "genuine consequence [*eigentliche Folge*]" (§200)?

Part 3: The Chasm

Throughout his logical investigations, Bolzano's considerably more sustained focus was devoted, not to the *Ableitbarkeit* relation, but to the theory of an objectively significant consequence relation, a theory he called "*Grundlehre*."

Part 3: The Chasm

Bolzano's 1810 *Beyträge* is the definitive exposition of this theory of ground and consequence. In §2 of part II of that booklet, Bolzano wrote:

Part 3: The Chasm

"[I]n the realm of truth, i.e. in the sum total of all true judgments, a certain *objective connection* prevails which is independent of our actual and *subjective recognition* of it. As a consequence of this some of these judgements are the grounds of others and the latter are the consequences of the former. To represent this objective connection of judgements, i.e. to choose a set of judgements and arrange them one after another so that a consequence is represented as such and conversely, seems to me to be the proper *purpose* to pursue in a scientific exposition. Instead of this, the purpose of a scientific exposition is *usually* imagined to be the greatest possible *certainty* and *strength of conviction*."

Part 3: The Chasm

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A proof, according to Bolzano, must track the objective *Abfolgen* between propositions. Individual mathematical truths therefore have at most one proof (§5).

Part 3: The Chasm

This consequence relation, which Bolzano called *Abfolge*, is at the center of a robust philosophical account of mathematical truth. Its influences are legion.

Moreover, the division of mathematical truths into those that have proofs and those basic truths for which no proof can be given is not a matter of convention but is objectively determined and there for us to discover (§13).

Part 3: The Chasm

This consequence relation, which Bolzano called *Abfolge*, is at the center of a robust philosophical account of mathematical truth. Its influences are legion.

The distinguishing features of an axiom are, accordingly, not its self-evidence, but its ontological role as ground for other truths and the absence of any proposition serving in the capacity of its ground (§14).

Part 3: The Chasm

This consequence relation, which Bolzano called *Abfolge*, is at the center of a robust philosophical account of mathematical truth. Its influences are legion.

Conversely, and most importantly for Bolzano, the self-evidence of a mathematical fact is no reason not to seek a proof for it, for a proof will uncover its grounds, which are typically unrelated to the (good) reasons we might have for accepting the fact as true (§7).

Part 3: The Chasm

One might reasonably wonder why this hypothesized network of objective relationships should more properly be the focal point of “scientific exposition” than the simple discovery of mathematical facts. Bolzano provided several justifications for the shift in perspective.

Part 3: The Chasm

Primarily, and most often, Bolzano points to an inherent value in coming to understand the structure of the hierarchy of facts.

This hierarchy is a feature of the world forever off limits to researchers who "stop short" at certainty. Behind this incentive is the idea that proofs, of the special sort that Bolzano seeks, are explanatory: A fact's grounds are the reason why that fact is true. In some sense they constitute their consequences, and therefore being more than "*Gewissmachungen*" that assure us of a truth, proper proofs are "*Begründungen*, i.e., presentations of the objective reason for the truth concerned" (*Bolzano 1817*, Preface, §I).

Science should not simply record but also explain facts.

Part 3: The Chasm

There is also an aesthetic value to Bolzano's proofs. Through them, one is able to see one's way to a mathematical truth without recourse to ideas and terms that are "off topic."

"[I]f there appear in a proof *intermediate concepts* which are, for example, *narrower* than the subject, then the proof is obviously defective; it is what is usually otherwise called a μετάβασις εὶς `άλλο γένος" (*1810*, II.§29).

Thus, although Bolzano did not actually define the *Abfolge* relation or specify, in any but a few select cases, what the unprovable basic truths are, he disclosed a highly non-trivial fact about the *Grundlehre*: Every non-basic fact is grounded in other facts about one and the same concepts that the consequent, non-basic fact is about.

Part 3: The Chasm

Bolzano further hinted that the conceptual purity of his proofs affords a scientific advantage, in that it will facilitate the discovery of new truths.

Part 3: The Chasm

In developing the *Grundlehre*, Bolzano advanced logical theory in ways comparable in scope to his work on *Ableitbarkeit* but oriented in a different direction.

In his youthful *1804* pamphlet he wrote, “I must point out that I believed I could not be satisfied with a completely strict proof *if it were not even derived from concepts* which the thesis proved contained, but rather made use of some fortuitous alien, *intermediate concept* [*Mittelbegriff*], which is always an erroneous μετάβασις εὶς `άλλο γένος” (Preface, par. 4).

Part 3: The Chasm

That proofs should be free from such intermediate concepts and the concomitant “atrocious detours” in reasoning (attributed to Euclidean methods) was inspired by the desire to capture the objective ground and consequence relations in the world, to produce proofs that were topically pure and therefore free from circularity.

But the significance of the notion of analyticity that Bolzano developed is not tied down to those ambitions.

Part 3: The Chasm

In §17 of part II of the *Beyträge*, Bolzano had distinguished analytic and synthetic truths according to the Kantian criterion of conceptual containment (the predicate of an analytic, and not a synthetic, truth contains its subject.)

In §31 he extended this to a distinction between analytic and synthetic *proofs*.

Part 3: The Chasm

A proof is analytic if its derived formula contains, in its compound concepts, all the simple concepts that appear elsewhere in the proof.

Remarkably, Bolzano suggested that "the whole difference between these two kinds of proof [analytic and synthetic] is based simply on the *order and sequence* of the propositions in the exposition."

Thus Bolzano rediscovered the formidable ontological burden that he placed on proofs reflected in a rather mundane feature of those proofs' written appearance.

Part 3: The Chasm

This observation is supported by the rudiments of a theory of proof transformation, outlined in §20.

Because every compound proposition is built out of a subject and predicate which depend on the individual concepts of which it is composed, the proposition itself, if true, “is actually also a derivable, i.e. provable proposition.”

Moreover, its single proof begins with only simple propositions about the simple concepts contained in the compound, proved proposition.

Part 3: The Chasm

One of Bolzano's great discoveries is that the rules of inference in a proper analytic proof that lead from these simple propositions to the proved proposition are other than the patterns of syllogistic reasoning to which logicians in his day devoted so much attention.

Part 3: The Chasm

He wrote: “I believe that there are some *simple kinds of inference* apart from the syllogism.”

Among his examples is the inference from “*A is* (or *contains*) *B*” and “*A is* (or *contains*) *C*” to “*A is* (or *contains*) [*B et C*].”

“[I]t is also obvious,” Bolzano claimed “that according to the necessary laws of our thinking the first two propositions can be considered as *ground* for the third, and not conversely” (§12).

Part 3: The Chasm

After illustrating a couple of other such rules, which similarly establish compound clauses within the sub-sentential structure of propositions, Bolzano noted a crucial difference between his new "analytical" rules and the syllogism, clearly based on his rich notion of *Abfolge*:

The syllogism rule is not reversible—its premises in no way follow from its conclusion—but the analytical rules each are. For this reason, the reverse of each analytical inference "could seem like an example of another kind of inference . . . .

Part 3: The Chasm

”But I do not believe that this is a [proper] *inference* . . . . I can perhaps *recognize* subjectively from the truth of the *first* of these three propositions the truth of the two others, but I cannot view the first *objectively* as the *ground* of the others.” (§12)

Part 3: The Chasm

Thus propositions with compound concepts can be proved in a way that charts the *Abfolge* hierarchy, i.e., purely analytically, from propositions containing only simple concepts.

"On the other hand," Bolzano wrote in a long note to §20, "how propositions with simple concepts could be proved other than through a syllogism, I really do not know."

Part 3: The Chasm

In §27, drawing from the observed features of the analytical inference rules, Bolzano argued for the following claim: “If several propositions appearing in a scientific system have the same subject, then the proposition with the more compound predicate must follow that with the simpler predicate and not conversely.”

Part 3: The Chasm

"Moreover, it is obvious here that we cannot extend our assertion *further*, and instead of the expression, "*the proposition with the more compound predicate*," put the more general one, "*the proposition with the narrower predicate*."

Part 3: The Chasm

In other words, Bolzano recognized that his proofs, because their propositions are ordered so as to track the objective *Abfolgen* in the world, would have a form of what modern logicians call a subformula property were it not for the ubiquity of the syllogism rule.

Even with this rule, though, every proof has a related property. Given the normalizing techniques discussed in §20, typical proofs may generally be written so that they begin with several syllogisms devoted to establishing the needed simple truths from which to infer, purely from analytic rules, their more compound consequence.

Part 3: The Chasm

In the preface of *1817a* Bolzano describes a “purely analytic procedure” differently, as one in which a derivation is performed “just through certain changes and combinations which are expressed *by a rule completely independent* of the nature of the designated quantities.”

This description points to the features of analyticity emphasized by the Eighteenth Century algebraists, who sought to extend algebraic techniques to mechanics, geometry, and other disciplines.

Part 3: The Chasm

Laplace, for example, in chapter 5 of book V of his *Exposition du système du monde* had written:

“The algebraic analysis soon makes us forget the main object [of our researches] by focusing our attention on abstract combinations and it is only at the end that we return to the original objective. But in abandoning oneself to the operations of analysis, one is led by the generality of this method and the inestimable advantage of transforming the reasoning by mechanical procedures to results inaccessible to geometry.” (*Kline 1972*, p. 615)

Part 3: The Chasm

Similarly Lagrange (*1788*, preface) declared, “The methods which I expound in [*Mécanique analytique*] demand neither constructions nor geometrical or mechanical reasonings, but solely algebraic operations subjected to a uniform and regular procedure.”

Part 3: The Chasm

Nowadays one reflexively associates these “mechanical procedures” with derivability and conceives of logical consequence as residing in the “object of our researches”—on the semantic half of this divide.

One wonders whether one’s formulas are adequate to their intended interpretations, whether these mechanical procedures in fact trace the interrelationships among the objects of our researches, these latter being “the original objective.”

Part 3: The Chasm

Bolzano could only have it the other way around: His *Grundlehre* revealed that the analytic calculus traces facts back their their ultimate, constitutive grounds, and he shared Laplace's suspicion that these same dependencies might be inaccessible by geometrical or other traditional mathematical inferences.

Should one side of this divide prove inadequate, it could only be the latter—this being rightly designated as mere derivability—because "by abandoning oneself to the operations of analysis" one accesses the objective dependencies among truths.

Part 4: The Question

Whereas the *Abfolge* relation holds only between truths, false propositions may stand in the relationship of *Ableitbarkeit* with one another so long as (1) under some substitution of ideas they all are true and (2) under all substitutions that make some one part of them true, so too is the second part.

For this simple reason, one cannot conclude from the derivability of some proposition that one has uncovered the grounds, in the premises of this derivation, of a proposition.

Part 4: The Question

More crucially, the same conclusion cannot be drawn even when all the propositions in the derivation are true. This is evident from the fact that derivability is obviously reflexive and often symmetric, whereas according to Bolzano no truth is its own ground (*1837*, §204), and no two truths could mutually ground one another (§211).

The *Ableitbarkeit* relation is not, in Bolzano's idiom, "subordinate" to the *Abfolge* relation.

Part 4: The Question

Given the nature of these relations, the converse question seems more relevant: Is every *Abfolge* relation representable with a derivation?

If not, then the very idea of placing this conception of logical consequence at the center of all scientific exposition is puzzling. Whatever the merits of knowing the objective grounds of a mathematical fact, no science can be devoted to this task without some sort of method for the discovery of such grounds.

Part 4: The Question

On the other hand, if from its grounds a truth can always, in principle, be derived, then the process-centered *Ableitbarkeit* relation is seen to be adequate to trace the objective dependencies of truths on one another.

Part 4: The Question

Bolzano devoted §200 to this question. The section is entitled, "Is the relation of ground and consequence subordinate to that of derivability?" Here is how he explained the point:

"If truths are supposed to be related to each other as ground and consequence, they must always, one might believe, be derivable from one another as well. The relation of ground and consequence would then be such as to be considered a particular species of the relation of derivability; the first concept would be subordinate to the second."

Part 4: The Question

A little reflection suggests that this is unlikely.

Why should the ultimate reasons for the myriad truths of mathematics always be related to them so that from consideration of the variation of ideas in each, one can reliable infer from them their objective dependencies?

Why should the formal features of propositions give us any access to the shimmering reality beyond? Few contemporary writers share Bolzano's confidence in our intuitions about the realm of objective dependencies, but even Bolzano recognized the unconvincingness of speculation on this issue.

Part 4: The Question

"Probable as this seems to me," Bolzano concluded, "I know no proof that would justify me in looking upon it as settled."

Part 5: Coordination

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Logic is blind to considerations of truth, to say nothing of ultimate explanations for why some statements are in fact true.

Indeed, Bolzano’s own development of *Ableitbarkeit* was a turn away from factual truth, towards distinguishing those statements that could be true from those that could not, towards identifying statements that rise and fall together no matter what the world is like.

Part 5: Coordination

"Mathematics," he wrote, "concerns itself with the question, *how must things be made in order that they should be possible?*" unlike metaphysics which "raises the question, *which things are real?*" (*1810*, I.§9).

Part 5: Coordination

"Mathematics," he wrote, "concerns itself with the question, *how must things be made in order that they should be possible?*" unlike metaphysics which "raises the question, *which things are real?*" (*1810*, I.§9).

But in the end, it was the objective grounding of truths that drove him, and if the theory of *Ableitbarkeit* cannot be shown to trace the world's *Abfolgen*, it loses much of its scientific interest.

Part 5: Coordination

Modern logicians, by contrast, have no expectation that their craft will uncover ultimate grounds. Many do not even believe in such things.

Part 5: Coordination

What remains of Bolzano's intricate scheme for writers who do not share his metaphysical aspirations?

Part 5: Coordination

There are at least two distinct ways forward from the impasse that Bolzano found himself in.

Part 5: Coordination

First Path

One of them involves reversing Bolzano's entire dialectic: Uproot the relations he called "derivability" and "consequence" from their metaphysical setting and ask, not about the adequacy of the first to capture the second, but whether the proof calculus (no longer subject to constraints of analyticity) can always be used to deduce, from a set of *formulas* those *formulas* that are true in every interpretation in which each of the first are true.

This way forward—Gödel's way forward—requires considerable modification of Bolzano's fundamental notions.

Part 5: Coordination

Second Path

The other way forward is a more natural extension of Bolzano's framework: Think again of the relation Bolzano called "derivability" (modified slightly) as an abstract consequence relation.

First project this relation into the proof system itself, so that a single logical system has analytical logical rules living side by side with rules encoding the abstract consequence relation.

Then continue Bolzano's project of proof transformation to see if the steps in a given proof of this system can be systematically rearranged until the entire proof is transformed into an object with no occurrences of the second type of rule.

Part 5: Coordination

Whereas on the first path, completeness is proved by bridging the divide between Bolzano's two realms, now re-conceived as the syntactic and semantic . . .

On the second path, the chasm Bolzano dug is filled in *before* the completeness problem is made precise, by formalizing the consequence relation within the same deductive system that houses the logical rules.

Part 5: Coordination

Gerhard Gentzen's first two fundamental theorems together constitute a completeness result according to the scheme of "projection and elimination" just described.

Part 5: Coordination

The principal result [*Hauptsatz*] of the *Untersuchungen* is the normalization technique for quantification theory known today as cut-elimination.

In subsection 2 of the synopsis, Gentzen explained that "[t]he *Hauptsatz* says that every purely logical proof can be reduced to a definite, though not unique, normal form," and added: "Perhaps we may express the essential properties of such a normal proof by saying: it makes no detour [*er macht keine Umwege]*."

Thus everything provable in predicate logic turns out in fact to have a direct proof into which "[n]o concepts enter . . . other than those contained in its final result."

Part 5: Coordination

Gentzen's formal analysis of logical consequence

"Sentences"' of the form $M \longrightarrow v$, where v is an "element" and M is a "complex" (a non-empty set of finitely many elements).

Sentences can also be written with the elements of a complex displayed: $u_1, u_2, \ldots u_n \longrightarrow v$.

Part 5: Coordination

Gentzen's formal analysis of logical consequence

Because complexes are sets, the same element cannot appear multiple times in the same complex, and the order in which the elements of a complex are listed is immaterial.

Gentzen referred to the complex of a sentence as its antecedent and to the lone element on the right of the arrow symbol as the succedent.

He defined tautologies to be those sentences whose antecedent is the singleton set containing the same element that appears in the sentence's succedent.

Part 5: Coordination

Gentzen's formal analysis of logical consequence

"We say that a complex of elements *satisfies* a given sentence if it either does not contain all antecedent elements of the sentence, or alternatively, contains all of them and also the succedent of that sentence. . . . We now look at the complex K of all (finitely many) elements of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ and $\mathfrak{q}$ and call $\mathfrak{q}$ a *consequence* of $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ if (and only if) every subcomplex of K which satisfies the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ also satisfies $\mathfrak{q}$."

Part 5: Coordination

Gentzen's formal analysis of logical consequence

Gentzen specified two inference rules for his system, which he called "thinning" and "cut":

$$\frac{L \longrightarrow v}{ML \longrightarrow v}\ \textit{thinning} \qquad\qquad \frac{L \longrightarrow u \qquad Mu \longrightarrow v}{LM \longrightarrow v}\ \textit{cut}$$

Part 5: Coordination

Gentzen's formal analysis of logical consequence

Then he defined a "proof" of a sentence $\mathfrak{q}$ from the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ to be

"an ordered succession of inferences (i.e., thinnings and cuts) arranged in such a way that the conclusion of the last inference is $\mathfrak{q}$ and that its premises are either premises of the $\mathfrak{p}$'s or tautologies."

Part 5: Coordination

Theorem I of *Gentzen 1932* states that the proof system is "correct": "if a sentence $\mathfrak{q}$ is 'provable' from the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$ then it is a 'consequence' of them."

Part 5: Coordination

As one would expect, Gentzen's statement of "informal completeness" is the converse of this result: "If a sentence $\mathfrak{q}$ is a 'consequence' of the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$, then it is also 'provable' from them."

Part 5: Coordination

Gentzen established the informal completeness of his sentence system in **Theorem II**, where he in fact showed that proofs of a specific “normal form” alone suffice to exhibit all the consequences among sentences.

Part 5: Coordination

Normal proofs are proofs of the form:

$$\dfrac{\mathfrak{r}_2 \qquad \dfrac{\mathfrak{r}_1 \qquad \mathfrak{s}_1}{\mathfrak{s}_2}\,\textit{cut}}{\vdots}\,\textit{cut}$$

$$\dfrac{\dfrac{\mathfrak{r}_{n-1} \qquad \dfrac{\mathfrak{r}_{n-2} \qquad \mathfrak{s}_{n-2}}{\mathfrak{s}_{n-1}}\,\textit{cut}}{\mathfrak{s}_n}\,\textit{cut}}{\mathfrak{q}}\,\textit{thinning}$$

That is, such proofs are chains of applications of CUT followed by a single, terminal application of THINNING.

Part 5: Coordination

So **Theorem II** states: If a sentence $\mathfrak{q}$ is a consequence of the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$, then there exists a normal proof of $\mathfrak{q}$ from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$.

Part 5: Coordination

Gentzen's point was that the pure notion of logical consequence is at once simple, uncontroversial, and easily enough specified to be captured formally by a logical calculus, and the soundness and completeness results of *1932* are the proof that his "formal definition of provability" does just that.

In other words, the system of *1932* based on "cut" and "thinning" is a fully adequate systematization of the pure notion of logical consequence.

Gentzen described this result, not as a coordination of the semantic notion of consequence and the syntactic notion of derivability, but as the formalization of the informal notion.

Part 5: Coordination

This analysis showcases logical consequence as synthetic in Bolzano's sense, in fact as being fully captured by his synthetic syllogism rule.

If one knows only that a sentence $u_1, \ldots u_n \longrightarrow v$ was obtained from an application of "cut," it is not possible to determine what sentences were used as premises for that inference, because the "cut element" vanishes in the course of the inference.

Conversely, however, given a collection of truths, presented as sentences in the style of *Gentzen 1932*, from some field of inquiry, it is possible to attempt various pairings of sentences from this collection as premises of a cut inference in order to obtain new sentences, thereby expanding the size of the collection.

Part 5: Coordination

What Gentzen took himself to have proved is that *all* purely logical reasoning that does not turn on a specific understanding of the meanings of any components of individual expressions can be recovered in just this way.

Part 5: Coordination

The idea of the **sequent calculus** was to map the expressions of pure predicate logic onto the “elements” of the *1932* formal definition of provability.

To the rules “cut” and “thinning” Gentzen added new rules for the analysis of the logical symbols that appear within individual elements.

Now that the synthetic notion of logical consequence was already a native structural rule of the system, the rules for the logical symbols (i.e., the conditional) did not have to reproduce its effects.

Structural Rules of LK

$$\frac{\Gamma \longrightarrow \Theta}{\mathfrak{D}, \Gamma \longrightarrow \Theta}\ \textit{thinning(L)} \qquad \frac{\Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \mathfrak{D}}\ \textit{thinning(R)}$$

$$\frac{\mathfrak{D}, \mathfrak{D}, \Gamma \longrightarrow \Theta}{\mathfrak{D}, \Gamma \longrightarrow \Theta}\ \textit{contraction(L)} \qquad \frac{\Gamma \longrightarrow \Theta, \mathfrak{D}, \mathfrak{D}}{\Gamma \longrightarrow \Theta, \mathfrak{D}}\ \textit{contraction(R)}$$

$$\frac{\Delta, \mathfrak{D}, \mathfrak{E}, \Gamma \longrightarrow \Theta}{\Delta, \mathfrak{E}, \mathfrak{D}, \Gamma \longrightarrow \Theta}\ \textit{exchange(L)} \qquad \frac{\Gamma \longrightarrow \Theta, \mathfrak{E}, \mathfrak{D}, \Delta}{\Gamma \longrightarrow \Theta, \mathfrak{D}, \mathfrak{E}, \Delta}\ \textit{exchange(R)}$$

$$\frac{\Gamma \longrightarrow \Theta, \mathfrak{D} \qquad \mathfrak{D}, \Delta \longrightarrow \Lambda}{\Gamma, \Delta \longrightarrow \Theta, \Lambda}\ \textit{cut}$$

Operational Rules of LK

$$\frac{\Gamma \longrightarrow \Theta, \mathfrak{A} \qquad \Gamma \longrightarrow \Theta, \mathfrak{B}}{\Gamma \longrightarrow \Theta, \mathfrak{A} \,\&\, \mathfrak{B}}\ \&\textit{(R)} \qquad \frac{\mathfrak{A}, \Gamma \longrightarrow \Theta \qquad \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \vee \mathfrak{B}, \Gamma \longrightarrow \Theta}\ \vee\textit{(L)}$$

$$\frac{\mathfrak{A}, \Gamma \longrightarrow \Delta \qquad \mathfrak{B}, \Gamma \longrightarrow \Delta}{\mathfrak{A} \,\&\, \mathfrak{B}, \Gamma \longrightarrow \Delta}\ \&\textit{(L)} \qquad \frac{\Gamma \longrightarrow \Delta, \mathfrak{A} \qquad \Gamma \longrightarrow \Delta, \mathfrak{B}}{\Gamma \longrightarrow \Delta, \mathfrak{A} \vee \mathfrak{B}}\ \vee\textit{(R)}$$

$$\frac{\Gamma \longrightarrow \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \longrightarrow \Theta, \forall \mathfrak{x}\mathfrak{F}\mathfrak{x}}\ \forall\textit{(R)} \qquad \frac{\mathfrak{F}\mathfrak{a}, \Gamma \longrightarrow \Theta}{\exists \mathfrak{x}\mathfrak{F}\mathfrak{x}, \Gamma \longrightarrow \Theta}\ \exists\textit{(L)}$$

$$\frac{\mathfrak{F}\mathfrak{a}, \Gamma \longrightarrow \Theta}{\forall \mathfrak{x}\mathfrak{F}\mathfrak{x}, \Gamma \longrightarrow \Theta}\ \forall\textit{(L)} \qquad \frac{\Gamma \longrightarrow \Theta, \mathfrak{F}\mathfrak{a}}{\Gamma \longrightarrow \Theta, \exists \mathfrak{x}\mathfrak{F}\mathfrak{x}}\ \exists\textit{(R)}$$

$$\frac{\mathfrak{A}, \Gamma \longrightarrow \Theta}{\Gamma \longrightarrow \Theta, \neg\mathfrak{A}}\ \neg\textit{(R)} \qquad \frac{\Gamma \longrightarrow \Theta, \mathfrak{A}}{\neg\mathfrak{A}, \Gamma \longrightarrow \Theta}\ \neg\textit{(L)}$$

$$\frac{\Gamma \longrightarrow \Theta, \mathfrak{A} \qquad \mathfrak{B}, \Gamma \longrightarrow \Theta}{\mathfrak{A} \supset \mathfrak{B}, \Gamma \longrightarrow \Theta}\ \supset\textit{(L)} \qquad \frac{\mathfrak{A}, \Gamma \longrightarrow \Theta, \mathfrak{B}}{\Gamma \longrightarrow \Theta, \mathfrak{A} \supset \mathfrak{B}}\ \supset\textit{(R)}$$

Part 5: Coordination

According to Gentzen, the cut rule is a complete formalization of the pure notion of logical consequence.

And it is by far the most widely emulated pattern of reasoning used in mathematics, so that mathematical thought is predominantly synthetic in nature.

But the cut-elimination theorem shows that all mathematical thought that does not rest on the principles of any specific mathematical theory can be simulated with purely analytical reasoning.

Part 5: Coordination

This is the conceptual significance of the subformula property, the fact that "[i]n an LI- or LK-derivation without cuts, all [formula occurrences in a sequent that occurs in the derivation] are *subformulae* of the [formula-occurrences] that occur in [its] endsequent."

Cut-elimination shows that everything provable in the predicate calculus can in fact be proved with a derivation exhibiting this subformula property.

Gentzen added that "[t]he final result [of such a derivation] is, as it were, gradually built up from its constituent elements"—i.e., that the derivation is an analysis, in Bolzano's sense, of the truth of the derived sequent.

Part 5: Coordination

The synthetic nature of "cut" as opposed to the analytic nature of the operational rules is reflected in Gentzen's formulation of LK.

In *1932* Gentzen had already presented a context-free version of "cut," so one might expect to find fully context-free calculi in the *Untersuchungen*.

Oddly though, Gentzen did not treat context consistently in his presentation of LK: He gave $\vee$*(L)*, &*(R)*, and $\supset$*(L)* context-sharing presentations alongside a context-free presentation of "cut."

Part 5: Coordination

This lack of uniformity does not evidently simplify his proof of the *Hauptsatz* and demands explanation.

If one distinguishes, as Aristotle did, between the two methods of logical discovery $\sigma\acute{\upsilon}\nu\theta\varepsilon\sigma\iota\varsigma$ and $\grave{\alpha}\nu\acute{\alpha}\lambda\upsilon\sigma\iota\varsigma$, then the explanation is forthcoming.

Part 5: Coordination

In synthesis, one generates new theorems by methodically combining previously established results that may be obtained from disparate sources—i.e., they may have different contexts.

In analysis, context is determined in advance and can only be narrowed: One begins with a claim and successively breaks it down to components in order either to refute the claim through the discovery that it rests on some untenable premise or to uncover the elementary facts that attest to the claim's truth.

Part 5: Coordination

Thus sequent calculus rules designed to analyze logical particles should be read upwards from the bottom sequent to its analyzing upper sequent(s). This reading necessitates a context-sharing presentation.

But logical synthesis is more naturally understood “downward from the top”: The premises of a “cut” are typically drawn from distant quarters, and the inference generates new information about how their contexts are related simultaneously with the dispensation of the cut formula. Only a context-free presentation brings out this reading.

Part 5: Coordination

We now see how the sequent calculus houses a highly non-trivial completeness question, about the adequacy of its purely analytic, cut-free fragment: Is this fragment closed under the synthetic operation of logical consequence (cut)?

If so, then the analytic, “logical” rules of this system fully capture the meanings of the particles they govern in a very strong sense: All logical consequences of sentences governed by those particles can be derived with these analytical rules alone.

And this is the question answered by the cut-elimination theorem.

Part 5: Coordination

Rather than a question about the correspondence of realms, this question is about the coordination of methods—analysis and synthesis.

Logical consequence is on the synthetic side of this *methodological* divide, but both analytic and synthetic styles of reasoning live side by side in the immanent features of the proof system.

Part 5: Coordination

The eliminability of the “cut” rule follows immediately from two straightforward observations one can make from the point of view of syntax and semantics coordination:

First, prove that with respect to the usual quantificational semantics the cut-free fragment of LK is complete.

Then verify the full calculus’s soundness with respect to that same semantics.

The possibility that a formula could be provable in the full calculus but not in its cut-free fragment is thereby ruled out.

Part 5: Coordination

Theorem III of Gentzen's *1932* paper states: "If a nontrivial sentence $\mathfrak{q}$ is provable from the sentences $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$, then there exists a normal proof for $\mathfrak{q}$ from $\mathfrak{p}_1, \ldots, \mathfrak{p}_\mathfrak{v}$."

It is illuminating to contrast Gentzen's comments about this normalization result with his approach in the *Untersuchungen*. He wrote:

Part 5: Coordination

"This follows at once from Theorems I [informal soundness] and II [informal completeness of normal-form proofs] together. The theorem can also be obtained directly without reference to the notion of consequence by taking an arbitrary proof and transforming it step by step into a normal proof. The reason for the approach chosen in this paper is that it involves little extra effort and yet provides us with important additional results, viz., the correctness and completeness of our forms of inference."

Part 5: Coordination

By following this same line of thought in the *Untersuchungen*, Gentzen could have established the correctness and completeness of the forms of inference of the calculus LK simultaneously with the eliminability of its cut rule.

Part 5: Coordination

The relevant observations were *technically* and *methodologically* within Gentzen's reach in 1935:

The fact that LK and its cut-free fragment are sound and complete is guaranteed by the proofs in section V of their deductive equivalence with the Hilbert calculus (for which Gödel had proved semantic completeness five years earlier) and with NK,

and the template of inferring normalization from such results appeared already in Gentzen's own earlier work.

Part 5: Coordination

But on Gentzen's *conception* of logic, the parallel between the sequence $\langle$Theorem I, Theorem II, Theorem III$\rangle$ of the *1932* paper and the sequence $\langle$LK soundness, cut-free completeness, cut-elimination$\rangle$ breaks down.

Part 5: Coordination

Theorems I and II of *Gentzen 1932* were Gentzen's verification that the notion of logical consequence is fully analyzed by the formal rule "cut."

Because the notion of logical consequence appears again in this exact form in the immanent features of the calculus LK, the question of the completeness of that logical system was not for Gentzen a question about how the system corresponds with something beyond itself, but a question about the ability of its analytic fragment to keep pace with its internal consequence relation.

Part 5: Coordination

If one thinks of cut-elimination as the completeness of the analytic methods with respect to the synthetic notion of logical consequence, then the idea of inferring the *Hauptsatz* from the "semantic completeness" of those methods does not arise.

Conclusion: Hindsight, Foresight

"As mathematics progresses, notions that were obscure and perplexing become clear and straightforward, sometimes even achieving the status of 'obvious.' Then hindsight can make us all wise after the event. But we are separated from the past by our knowledge of the present, which may draw us into 'seeing' more than was really there at the time."

Goldblatt 2005, section 4.2

Thank you!