Modern Alternatives to Cantor's Theory of Infinity

Piotr Błaszczyk
Institute of Mathematics
Pedagogical University of Cracow
Kraków, Poland

Brno, 13-15 January 2020

Cantor's infinite numbers

$$\overline{\overline{\mathbb{N}}} = \aleph_0, \quad \overline{(\mathbb{N}, <)} = \omega$$

Cantor's finite numbers
Laws for finite numbers

$$(\mathbb{N}, +, \cdot, 0, 1, <)$$

$$x + y = y + x,\ \ xy = yx,$$
$$x < y \Rightarrow x + z < y + z,$$
$$x < y, 0 < z \Rightarrow xz < yz$$

Flaws of Cantor's arithmetic of inifnite numbers

$$1 + \omega \neq \omega + 1, \quad 2 \cdot \omega \neq \omega \cdot 2$$

$$\neg(2 < 3 \Rightarrow 2 + \aleph_0 < 3 + \aleph_0), \quad \neg(2 < 3 \Rightarrow 2 \cdot \aleph_0 < 3 \cdot \aleph_0)$$

K. Gödel, *What is Cantor's continuum problem?* 1947

Gödel presents Cantor's cardinal numbers as extending the system of natural numbers, $(\mathbb{N}, +, \cdot, 0, 1, <)$, and seeks to show that "this extension can be effected in a uniquely determined manner".
To this end, Gödel discusses (1) definition of cardinal numbers, (2) their equality, (3) total order, (4) operations of sum and product.

(Gödel, 1947)

(Ad 1) Gödel claims that “Cantor's definition of infinite numbers really has this character of uniqueness", since “whatever ‘number' as applied to infinite sets may mean" it has to be based on the one-to-one correspondence.

(Ad 2, 3) Gödel claims that “there is hardly any choice left but to accept Cantor's definition of equality between numbers, which can easily be extended to a definition of ‘greater‘ and 'less' for infinite numbers".

(Ad 4) As for the sum and product of ordinal numbers, Gödel writes: “it becomes possible to extend (again without any arbitrariness) the arithmetical operations to infinite numbers (including sums and products with any infinite number of terms or factors) and to prove practically all ordinary rules of computation".

Alternatives to Cantor's theory of infinite numbers

- Hessenberg's normal sums and products; $(Ord, +_n, \cdot_n, 0, 1, <)$
- Conway numbers; *ONAG*
- Benci and Di Nasso's numerosities (and Euclidean numbers)

Alternatives to Cantor's theory of infinite numbers

- Hessenberg's normal sums and products; $(Ord, +_n, \cdot_n, 0, 1, <)$
- Conway numbers; *ONAG*
- Benci and Di Nasso's numerosities (and Euclidean numbers)

$$\begin{aligned} (Ord, +_n, \cdot_n, 0, 1, <) &\subset ONAG \\ \text{numerosities} &\subset ONAG \\ \text{Euclidean numbers} &\subset ONAG \end{aligned}$$

Alternative 1. Normal sums and products of ordinal numbers

Normal form theorem (Cantor, 1897); $\alpha \in Ord$

$$\alpha = \omega^{\eta_1} \cdot p_1 + \ldots + \omega^{\eta_h} \cdot p_h,$$

where $\eta_1 > \ldots > \eta_h$, $\eta_i \in Ord$, $h, p_i \in \mathbb{N}$

(Hessenberg, 1906)

$$\alpha = \omega^{\eta_1} \cdot p_1 + \ldots + \omega^{\eta_h} \cdot p_h,$$
$$\beta = \omega^{\eta_1} \cdot q_1 + \ldots + \omega^{\eta_h} \cdot q_h$$

$$\begin{aligned} \alpha +_n \beta &=_{df} \omega^{\eta_1} \cdot (p_1 + q_1) + \ldots + \omega^{\eta_h} \cdot (p_h + q_h) \\ \alpha \cdot_n \beta &=_{df} \sum_{1 \leq i,j \leq h} \omega^{\eta_i +_n \eta_j} \cdot p_i q_j \end{aligned}$$

$$\alpha +_n \beta = \beta +_n \alpha, \quad \alpha \cdot_n \beta = \beta \cdot_n \alpha$$
$$\alpha < \beta \Rightarrow \alpha +_n \gamma < \beta +_n \gamma, \quad \alpha < \beta \Rightarrow \alpha \cdot_n \gamma < \beta \cdot_n \gamma$$

$$(Ord, +_n, \cdot_n, 0, 1, <)$$

J. Conway (1976, 2001). *On Numbers and Games*.
The ordered *field* ONAG

$ONAG = \{a : a \text{ is a surreal number}\}$
$(ONAG, +, \cdot, 0, 1, <)$

Alternative 2. The ordered field $ONAG$

$$(Ord, +_n, \cdot_n, 0, 1, <) \;\mapsto\; (ONAG, +, \cdot, 0, 1, <)$$

$Ord \subset ONAG$

Some surreal numbers

$$\omega, \quad -\omega, \quad \omega - 1, \quad \frac{\omega}{2}, \quad \frac{1}{\omega}, \quad \sqrt{\omega}$$

H. Gonshor (1986). *An Introduction to the Theory of Surreal Numbers*.

Df A surreal number is a function **a** from an ordinal α, $\alpha \in$ *Ord*, into the set $\{+, -\}$,

$$\mathbf{a} : \alpha \mapsto \{+, -\}.$$

$$\alpha \ \sim \ (\underbrace{+ + ...}_{\alpha})$$

$$\omega = (\underbrace{+ + \ldots}_{\omega})$$

$$-\omega = (\underbrace{- - \ldots.}_{\omega})$$

$$\omega - 1 = (\underbrace{+ + +\ldots}_{\omega} -)$$

$$\frac{\omega}{2} = (\underbrace{+ + +\ldots}_{\omega}\underbrace{- - -\ldots}_{\omega})$$

$$\frac{1}{\omega} = (\underbrace{+ - - - \ldots}_{\omega})$$

J. Conway, *On numbers and Games*

$ONAG = \{a : a \text{ is a surreal number}\}$
$(ONAG, +, \cdot, 0, 1, <)$

$\mathbb{R} \subset \mathbb{R}^* \subset ONAG, \quad Ord \subset ONAG, \quad \mathbb{N}^* \subset ONAG$

$$-\omega, \quad \omega - 1, \quad \frac{\omega}{2}, \quad \frac{1}{\omega}, \quad \sqrt{\omega}$$

Alternative 3. Numerosities

V. Benci, M. Di Nasso (2019). *How to Measure the Infinite: Mathematics with Infinite and Infinitesimal Numbers*. Singapore.
V. Benci, M. Forti,(2017). The Euclidean numbers. arXiv:1702.04163.

What are numerosities?

$(\mathbb{N}, +, \cdot, 0, 1, <)$
$\mathbb{N}^* = \mathbb{N}^{\mathbb{N}}/\mathcal{U}$
$(\mathbb{N}^*, +, \cdot, 0, 1, <)$

$(\mathbb{R}, +, \cdot, 0, 1, <)$
$\mathbb{R}^* = \mathbb{R}^{\mathbb{N}}/\mathcal{U}$
$(\mathbb{R}^*, +, \cdot, 0, 1, <)$

$$(r_j) \equiv (s_j) \Leftrightarrow \{j \in \mathbb{N} : r_j = s_j\} \in \mathcal{U}$$

$$[(r_j)] +^* [(s_j)] = [(r_j + s_j)], \quad [(r_j)] \cdot^* [(s_j)] = [(r_j \cdot s_j)]$$

$$[(r_j)] <^* [(s_j)] \Leftrightarrow \{j \in \mathbb{N} : r_j < s_j\} \in \mathcal{U}$$

Standard numbers in nonstandard framework

$$r = [(r, r, r, ...)]$$
$$2 = [(2, 2, 2, ...)] = [(0, 0, 2, 2, 2, 2...)]$$

If the sequence (n_j) representing a hypereal number is such that

$$\{j \in \mathbb{N} : n_j = 2\} = \{j \in \mathbb{N} : n_0 < j\} = \mathbb{N} \setminus \{1, 2, ..., n_0\},$$

then $[(n_j)] = 2$.

Number α

$$\alpha = [(1,2,3,\ldots)] = [(n)].$$

$$\alpha^2 = [(1,2,3,\ldots)]\cdot[(1,2,3,\ldots)] = [(1^1,2^2,3^2,\ldots)] = [(n^2)].$$

$$\frac{\alpha}{2} = [(1,2,3,\ldots)]\cdot[(\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\ldots)] = \left[\left(\tfrac{1}{2},\tfrac{2}{2},\tfrac{3}{2},\ldots\right)\right] = \left[\left(\tfrac{n}{2}\right)\right].$$

$$\sqrt{\alpha} = \left[\left(\sqrt{1},\sqrt{2},\sqrt{3},\ldots\right)\right] = \left[\left(\sqrt{n}\right)\right].$$

$$\left\lfloor\frac{\alpha}{2}\right\rfloor = \left[\left(\left\lfloor\tfrac{n}{2}\right\rfloor\right)\right], \quad \left\lfloor\sqrt{\alpha}\right\rfloor = \left[\left(\left\lfloor\sqrt{n}\right\rfloor\right)\right].$$

How to assign numerosity to subset of $\mathbb{N}$

Let A be a subset of $\mathbb{N}$. We define a function $\varphi_A : \mathbb{N} \mapsto \mathbb{N}$, by

$$\varphi_A(n) = \overline{\overline{\{a \in A \mid a \leq n\}}}. \tag{1}$$

The numerosity of the set A is the nonstandard natural number $\nu_\alpha(A)$ represented by the sequence $(\varphi_A(n))$, that is

$$\begin{aligned} \nu_\alpha(A) &= [(\varphi_A(n))] \\ &= [(\varphi_A(1), \varphi_A(2), \varphi_A(3), ...)]. \end{aligned}$$

Some examples

1) Let us start with finite sets, e.g. a two elements set $A = \{k, l\}$, with $k < l$. We have,

$$\varphi_A(n) = \begin{cases} 0, & \text{for } n < k, \\ 1, & \text{for } k \leq n < l, \\ 2, & \text{for } l \leq n. \end{cases}$$

$$[(\varphi_A(n))] = [(000011111111111122222222...)] = 2$$

$$\nu_\alpha(A) = 2$$

2) When $A = \{a_1, ..., a_k\}$, $\nu_\alpha(A) = k$.

Some examples

$$\nu_\alpha(\mathbb{N}) = [(1,2,3,4,...)] = \alpha$$

$$\nu_\alpha(\{2,4,6,8,...\}) = [(0,1,1,2,2,3,3,4,4,...)] = \left\lfloor \frac{\alpha}{2} \right\rfloor$$

$$\nu_\alpha(\{1,4,9,16,25...\}) = [(1,1,1,2,2,2,...)] = \left\lfloor \sqrt{\alpha} \right\rfloor$$

Some general rules

$$A \subsetneq B \Rightarrow \nu_\alpha(A) < \nu_\alpha(B). \tag{2}$$

$$\nu_\alpha(A \cup B) = \nu_\alpha(A) + \nu_\alpha(B), \quad \text{whenever} \quad A \cap B = \emptyset. \tag{3}$$

$$\nu_\alpha(A \cup B) = \nu_\alpha(A) + \nu_\alpha(B) - \nu_\alpha(A \cap B). \tag{4}$$

Numerosities vs numbers $\aleph_0$ and ω

- In Cantor's theory any subset of $\mathbb{N}$ can be either finite or of cardinality $\aleph_0$. Similarly, there are no ordinal numbers in-between finite numbers and ω.
- The numerosity of any infinite subset of $\mathbb{N}$ is less than α, and greater than any finite number.

$$\sqrt{\textit{The End}}$$