Informal Provability and its Logics

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14 Jan 2020

Tribute to Kurt Gödel 2020

Outline

- Define concepts related to that of informal provability as individuated by S4 modal operators
- Employ the concepts both from a formal (embeddings in S4) and an informal (BHK-like interpretation) point of view
- Define logics out of such characterisations
- Define a BHK-like interpretation for classical logic validities and their embedding in S4
- Employ our intuitions in a natural deduction calculus

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Informal ("Absolute") Provability

Gödel (1933), Myhill (1960), Leitgeb (2009)

S4 Modal logic:

- *Necessitation rule:* $\dfrac{A}{\Box A}$
- *K axiom:* $\Box(A \to B) \to (\Box A \to \Box B)$
- *T axiom:* $\Box A \to A$
- *4 axiom:* $\Box A \to \Box\Box A$

Intuitive reading of $\Box A$: $\vdash A$, that is, there is a proof of A.

What about other operators $\Diamond$, $\neg\Box$, $\neg\Diamond$?

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Embeddings

Fitting (1970): $\Box\Diamond$-translation for classical *theorems* in S4:

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P^{\Box\Diamond} & := & \Box\Diamond P \\
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Square of opposition for informal provability

- *if $\Box A$, then $\vdash A$, that is, there is a proof of A.*
- *if $\Diamond A$, then $\nvdash \neg A$, that is, there is no proof of $\neg A$.*
- *if $\neg\Diamond A$, then $\vdash \neg A$, that is, there is a proof of $\neg A$.*
- *if $\neg\Box A$, then $\nvdash A$, that is, there is no proof of A.*

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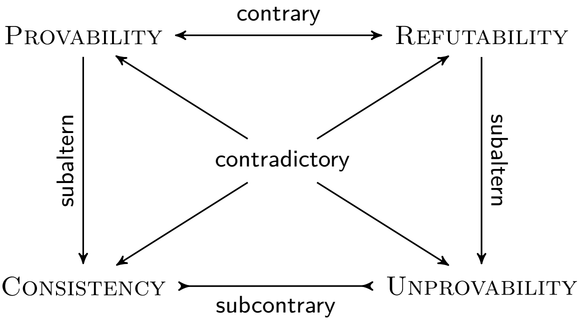

BHK interpretations

- Is it possible to characterise a logic for every point of the square?
- Is it possible to give a BHK interpretation for each one of them?

Example:

$\mathrm{BHK}_{\mathrm{I}}$ elements:

- (Successful) constructions
- Provability
- Logical constant clauses
- Species

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Employing the concepts

Characterising the logic of Provability:

- $\mathrm{BHK_I}$ interpr.
- $\Box$-translation

- *the construction* $\mathbf{c}$ *proves* $A \wedge B$ *iff* $\mathbf{c}$ *is of the form* $\langle \mathbf{c}', \mathbf{c}'' \rangle$ *and* $\mathbf{c}'$ *proves* A *and* $\mathbf{c}''$ *proves* B.
- *the construction* $\mathbf{c}$ *proves* $A \vee B$ *iff* $\mathbf{c}$ *is of the form* $\langle \mathbf{i}, \mathbf{c}' \rangle$ *with* $\mathbf{i}$ *either* $\mathbf{0}$ *or* $\mathbf{1}$, *such that for* $\mathbf{i} = \mathbf{0}$ *then* $\mathbf{c}'$ *proves* A *and for* $\mathbf{i} = \mathbf{1}$ *then* $\mathbf{c}'$ *proves* B.
- *the construction* $\mathbf{c}$ *proves* $A \rightarrow B$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical construct* $\mathbf{c}'$ *that proves* A, $\mathbf{c}(\mathbf{c}')$ *proves* B.
- *the construction* $\mathbf{c}$ *proves* $\neg A$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical construct* $\mathbf{c}'$ *that proves* A, $\mathbf{c}(\mathbf{c}')$ *proves* $\bot$.
- *the construction* $\mathbf{c}$ *proves* $\forall x A$ *iff* $\mathbf{c}$ *is a general method of construction such that given any individual* a *from the species under consideration,* $\mathbf{c}(a)$ *proves* $A(a/x)$.
- *the construction* $\mathbf{c}$ *proves* $\exists x A$ *iff* $\mathbf{c}$ *is of the form* $\langle a, \mathbf{c}' \rangle$, *where* a *is an individual such that* $\mathbf{c}'$ *proves* $A(a/x)$.
- *no construction* $\mathbf{c}$ *proves* $\bot$.
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System LJ

$$P \Rightarrow P \qquad \bot \Rightarrow \qquad \Rightarrow \top \qquad \frac{\Gamma \Rightarrow \Psi}{\Gamma,\, A \Rightarrow \Psi}\ \mathsf{L}W \qquad \frac{\Gamma \Rightarrow}{\Gamma \Rightarrow A}\ \mathsf{R}W$$

$$\frac{A,\, A,\, \Gamma \Rightarrow \Psi}{A,\, \Gamma \Rightarrow \Psi}\ \mathsf{L}C \qquad \frac{\Gamma \Rightarrow A \qquad A,\, \Pi \Rightarrow \Psi}{\Gamma,\, \Pi \Rightarrow \Psi}\ Cut$$

$$\frac{A,\, \Gamma \Rightarrow \Psi}{A \wedge B,\, \Gamma \Rightarrow \Psi}\ \mathsf{L}\wedge_1 \qquad \frac{B,\, \Gamma \Rightarrow \Psi}{A \wedge B,\, \Gamma \Rightarrow \Psi}\ \mathsf{L}\wedge_2 \qquad \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \mathsf{R}\wedge$$

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$$\frac{\Gamma \Rightarrow A}{\Gamma,\, \neg A \Rightarrow \Psi}\ \mathsf{L}\neg \qquad \frac{A,\, \Gamma \Rightarrow}{\Gamma \Rightarrow \neg A}\ \mathsf{R}\neg$$

$$\frac{A(a/x),\, \Gamma \Rightarrow \Psi}{\forall x A,\, \Gamma \Rightarrow \Psi}\ \mathsf{L}\forall \qquad \frac{\Gamma \Rightarrow A(a/x)}{\Gamma \Rightarrow \forall x A}\ \mathsf{R}\forall^* \qquad \frac{A(a/x),\, \Gamma \Rightarrow \Psi}{\exists x A,\, \Gamma \Rightarrow \Psi}\ \mathsf{L}\exists^* \qquad \frac{\Gamma \Rightarrow A(a/x)}{\Gamma \Rightarrow \exists x A}\ \mathsf{R}\exists$$

General method for proving embeddings

$\Gamma^{\Box} \vdash_{\mathsf{S4}} A^{\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}} A$

$\Leftarrow$: Show that every rule of LJ is derivable in LKS4 modulo $\Box$-translation.

$\Rightarrow$: Show that every LKS4 derivation of a $\Box$-translated sequent is equivalent to a single-conclusion derivation, given cut-elimination and subformula property. Therefore, the only $\Box$-translated provable sequents are single-conclusion ones. Only non-trivial case regards rule RC.

As a corollary, this entails $\Gamma^{\Box} \vdash_{\mathsf{IS4}} A^{\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}} A$.

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$$P \Rightarrow P \qquad \bot \Rightarrow \qquad \Rightarrow \top \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma,\, A \Rightarrow \Delta}\ \mathsf{L}W \qquad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta,\, A}\ \mathsf{R}W$$

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$$\frac{A(a/x),\ \Gamma \Rightarrow \Delta}{\forall x A,\ \Gamma \Rightarrow \Delta}\ \mathrm{L}\forall \qquad \frac{\Gamma \Rightarrow \Delta,\ A(a/x)}{\Gamma \Rightarrow \Delta,\ \forall x A}\ \mathrm{R}\forall^* \qquad \frac{A(a/x),\ \Gamma \Rightarrow \Delta}{\exists x A,\ \Gamma \Rightarrow \Delta}\ \mathrm{L}\exists^* \qquad \frac{\Gamma \Rightarrow \Delta,\ A(a/x)}{\Gamma \Rightarrow \Delta,\ \exists x A}\ \mathrm{R}\exists$$

Employing the concepts

Characterising the logic of Refutability:

- $\neg\Diamond$-translation

Propositional part: Shramko (2016)

$$
\begin{array}{ccc}
\mathcal{L} & & \mathcal{L}_{\Box} \\
P^{\neg\Diamond} & := & \Diamond P \\
(\neg A)^{\neg\Diamond} & := & \Diamond\neg A^{\neg\Diamond} \\
(A \wedge B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \wedge B^{\neg\Diamond}) \\
(A \vee B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \vee B^{\neg\Diamond}) \\
(A \prec B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \prec B^{\neg\Diamond}) \\
(\forall x\, A)^{\neg\Diamond} & := & \Diamond\forall x\, A^{\neg\Diamond} \\
(\exists x\, A)^{\neg\Diamond} & := & \Diamond\exists x\, A^{\neg\Diamond}
\end{array}
$$

In Classical logic, $A \prec B := (\neg B \wedge A)$, that is, $\neg(B \rightarrow A)$.

Employing the concepts

Characterising the logic of Refutability:

- $\neg\Diamond$-translation

Propositional part: Shramko (2016)

$\mathcal{L}$		$\mathcal{L}_{\Box}$
$P^{\neg\Diamond}$	$:=$	$\Diamond P$
$(\neg A)^{\neg\Diamond}$	$:=$	$\Diamond\neg A^{\neg\Diamond}$
$(A \wedge B)^{\neg\Diamond}$	$:=$	$\Diamond(A^{\neg\Diamond} \wedge B^{\neg\Diamond})$
$(A \vee B)^{\neg\Diamond}$	$:=$	$\Diamond(A^{\neg\Diamond} \vee B^{\neg\Diamond})$
$(A \prec B)^{\neg\Diamond}$	$:=$	$\Diamond(A^{\neg\Diamond} \prec B^{\neg\Diamond})$
$(\forall x A)^{\neg\Diamond}$	$:=$	$\Diamond\forall x A^{\neg\Diamond}$
$(\exists x A)^{\neg\Diamond}$	$:=$	$\Diamond\exists x A^{\neg\Diamond}$

In Classical logic, $A \prec B := (\neg B \wedge A)$, that is, $\neg(B \to A)$.

Employing the concepts

Characterising the logic of Refutability:

- $\neg\Diamond$-translation

Propositional part: Shramko (2016)

$$
\begin{array}{ccc}
\mathcal{L} & & \mathcal{L}_{\Box} \\
P^{\neg\Diamond} & := & \Diamond P \\
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(A \vee B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \vee B^{\neg\Diamond}) \\
(A \prec B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \prec B^{\neg\Diamond}) \\
(\forall x A)^{\neg\Diamond} & := & \Diamond\forall x A^{\neg\Diamond} \\
(\exists x A)^{\neg\Diamond} & := & \Diamond\exists x A^{\neg\Diamond}
\end{array}
$$

In Classical logic, $A \prec B := (\neg B \wedge A)$, that is, $\neg(B \rightarrow A)$.

System LDJ

$$P \Rightarrow P \qquad \bot \Rightarrow \qquad \Rightarrow \top \qquad \frac{\Rightarrow \Delta}{A \Rightarrow \Delta}\ \mathrm{L}W \qquad \frac{\Phi \Rightarrow \Delta}{\Phi \Rightarrow A,\ \Delta}\ \mathrm{R}W$$

$$\frac{\Phi \Rightarrow \Delta,\ A,\ A}{\Phi \Rightarrow \Delta,\ A}\ \mathrm{R}C \qquad \frac{\Phi \Rightarrow \Delta,\ A \qquad A \Rightarrow \Lambda}{\Phi \Rightarrow \Delta,\ \Lambda}\ Cut$$

$$\frac{A \Rightarrow \Delta}{A \wedge B \Rightarrow \Delta}\ \mathrm{L}\wedge_1 \qquad \frac{B \Rightarrow \Delta}{A \wedge B \Rightarrow \Delta}\ \mathrm{L}\wedge_2 \qquad \frac{\Phi \Rightarrow \Delta,\ A \qquad \Phi \Rightarrow \Delta,\ B}{\Phi \Rightarrow \Delta,\ A \wedge B}\ \mathrm{R}\wedge$$

$$\frac{A \Rightarrow \Delta \qquad B \Rightarrow \Delta}{A \vee B \Rightarrow \Delta}\ \mathrm{L}\vee \qquad \frac{\Phi \Rightarrow \Delta,\ A}{\Phi \Rightarrow \Delta,\ A \vee B}\ \mathrm{R}\vee_1 \qquad \frac{\Phi \Rightarrow \Delta,\ B}{\Phi \Rightarrow \Delta,\ A \vee B}\ \mathrm{R}\vee_2$$

$$\frac{A \Rightarrow B,\ \Delta}{B \prec A \Rightarrow \Delta}\ \mathrm{L}\prec \qquad \frac{\Phi \Rightarrow A \qquad B \Rightarrow \Delta}{\Phi \Rightarrow \Delta,\ B \prec A}\ \mathrm{R}\prec$$

$$\frac{\Rightarrow \Delta,\ A}{\neg A \Rightarrow \Delta}\ \mathrm{L}\neg \qquad \frac{A \Rightarrow \Delta}{\Rightarrow \Delta,\ \neg A}\ \mathrm{R}\neg$$

$$\frac{A(a/x) \Rightarrow \Delta}{\forall x A \Rightarrow \Delta}\ \mathrm{L}\forall \qquad \frac{\Phi \Rightarrow \Delta,\ A(a/x)}{\Phi \Rightarrow \Delta,\ \forall x A}\ \mathrm{R}\forall^* \qquad \frac{A(a/x) \Rightarrow \Delta}{\exists x A \Rightarrow \Delta}\ \mathrm{L}\exists^* \qquad \frac{\Phi \Rightarrow \Delta,\ A(a/x)}{\Phi \Rightarrow \Delta,\ \exists x A}\ \mathrm{R}\exists$$

Characterising the logic of Consistency and Unprovability:

- $\Diamond$-translation and $\neg\Box$-translation

$$
\begin{array}{ccc}
\mathcal{L} & & \mathcal{L}_{\Box} \\
P^{\Diamond} & := & \Diamond P \\
(\neg A)^{\Diamond} & := & \Diamond A^{\Diamond} \\
(A \wedge B)^{\Diamond} & := & \Diamond(A^{\Diamond} \wedge B^{\Diamond}) \\
(A \vee B)^{\Diamond} & := & \Diamond(A^{\Diamond} \vee B^{\Diamond}) \\
(A \to B)^{\Diamond} & := & \Diamond(A^{\Diamond} \to B^{\Diamond}) \\
(\forall x A)^{\Diamond} & := & \Diamond\forall x A^{\Diamond} \\
(\exists x A)^{\Diamond} & := & \Diamond\exists x A^{\Diamond}
\end{array}
\qquad
\begin{array}{ccc}
\mathcal{L} & & \mathcal{L}_{\Box} \\
P^{\neg\Box} & := & \Box P \\
(\neg A)^{\neg\Box} & := & \Box\neg A^{\neg\Box} \\
(A \wedge B)^{\neg\Box} & := & \Box(A^{\neg\Box} \wedge B^{\neg\Box}) \\
(A \vee B)^{\neg\Box} & := & \Box(A^{\neg\Box} \vee B^{\neg\Box}) \\
(A \prec B)^{\neg\Box} & := & \Box(A^{\neg\Box} \prec B^{\neg\Box}) \\
(\forall x A)^{\neg\Box} & := & \Box\forall x A^{\neg\Box} \\
(\exists x A)^{\neg\Box} & := & \Box\exists x A^{\neg\Box}.
\end{array}
$$

System $\mathbf{LJ}^{\prec}$

Same as LJ, *minus* L$\to$ and R$\to$, *plus*

$$\frac{A, \Gamma \Rightarrow \Psi}{B \prec A, \Gamma \Rightarrow \Psi}\ \mathsf{L}{\prec_1} \qquad \frac{\Gamma \Rightarrow B}{B \prec A, \Gamma \Rightarrow \Psi}\ \mathsf{L}{\prec_2} \qquad \frac{\Gamma \Rightarrow A \qquad B \Rightarrow}{\Gamma \Rightarrow B \prec A}\ \mathsf{R}{\prec}$$

System $\mathbf{LDJ}^{\to}$

Same as LDJ, *minus* L$\prec$ and R$\prec$, *plus*

$$\frac{\Rightarrow \Delta, A \qquad B \Rightarrow \Delta}{A \to B \Rightarrow \Delta}\ \mathsf{L}{\to} \qquad \frac{A \Rightarrow \Delta}{\Rightarrow A \to B, \Delta}\ \mathsf{R}{\to_1} \qquad \frac{\Phi \Rightarrow B}{\Phi \Rightarrow A \to B}\ \mathsf{R}{\to_2}$$

Translations schemes

$$
\begin{array}{rcl}
P^{\Box} & := & \Box P \\
(\neg A)^{\Box} & := & \Box\neg A^{\Box} \\
(A \wedge B)^{\Box} & := & \Box(A^{\Box} \wedge B^{\Box}) \\
(A \vee B)^{\Box} & := & \Box(A^{\Box} \vee B^{\Box}) \\
(A \to B)^{\Box} & := & \Box(A^{\Box} \to B^{\Box}) \\
(\forall x A)^{\Box} & := & \Box\forall x A^{\Box} \\
(\exists x A)^{\Box} & := & \Box\exists x A^{\Box}
\end{array}
\qquad
\begin{array}{rcl}
P^{\neg\Diamond} & := & \Diamond P \\
(\neg A)^{\neg\Diamond} & := & \Diamond\neg A^{\neg\Diamond} \\
(A \wedge B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \wedge B^{\neg\Diamond}) \\
(A \vee B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \vee B^{\neg\Diamond}) \\
(A \prec B)^{\neg\Diamond} & := & \Diamond(A^{\neg\Diamond} \prec B^{\neg\Diamond}) \\
(\forall x A)^{\neg\Diamond} & := & \Diamond\forall x A^{\neg\Diamond} \\
(\exists x A)^{\neg\Diamond} & := & \Diamond\exists x A^{\neg\Diamond}
\end{array}
$$

$$
\begin{array}{rcl}
P^{\Diamond} & := & \Diamond P \\
(\neg A)^{\Diamond} & := & \Diamond A^{\Diamond} \\
(A \wedge B)^{\Diamond} & := & \Diamond(A^{\Diamond} \wedge B^{\Diamond}) \\
(A \vee B)^{\Diamond} & := & \Diamond(A^{\Diamond} \vee B^{\Diamond}) \\
(A \to B)^{\Diamond} & := & \Diamond(A^{\Diamond} \to B^{\Diamond}) \\
(\forall x A)^{\Diamond} & := & \Diamond\forall x A^{\Diamond} \\
(\exists x A)^{\Diamond} & := & \Diamond\exists x A^{\Diamond}
\end{array}
\qquad
\begin{array}{rcl}
P^{\neg\Box} & := & \Box P \\
(\neg A)^{\neg\Box} & := & \Box\neg A^{\neg\Box} \\
(A \wedge B)^{\neg\Box} & := & \Box(A^{\neg\Box} \wedge B^{\neg\Box}) \\
(A \vee B)^{\neg\Box} & := & \Box(A^{\neg\Box} \vee B^{\neg\Box}) \\
(A \prec B)^{\neg\Box} & := & \Box(A^{\neg\Box} \prec B^{\neg\Box}) \\
(\forall x A)^{\neg\Box} & := & \Box\forall x A^{\neg\Box} \\
(\exists x A)^{\neg\Box} & := & \Box\exists x A^{\neg\Box}.
\end{array}
$$

Square of opposition for informal provability logics

- $\Gamma^{\Box} \vdash_{\mathsf{S4}} A^{\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}} A$
- $\Gamma^{\Diamond} \vdash_{\mathsf{S4}} A^{\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}^{\rightarrow}} A$
- $\Gamma^{\neg\Diamond} \vdash_{\mathsf{S4}} A^{\neg\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}} A$
- $\Gamma^{\neg\Box} \vdash_{\mathsf{S4}} A^{\neg\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}^{\prec}} A$

Square of opposition for informal provability logics

- $\Gamma^{\Box} \vdash_{\mathsf{S4}} A^{\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}} A$
- $\Gamma^{\Diamond} \vdash_{\mathsf{S4}} A^{\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}^{\rightarrow}} A$
- $\Gamma^{\neg\Diamond} \vdash_{\mathsf{S4}} A^{\neg\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}} A$
- $\Gamma^{\neg\Box} \vdash_{\mathsf{S4}} A^{\neg\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}^{\prec}} A$

$$
\begin{tikzcd}
\mathsf{I} \arrow[dd, "\text{prop. theorems}"'] & \star \arrow[l] \arrow[r] & \mathsf{DI} \arrow[dd, "\text{prop. antitheorems}"] \\
 & \text{duality} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
\mathsf{DI}^{\rightarrow} \arrow[r, tail, no head] & \star & \mathsf{I}^{\prec} \arrow[l, tail, no head]
\end{tikzcd}
$$

$$
\begin{array}{rcl}
(\Gamma \Rightarrow \Delta)^\star & = & \Delta^\star \Rightarrow \Gamma^\star \\[6pt]
P^\star & = & P \\
(\neg A)^\star & = & \neg A^\star \\
(A \wedge B)^\star & = & A^\star \vee B^\star \\
(A \vee B)^\star & = & A^\star \wedge B^\star \\
(A \rightarrow B)^\star & = & A^\star \prec B^\star \\
(A \prec B)^\star & = & A^\star \rightarrow B^\star \\
(\forall x A)^\star & = & \exists x (A)^\star \\
(\exists x A)^\star & = & \forall x (A)^\star
\end{array}
$$

Dual-Glivenko theorem for $\mathrm{DI}^{\rightarrow}$: $\vdash_{\mathrm{DI}^{\rightarrow}} A$ *iff* $\vdash_{\mathrm{C}} A$ for A propositional

Corollaries:

- $\Gamma^{\Box} \vdash_{\mathrm{IS4}} A^{\Box}$ *iff* $\Gamma \vdash_{\mathrm{I}} A$
- $\Gamma^{\Diamond} \vdash_{\mathrm{DIS4}^{\rightarrow}} A^{\Diamond}$ *iff* $\Gamma \vdash_{\mathrm{DI}^{\rightarrow}} A$
- $\Gamma^{\neg\Diamond} \vdash_{\mathrm{DIS4}} A^{\neg\Diamond}$ *iff* $\Gamma \vdash_{\mathrm{DI}} A$
- $\Gamma^{\neg\Box} \vdash_{\mathrm{IS4}^{\prec}} A^{\neg\Box}$ *iff* $\Gamma \vdash_{\mathrm{I}^{\prec}} A$.

$$
\begin{array}{rcl}
(\Gamma \Rightarrow \Delta)^\star & = & \Delta^\star \Rightarrow \Gamma^\star \\
P^\star & = & P \\
(\neg A)^\star & = & \neg A^\star \\
(A \wedge B)^\star & = & A^\star \vee B^\star \\
(A \vee B)^\star & = & A^\star \wedge B^\star \\
(A \to B)^\star & = & A^\star \prec B^\star \\
(A \prec B)^\star & = & A^\star \to B^\star \\
(\forall x A)^\star & = & \exists x (A)^\star \\
(\exists x A)^\star & = & \forall x (A)^\star
\end{array}
$$

Dual-Glivenko theorem for $\mathsf{DI}^{\to}$: $\vdash_{\mathsf{DI}^{\to}} A$ *iff* $\vdash_{\mathsf{C}} A$ for A propositional

Corollaries:

- $\Gamma^{\Box} \vdash_{\mathsf{IS4}} A^{\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}} A$
- $\Gamma^{\Diamond} \vdash_{\mathsf{DIS4}^{\to}} A^{\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}^{\to}} A$
- $\Gamma^{\neg\Diamond} \vdash_{\mathsf{DIS4}} A^{\neg\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}} A$
- $\Gamma^{\neg\Box} \vdash_{\mathsf{IS4}^{\prec}} A^{\neg\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}^{\prec}} A$.

$$
\begin{array}{rcl}
(\Gamma \Rightarrow \Delta)^\star & = & \Delta^\star \Rightarrow \Gamma^\star \\[1ex]
P^\star & = & P \\
(\neg A)^\star & = & \neg A^\star \\
(A \wedge B)^\star & = & A^\star \vee B^\star \\
(A \vee B)^\star & = & A^\star \wedge B^\star \\
(A \rightarrow B)^\star & = & A^\star \prec B^\star \\
(A \prec B)^\star & = & A^\star \rightarrow B^\star \\
(\forall x A)^\star & = & \exists x (A)^\star \\
(\exists x A)^\star & = & \forall x (A)^\star
\end{array}
$$

Dual-Glivenko theorem for $\mathsf{DI}^{\rightarrow}$: $\vdash_{\mathsf{DI}^{\rightarrow}} A$ *iff* $\vdash_{\mathsf{C}} A$ for A propositional

Corollaries:

- $\Gamma^{\Box} \vdash_{\mathsf{IS4}} A^{\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}} A$
- $\Gamma^{\Diamond} \vdash_{\mathsf{DIS4}^{\rightarrow}} A^{\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}^{\rightarrow}} A$
- $\Gamma^{\neg\Diamond} \vdash_{\mathsf{DIS4}} A^{\neg\Diamond}$ *iff* $\Gamma \vdash_{\mathsf{DI}} A$
- $\Gamma^{\neg\Box} \vdash_{\mathsf{IS4}^{\prec}} A^{\neg\Box}$ *iff* $\Gamma \vdash_{\mathsf{I}^{\prec}} A$.

Semantic Intuition

For R reflexive and transitive and w, v belong to the set of points of evaluation

- $w \Vdash_{\mathsf{I}} A \rightarrow B$ iff $\forall v$ wRv then $v \Vdash_{\mathsf{I}} A$ implies $v \Vdash_{\mathsf{I}} B$
- $w \Vdash_{\mathsf{DI}} A \prec B$ iff $\exists v$ vRw then $v \Vdash_{\mathsf{DI}} A$ and not $v \Vdash_{\mathsf{DI}} B$
- $w \Vdash_{\mathsf{DI}^{\rightarrow}} A \rightarrow B$ iff $\exists v$ vRw then $v \Vdash_{\mathsf{DI}^{\rightarrow}} A$ implies $v \Vdash_{\mathsf{DI}^{\rightarrow}} B$
- $w \Vdash_{\mathsf{I}^{\prec}} A \prec B$ iff $\forall v$ wRv then $v \Vdash_{\mathsf{I}^{\prec}} A$ and not $v \Vdash_{\mathsf{I}^{\prec}} B$

- Increasing domain: I and $\mathsf{I}^{\prec}$
- Decreasing domain: DI and $\mathsf{DI}^{\rightarrow}$

Characterising the logic of Consistency:

- $\Diamond$-translation
- $\text{BHK}_{\text{DI}\rightarrow}$ interpr.

- *the abstraction* $\mathbf{a}$ *does not refute* $A \wedge B$ *iff* $\mathbf{s}$ *is of the form* $\langle \mathbf{i}, \mathbf{a}' \rangle$ *with* $\mathbf{i}$ *either* $\mathbf{0}$ *or* $\mathbf{1}$*, such that for* $\mathbf{i} = \mathbf{0}$ *then* $\mathbf{a}'$ *does not refute* A *and for* $\mathbf{i} = \mathbf{1}$ *then* $\mathbf{a}'$ *does not refute* B*.*
- *the abstraction* $\mathbf{a}$ *does not refute* $A \vee B$ *iff* $\mathbf{a}$ *is of the form* $\langle \mathbf{a}', \mathbf{a}'' \rangle$ *and* $\mathbf{a}'$ *does not refute* A *and* $\mathbf{a}''$ *does not refute* B*.*
- *the abstraction* $\mathbf{a}$ *does not refute* $A \rightarrow B$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for an hypothetical abstraction* $\mathbf{a}'$ *that does not refute* A*, the abstract* $\mathbf{a}(\mathbf{a}')$ *does not refute* B*.*
- *the abstraction* $\mathbf{a}$ *does not refute* $\neg A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for an hypothetical abstraction* $\mathbf{a}'$ *that does not refute* A*, the abstract* $\mathbf{a}(\mathbf{a}')$ *does not refute* $\perp$*.*
- *the abstraction* $\mathbf{a}$ *does not refute* $\forall x A$ *iff* $\mathbf{a}$ *is of the form* $\langle a, \mathbf{a}' \rangle$*, where* a *is an individual such that* $\mathbf{a}'$ *does not refute* $A(a/x)$*.*
- *the abstraction* $\mathbf{a}$ *does not refute* $\exists x A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that given any individual* a *from the species under consideration,* $\mathbf{a}(a)$ *does not refute* $A(a/x)$*.*
- *no abstraction does not refute* $\perp$*.*
- *every abstraction does not refute* $\top$*.*

Characterising the logic of Consistency:

- $\Diamond$-translation
- BHK$_{\mathsf{DI}\rightarrow}$ interpr.

- *the abstraction* $\mathbf{a}$ *does not refute* $A \wedge B$ *iff* $\mathbf{s}$ *is of the form* $\langle \mathbf{i}, \mathbf{a}' \rangle$ *with* $\mathbf{i}$ *either* $\mathbf{0}$ *or* $\mathbf{1}$, *such that for* $\mathbf{i} = \mathbf{0}$ *then* $\mathbf{a}'$ *does not refute* A *and for* $\mathbf{i} = \mathbf{1}$ *then* $\mathbf{a}'$ *does not refute* B.
- *the abstraction* $\mathbf{a}$ *does not refute* $A \vee B$ *iff* $\mathbf{a}$ *is of the form* $\langle \mathbf{a}', \mathbf{a}'' \rangle$ *and* $\mathbf{a}'$ *does not refute* A *and* $\mathbf{a}''$ *does not refute* B.
- *the abstraction* $\mathbf{a}$ *does not refute* $A \rightarrow B$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for an hypothetical abstraction* $\mathbf{a}'$ *that does not refute* A, *the abstract* $\mathbf{a}(\mathbf{a}')$ *does not refute* B.
- *the abstraction* $\mathbf{a}$ *does not refute* $\neg A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for an hypothetical abstraction* $\mathbf{a}'$ *that does not refute* A, *the abstract* $\mathbf{a}(\mathbf{a}')$ *does not refute* $\perp$.
- *the abstraction* $\mathbf{a}$ *does not refute* $\forall x A$ *iff* $\mathbf{a}$ *is of the form* $\langle a, \mathbf{a}' \rangle$, *where* a *is an individual such that* $\mathbf{a}'$ *does not refute* $A(a/x)$.
- *the abstraction* $\mathbf{a}$ *does not refute* $\exists x A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that given any individual* a *from the species under consideration,* $\mathbf{a}(a)$ *does not refute* $A(a/x)$.
- *no abstraction does not refute* $\perp$.
- *every abstraction does not refute* $\top$.

Classical Logic

$$
\begin{array}{rcl}
P^{\boxdot} & := & \Box P \\
(\neg A)^{\boxdot} & := & \Box \neg A^{\Diamonddot} \\
(A \wedge B)^{\Diamonddot} & := & \Box(A^{\boxdot} \wedge B^{\boxdot}) \\
(A \vee B)^{\boxdot} & := & \Box(A^{\boxdot} \vee B^{\boxdot}) \\
(A \to B)^{\boxdot} & := & \Box(A^{\color{red}\Diamonddot} \to B^{\boxdot}) \\
(\forall x\, A)^{\boxdot} & := & \Box \forall x\, A^{\boxdot} \\
(\exists A)^{\boxdot} & := & \Box \exists A^{\boxdot}
\end{array}
\qquad
\begin{array}{rcl}
P^{\Diamonddot} & := & \Diamond P \\
(\neg A)^{\Diamonddot} & := & \Diamond \neg A^{\boxdot} \\
(A \wedge B)^{\Diamonddot} & := & \Diamond(A^{\Diamonddot} \wedge B^{\Diamonddot}) \\
(A \vee B)^{\Diamonddot} & := & \Diamond(A^{\Diamonddot} \vee B^{\Diamonddot}) \\
(A \to B)^{\Diamonddot} & := & \Diamond(A^{\color{red}\boxdot} \to B^{\Diamonddot}) \\
(\forall x\, A)^{\Diamonddot} & := & \Diamond \forall x\, A^{\Diamonddot} \\
(\exists x\, A)^{\Diamonddot} & := & \Diamond \exists x\, A^{\Diamonddot}
\end{array}
$$

$\Gamma^{\boxdot} \vdash_{S4} A^{\Diamonddot}$ *iff* $\Gamma \vdash_{C} A$.

Classical Logic

$$
\begin{array}{rcl@{\qquad}rcl}
P^{\boxdot} & := & \Box P & P^{\dot{\Diamond}} & := & \Diamond P \\
(\neg A)^{\boxdot} & := & \Box\neg A^{\dot{\Diamond}} & (\neg A)^{\dot{\Diamond}} & := & \Diamond\neg A^{\boxdot} \\
(A \wedge B)^{\dot{\Diamond}} & := & \Box(A^{\boxdot} \wedge B^{\boxdot}) & (A \wedge B)^{\dot{\Diamond}} & := & \Diamond(A^{\dot{\Diamond}} \wedge B^{\dot{\Diamond}}) \\
(A \vee B)^{\boxdot} & := & \Box(A^{\boxdot} \vee B^{\boxdot}) & (A \vee B)^{\dot{\Diamond}} & := & \Diamond(A^{\dot{\Diamond}} \vee B^{\dot{\Diamond}}) \\
(A \to B)^{\boxdot} & := & \Box(A^{\dot{\Diamond}} \to B^{\boxdot}) & (A \to B)^{\dot{\Diamond}} & := & \Diamond(A^{\boxdot} \to B^{\dot{\Diamond}}) \\
(\forall x\, A)^{\boxdot} & := & \Box\forall x\, A^{\boxdot} & (\forall x\, A)^{\dot{\Diamond}} & := & \Diamond\forall x\, A^{\dot{\Diamond}} \\
(\exists A)^{\boxdot} & := & \Box\exists A^{\boxdot} & (\exists x\, A)^{\dot{\Diamond}} & := & \Diamond\exists x\, A^{\dot{\Diamond}}
\end{array}
$$

$$\Gamma^{\boxdot} \vdash_{\mathsf{S4}} A^{\dot{\Diamond}} \textit{ iff } \Gamma \vdash_{\mathsf{C}} A.$$

Classical logic

$\mathrm{BHK_I} + \mathrm{BHK_{DI^\rightarrow}}$, except for implication and negation, plus:

- *the construction* $\mathbf{c}$ *proves* $A \rightarrow B$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical abstraction* $\mathbf{a}$ *that that does not refute* A, $\mathbf{c}(\mathbf{a})$ *proves* B.
- *the construction* $\mathbf{c}$ *proves* $\neg A$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical abstraction* $\mathbf{a}$ *that that does not refute* A, $\mathbf{c}(\mathbf{a})$ *proves* $\bot$.
- *the abstraction* $\mathbf{a}$ *does not refute* $A \rightarrow B$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for* $\mathbf{c}$ *that proves* A, *the abstract* $\mathbf{a}(\mathbf{c})$ *does not refute* B.
- *the abstraction* $\mathbf{a}$ *does not refute* $\neg A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for* $\mathbf{c}$ *that proves* A, *the abstract* $\mathbf{a}(\mathbf{c})$ *does not refute* $\bot$.

Classical logic

$\mathsf{BHK}_{\mathsf{I}} + \mathsf{BHK}_{\mathsf{DI}^{\rightarrow}}$, except for implication and negation, plus:

- *the construction* $\mathbf{c}$ *proves* $A \rightarrow B$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical abstraction* $\mathbf{a}$ *that that does not refute* A, $\mathbf{c}(\mathbf{a})$ *proves* B.
- *the construction* $\mathbf{c}$ *proves* $\neg A$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical abstraction* $\mathbf{a}$ *that that does not refute* A, $\mathbf{c}(\mathbf{a})$ *proves* $\bot$.
- *the abstraction* $\mathbf{a}$ *does not refute* $A \rightarrow B$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for* $\mathbf{c}$ *that proves* A, *the abstract* $\mathbf{a}(\mathbf{c})$ *does not refute* B.
- *the abstraction* $\mathbf{a}$ *does not refute* $\neg A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for* $\mathbf{c}$ *that proves* A, *the abstract* $\mathbf{a}(\mathbf{c})$ *does not refute* $\bot$.

Relations with Natural deduction calculi

There is a quite clear correspondence between $\mathsf{BHK}_{\mathsf{I}}$ and natural deduction for I, that is, NJ. There is also one for $\mathsf{BHK}_{\mathsf{DI}^{\rightarrow}}$ which leads to a calculus $\mathsf{NDJ}^{\rightarrow}$ (Tranchini, 2012) "dual" to NJ in the same way in which LJ is dual to $\mathsf{LDJ}^{\rightarrow}$.

There is a way of combining the two structures in order to get Classical logic which looks very different from the standard natural deduction calculi for C, that is, NK, but seem to be in a natural correspondence with a (single-conclusion) sequent calculus for classical logic.

System NKM

$$\begin{array}{ccc} \multicolumn{3}{c}{C} \\ \hline A \land B & C & C \\ & \vdots\,\mathcal{D}_1 & \vdots\,\mathcal{D}_2 \\ & \langle A \rangle & \langle B \rangle \end{array}\ \mathsf{I}\land \qquad \frac{A \land B}{A}\ \mathsf{E}\land_1 \qquad \frac{A \land B}{B}\ \mathsf{E}\land_2$$

$$\frac{A}{A \lor B}\ \mathsf{I}\lor_1 \qquad \frac{B}{A \lor B}\ \mathsf{I}\lor_2 \qquad \begin{array}{ccc} & [A] & [B] \\ & \vdots\,\mathcal{D}_1 & \vdots\,\mathcal{D}_2 \\ A \lor B & C & C \\ \hline \multicolumn{3}{c}{C} \end{array}\ \mathsf{E}\lor$$

$$\begin{array}{cc} \multicolumn{2}{c}{\top} \\ \hline A \to B & A \\ & \vdots\,\mathcal{D} \\ & \langle B \rangle \end{array}\ \mathsf{I}{\to_1} \qquad \frac{B}{A \to B}\ \mathsf{I}{\to_2} \qquad \frac{A \to B \quad A}{B}\ \mathsf{E}{\to} \qquad \frac{A}{\top}\ \top \qquad \frac{\bot}{A}\ \bot$$

$$\begin{array}{cc} \multicolumn{2}{c}{C} \\ \hline \forall x\, A & C \\ & \vdots\,\mathcal{D} \\ & \langle A(a) \rangle \end{array}\ \mathsf{I}\forall^* \qquad \frac{\forall x\, A}{A(a)}\ \mathsf{E}\forall \qquad \frac{A(a)}{\exists x\, A}\ \mathsf{I}\exists \qquad \begin{array}{cc} & [A(a)] \\ & \vdots\,\mathcal{D} \\ \exists x\, A & C \\ \hline \multicolumn{2}{c}{C} \end{array}\ \mathsf{E}\exists^*$$

Proof of $\neg A \to A \vdash_C A$:

$$
\mathrm{E}{\to}\ \dfrac{\neg A \to A \qquad \dfrac{\top}{\neg A \qquad\quad A}\ \mathrm{I}{\to}_1}{A}
$$

Proof of $\vdash_C \neg\neg\forall x(A(x) \vee \neg A(x))$:

$$
\mathrm{I}{\to}_1\ \dfrac{\top}{\neg\neg\forall x(A(x) \vee \neg A(x)) \qquad \dfrac{\bot}{\top}\ \bot}
$$

$$
\dfrac{\qquad\qquad}{\langle\cancel{\neg\forall x(A(x) \vee \neg A(x))}\rangle \qquad \dfrac{\bot}{\top}\ \bot}\ \mathrm{I}{\to}_1
$$

$$
\dfrac{\top}{\langle\cancel{\forall x(A(x) \vee \neg A(x))}\rangle}\ \mathrm{I}{\to}_1
$$

$$
\mathrm{I}{\to}_1\ \dfrac{\top}{\neg A(x) \qquad\qquad A(x)}
$$

$$
\mathrm{I}{\vee}\ \dfrac{\neg A(x)}{\langle\cancel{A(x) \vee \neg A(x)}\rangle} \qquad \mathrm{I}{\vee}\ \dfrac{A(x)}{\langle\cancel{A(x) \vee \neg A(x)}\rangle}
$$

Proof of $\neg A \to A \vdash_{\mathsf{C}} A$:

$$\mathrm{E}{\to}\ \dfrac{\neg A \to A \qquad \dfrac{\top}{\neg A \qquad\qquad A}\ \mathrm{I}{\to}_1}{A}$$

Proof of $\vdash_{\mathsf{C}} \neg\neg\forall x(A(x) \vee \neg A(x))$:

$$\mathrm{I}{\to}_1\ \dfrac{\top}{\neg\neg\forall x(A(x)\vee\neg A(x)) \qquad \dfrac{\bot}{\top}\ \bot}$$

$$\dfrac{\qquad\qquad\qquad}{\cancel{\langle\neg\forall x(A(x)\vee\neg A(x))\rangle} \qquad \dfrac{\bot}{\top}\ \bot}\ \mathrm{I}{\to}_1$$

$$\dfrac{\qquad\qquad\qquad}{\top \qquad \cancel{\langle\forall x(A(x)\vee\neg A(x))\rangle}}\ \mathrm{I}{\to}_1$$

$$\mathrm{I}{\to}_1\ \dfrac{\top}{\neg A(x) \qquad\qquad A(x)}$$

$$\mathrm{I}\vee\ \dfrac{\neg A(x)}{\cancel{\langle A(x)\vee\neg A(x)\rangle}} \qquad \mathrm{I}\vee\ \dfrac{A(x)}{\cancel{\langle A(x)\vee\neg A(x)\rangle}}$$

System LKS

$$\dfrac{\dfrac{\overline{\vdash P}}{\Gamma \vdash C}\,\vdots\,\mathcal{D}^1 \qquad \dfrac{\overline{P \vdash \bot}}{\Gamma \vdash C}\,\vdots\,\mathcal{D}^2}{\Gamma \vdash C}\; MC \qquad\qquad \Rightarrow \top$$

$$\dfrac{\Gamma \Rightarrow C}{\Gamma,\ A \Rightarrow C}\ \mathrm{L}W \qquad \dfrac{\Gamma \Rightarrow \bot}{\Gamma \Rightarrow A}\ \mathrm{Exp} \qquad \dfrac{A,\ A,\ \Gamma \Rightarrow C}{A,\ \Gamma \Rightarrow C}\ \mathrm{L}C \qquad \dfrac{\Gamma \Rightarrow A \qquad A,\ \Pi \Rightarrow C}{\Gamma,\ \Pi \Rightarrow C}\ Cut$$

$$\dfrac{A,\ \Gamma \Rightarrow C}{A \wedge B,\ \Gamma \Rightarrow C}\ \mathrm{L}\wedge_1 \qquad \dfrac{B,\ \Gamma \Rightarrow C}{A \wedge B,\ \Gamma \Rightarrow C}\ \mathrm{L}\wedge_2 \qquad \dfrac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}\ \mathrm{R}\wedge$$

$$\dfrac{A,\ \Gamma \Rightarrow C \qquad B,\ \Gamma \Rightarrow C}{A \vee B,\ \Gamma \Rightarrow C}\ \mathrm{L}\vee \qquad \dfrac{\Gamma \Rightarrow A}{\Gamma \Rightarrow A \vee B}\ \mathrm{R}\vee_1 \qquad \dfrac{\Gamma \Rightarrow B}{\Gamma \Rightarrow A \vee B}\ \mathrm{R}\vee_2$$

$$\dfrac{\Gamma \Rightarrow A \qquad B,\ \Gamma \Rightarrow C}{A \to B,\ \Gamma \Rightarrow C}\ \mathrm{L}\!\to \qquad \dfrac{\Gamma,\ A \Rightarrow B}{\Gamma \Rightarrow A \to B}\ \mathrm{R}\!\to$$

$$\dfrac{A(a/x),\ \Gamma \Rightarrow C}{\forall x A,\ \Gamma \Rightarrow C}\ \mathrm{L}\forall \qquad \dfrac{\Gamma \Rightarrow A(a/x)}{\Gamma \Rightarrow \forall x A}\ \mathrm{R}\forall^* \qquad \dfrac{A(a/x),\ \Gamma \Rightarrow C}{\exists x A,\ \Gamma \Rightarrow C}\ \mathrm{L}\exists^* \qquad \dfrac{\Gamma \Rightarrow A(a/x)}{\Gamma \Rightarrow \exists x A}\ \mathrm{R}\exists$$

Thank you!

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Employing the concepts

Characterising the logic of Refutability:

- $\neg\Diamond$-translation
- BHK$_{\mathrm{DI}}$ interpr.

- *the abstraction* $\mathbf{a}$ *refutes* $A \wedge B$ *iff* $\mathbf{a}$ *is of the form* $\langle \mathbf{i}, \mathbf{a}' \rangle$ *with* $\mathbf{i}$ *either* $\mathbf{0}$ *or* $\mathbf{1}$*, such that for* $\mathbf{i} = \mathbf{0}$ *then* $\mathbf{a}'$ *refutes* A *and for* $\mathbf{i} = \mathbf{1}$ *then* $\mathbf{a}'$ *refutes* B*.*
- *the abstraction* $\mathbf{a}$ *refutes* $A \vee B$ *iff* $\mathbf{a}$ *is of the form* $\langle \mathbf{a}', \mathbf{a}'' \rangle$ *and* $\mathbf{a}'$ *refutes* A *and* $\mathbf{a}''$ *refutes* B*.*
- *the abstraction* $\mathbf{a}$ *refutes* $A \prec B$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for* $\mathbf{a}'$ *refuting* A*, the hypothetical abstract* $\mathbf{a}(\mathbf{a}')$ *refutes* B*.*
- *the abstraction* $\mathbf{a}$ *refutes* $\neg A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that for* $\mathbf{a}'$ *refuting* A*, the hypothetical abstract* $\mathbf{a}(\mathbf{a}')$ *refutes* $\top$*.*
- *the abstraction* $\mathbf{a}$ *refutes* $\forall x A$ *iff* $\mathbf{a}$ *is of the form* $\langle a, \mathbf{a}' \rangle$*, where* a *is an individual such that* $\mathbf{a}'$ *refutes* $A(a/x)$*.*
- *the abstraction* $\mathbf{a}$ *refutes* $\exists x A$ *iff* $\mathbf{a}$ *is a general method of abstraction such that given any individual* a *from the species under consideration,* $\mathbf{a}(a)$ *refutes* $A(a/x)$*.*
- *every abstraction refutes* $\bot$*.*
- *no abstraction refutes* $\top$*.*

Employing the concepts

Characterising the logic of Unprovability:

- $\mathsf{BHK}_{\mathsf{I}^{\rightarrow}}$ interpr.
- $\Box$-translation
- $\mathsf{G3I}^{\rightarrow}$

- *the construction* $\mathbf{c}$ *does not prove* $A \wedge B$ *iff* $\mathbf{c}$ *is of the form* $\langle \mathbf{c}', \mathbf{c}'' \rangle$ *and* $\mathbf{c}'$ *does not prove* A *and* $\mathbf{c}''$ *does not prove* B*.*
- *the construction* $\mathbf{c}$ *does not prove* $A \vee B$ *iff* $\mathbf{c}$ *is of the form* $\langle \mathbf{i}, \mathbf{c}' \rangle$ *with* $\mathbf{i}$ *either* $\mathbf{0}$ *or* $\mathbf{1}$*, such that for* $\mathbf{i} = \mathbf{0}$ *then* $\mathbf{c}'$ *does not prove* A *and for* $\mathbf{i} = \mathbf{1}$ *then* $\mathbf{c}'$ *does not prove* B*.*
- *the construction* $\mathbf{c}$ *does not prove* $A \prec B$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical construct* $\mathbf{c}'$ *that does not prove* A*,* $\mathbf{c}(\mathbf{c}')$ *does not prove* B*.*
- *the construction* $\mathbf{c}$ *does not prove* $\neg A$ *iff* $\mathbf{c}$ *is a general method of construction such that applied to a hypothetical construct* $\mathbf{c}'$ *that does not prove* A*,* $\mathbf{c}(\mathbf{c}')$ *does not prove* $\top$*.*
- *the construction* $\mathbf{c}$ *does not prove* $\forall x A$ *iff* $\mathbf{c}$ *is a general method of construction such that given any individual* a *from the species under consideration,* $\mathbf{c}(a)$ *does not prove* $A(a/x)$*.*
- *the construction* $\mathbf{c}$ *does not prove* $\exists x A$ *iff* $\mathbf{c}$ *is of the form* $\langle a, \mathbf{c}' \rangle$*, where* a *is an individual such that* $\mathbf{c}'$ *does not prove* $A(a/x)$*.*
- *every construction* $\mathbf{c}$ *does not prove* $\bot$*.*
- *no construction* $\mathbf{c}$ *does not prove* $\top$*.*

A refined Square of opposition?

- KF axiom: $\neg\neg\forall x\,(A(x) \vee \neg A(x))$
- DKF axiom: $\neg\exists x\,(A(x) \wedge \neg A(x))$

$$
\begin{tikzcd}
\mathsf{I} \arrow[dd, "\text{theorems}"'] & \star \arrow[l] \arrow[r] & \mathsf{DI} \arrow[dd, "\text{antitheorems}"] \\
& \text{duality} \arrow[ul] \arrow[ur] \arrow[dl] \arrow[dr] & \\
\mathsf{DI}^{\to} + \mathsf{DKF} \arrow[r, tail, no head] & \star & \mathsf{I}^{\prec} + \mathsf{KF} \arrow[l, tail, no head]
\end{tikzcd}
$$