Calibration Of Photon Counting Detectors

Case of DK154

Filip Hroch

ÚTFA, MU

June 2013

Outline

Principles
photon detectors, photon fluxes, ...

Color transformations
approximation of filters, color images, ...

Poisson statistics
principles, maximum likelihood, ...

Robust methods
in use, ...

Extinction
ans many many graphs, ...

Magnitudes
the classical taste

Munipack
an implementation

Part I

Principles of Calibration

Photon Counting Detectors

Photon counting detectors n

- detects individual photons as a particle (by technical design),
- like CCD, channels on spacecrafts, photographic emulsion, photomultiplier, eye (?).

Calorimeters E

- energy-based detectors (measures energy)
- examples: bolometers, ...

Detector absorb energy (by Planck's law):

$$E = nh\nu \tag{1}$$

Important:

- Photon detectors collects all photons (energy doesn't matter).
- Calorimeters collects energy (amount of photons doesn't matter).

Principle Of The Calibration

Our device registers the counts — c (per seconds and area, single frequency)
A number of photons expected from calibration sources — n
The *crucial point of the calibration* is determination of the coefficient η:

$$\eta = \frac{c}{n} \tag{2}$$

General properties:

- η is probability of detection a photon on ν
- η characterizes of efficiency: $0 < \eta < 1$
- response of full apparatus (including optics, atmospheric conditions, ...)

Difficulties Of Photometric Calibration

Methods:

- laboratory: ideal for CCD, precise with calibrated lamp
- celestial: full apparatus, low precision, easy available

Theoretical difficulties:

- multi-frequency observing (finite frequency band)

Practical difficulties:

- atmospheric conditions
- calibration sources

Methods Of Photometric Calibration

Photons and photometric quantities

- How many photons is coming from Vega ?
- What are we exactly observing?

Multi-band calibration

- Color systems
- Conversions

Poisson's Nature of Photons

- Statistical methods for calibration
- We are robust!

Energy, fluxes, ...

- Energy conservation E

$$L = \frac{\mathrm{d}E}{\mathrm{d}t} = \int_V e\,\mathrm{d}V = \int_S \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}S$$

- Energy flux $\mathbf{F}$:
  - has direction
  - per second
  - per area
  - SI units: $\mathrm{W}\cdot\mathrm{m}^{-2}$
- Intensity:

$$F = \int_\Omega I\,\mathrm{d}\Omega$$

  - per second
  - per area
  - per cone
  - per frequency (wavelength)
  - SI units: $\mathrm{W}\cdot\mathrm{m}^{-2}\cdot\mathrm{sr}^{-1}$

Photon flux

Photon flux $\phi_\nu \cdot \mathrm{s}^{-1} \cdot \mathrm{m}\ ^{-2} \cdot \mathrm{sr}^{-1} \cdot \mathrm{Hz}$ [m]

$$\phi_\nu \equiv \frac{\Delta n}{\Delta t \Delta A \Delta \Omega \Delta \nu}$$

Photon flux

$$I_\nu = \phi_\nu h\nu$$

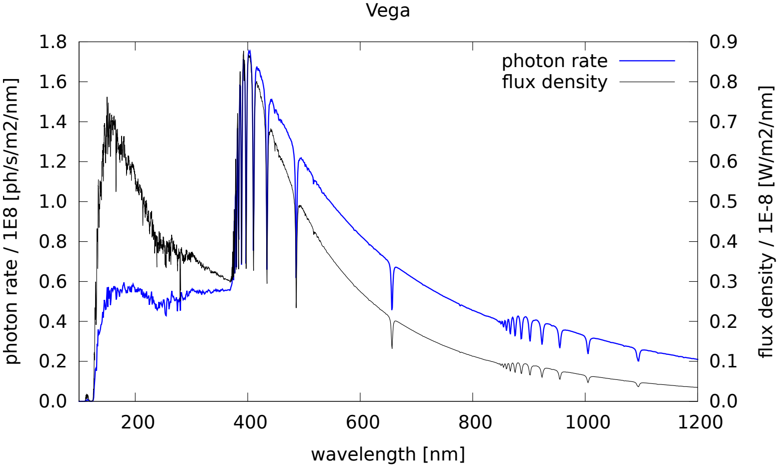
(equivalent o Planck's law for continuous quantities)

The Spectrum

- Basically, an energy spectrum are proper values of Hamiltonian operator $H|\psi_n\rangle = E_n|\psi_n\rangle$ (discrete).
- Basically, the spectrum are proper values of density matrix ??? (continuous).
- Observed as the spectral density flux f_ν, f_λ
  - has direction? No! it's rate[1], has no direction!
  - per second
  - per area
  - per frequency (wavelength)
  - SI units: $\mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{sr}^{-1} \cdot \mathrm{Hz}$ [m]

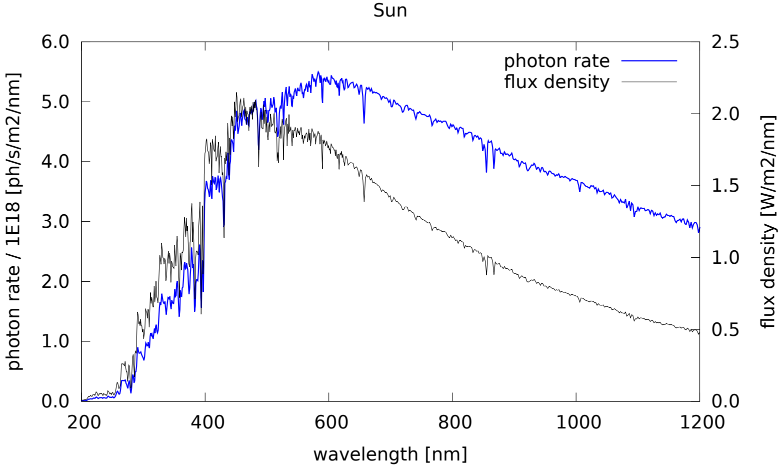
[1] A rate is a general normalized quantity. The flux is used historically as $\mathbf{F} = f \cdot \mathbf{n}$.

Spectrum of Vega



Synphot reference spectrum [2]

Spectrum of Sun



Synphot reference spectrum [2]

Energy And Photon Fluxes In A Filter

- observation via filter (band in radio, channel in HEA)
- has finite spectral width (mixes near frequencies)
- function: $f(\nu)$ is probability of "transmission" of a photon throughout the filter
- mathematically means conditional probability

Energy flux in filter ($\int_0^\infty f_F(\nu)\,\mathrm{d}\nu < \infty$):

$$F_F = \int_0^\infty F_\nu(\nu) f_F(\nu)\,\mathrm{d}\nu$$

Photon Flux

$$\phi_F = \int_0^\infty \frac{F_\nu(\nu) f_F(\nu)}{h\nu}\,\mathrm{d}\nu$$

Approximations Of Photon Flux In The Filter

- Usually, exact spectral transmisivity of filters is not known (need due precision).

Gauss–Hermite quadrature[2]

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} f(x)\,\mathrm{d}x \approx \sum_{n=1}^{N} w_n f(x_n)$$

where w_n are weights and x_n are roots of Hermite polynomial $H_n(x)$, $w_1 = 2(?)$.

- The interval of integration extended to $-\infty$.
- The weighting function approximates the real filter transmitivity

The filter approximation is

$$f(\nu) \approx f_{\nu_0} \mathrm{e}^{-(\nu-\nu_0)^2/2(\Delta\nu)^2}$$

with parameters ν_0 as center of filter, $\Delta\nu$ as "broadness" parameter and f_{ν_0} as the transmitivity at maximum.
The area under the graph is approximately

Vega

Our Hero

- many photometry systems foundation
- defined for V, $m = 0$, $B - V = 0$
- Luminosity $L = 40L_{\odot}$
- Distance $d = 7.68$ pc
- flux
$$F = \frac{L}{4\pi d^2}$$
 [W/m^2] $F = 2.27 \cdot 10^{-8}$ [W/m^2] (energy conservation, energy is spread over larger cone (surface)) $\mathbf{F} = (F, 0, 0)$ in spherical coordinates (r, θ, ϕ).

V filter

- V filter defines flux density at 1 Hz as $f_0 = 3600 \cdot 10^{-26}$ [W/m^2/Hz]
- V filter has effective wavelength $\nu_{\text{eff}} = 550 \cdot 10^{12}$ Hz and width $\Delta\nu = 89 \cdot 10^{12}$ Hz.
- Flux in filter with trnasmitivity $T_V(\nu)$ is

$$F_V = \int_0^\infty f(\nu) \cdot T_V(\nu) \mathrm{d}\nu \approx f(\nu_{\text{eff}}) T(\nu_{\text{eff}}) \Delta\nu$$

- For the ideal filter for Vega $f(\nu_{\text{eff}}) = f_0 = 3600 \cdot 10^{-26}$ [W/m^2/Hz], $T = 1$ and so $F_V = 3.2 \cdot 10^{-9}$ [W/m^2]

Photon flux

- Energie jednoho fotonu ve V filtru je $e = h\nu_{\mathrm{eff}} = 3.6 \cdot 10^{-19}$ [J]
- Energie nesená více fotony $E = ne = nh\nu_{\mathrm{eff}}$
- Pro fotonový tok ve filtru

$$\frac{E}{1\ \mathrm{s}\ \ 1\ \mathrm{Hz}} \approx \frac{F}{\Delta\nu} \approx nh\nu_{\mathrm{eff}}$$

$$n = \frac{F}{h\nu_{\mathrm{eff}}} = \frac{f_0 \Delta\nu}{h\nu_{\mathrm{eff}}}$$

pro Vegu vychází ve V filtru asi $8.8 \cdot 10^9$ [fotonů/s/m^2]

Pro zajímavost, plocha lidského oka je $\pi \cdot 0.003^2$ m^2 a tedy $n_{\mathrm{oko}} = 2.5 \cdot 10^5$ [fotonů/s], pro magnitudy $m = 7$ je $n_{\mathrm{oko}} = 400$ [fotonů/s] (ale vadí i pozadí)

Energy And Photon Fluxes For V Filter

Fluxes:

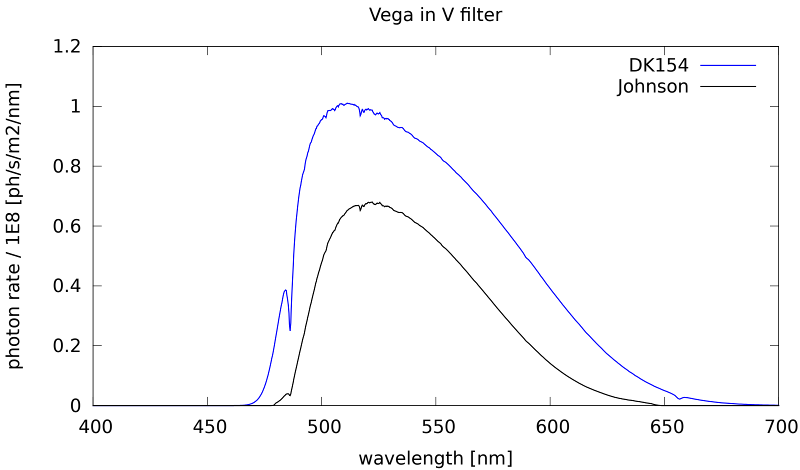
$$\phi = f_0 \Delta\nu 10^{0.4m}$$

Photon fluxes:

$$\phi = \frac{f_0 \Delta\nu}{h\nu_{\mathrm{eff}}} 10^{0.4m}$$

m	enery flux [W/m^2]	photon flux [ph/s/m^2]	
0	10^{-9}	10^{10}	Vega
5	10^{-11}	10^{8}	eye faint
10	10^{-13}	10^{6}	Perek's 2m
15	10^{-15}	10^{4}	CCD on telescope
20	10^{-17}	100	single-exposure limit
25	10^{-19}	0.9	full-night observation

Vega Photon Fluxes For V Filter



Photon fluxes for V filter: DK154 1.026E+10 Johnson 5.5E+09 (no CCD quantum sensitivity).

Interpretation: Photon fluxes for DK154 filters are approximately twice more than standard filters.

Photon fluxes for R filter

Vega with and without Hα

Photons (removed Hα): 8.621E+09 photons/s/m2
Photons (including Hα): 8.608E+09 photons/s/m2
To resolve between stars with and without, we need relative precision better than 1% (!).

Photon fluxes for DK154 filters

	filter	DK154† $\times 10^9$	Landolt†	DK154 / Landolt
	B	7.401	3.023	2.448
Vega	V	8.999	4.814	1.870
	R	7.858	6.894	1.140
	I	3.578	3.985	0.899

	filter	DK154† $\times 10^{20}$	Landolt†	DK154 / Landolt
	B	1.950	0.821	2.375
Sun	V	4.594	2.409	1.907
	R	5.647	4.867	1.160
	I	3.472	4.032	0.861

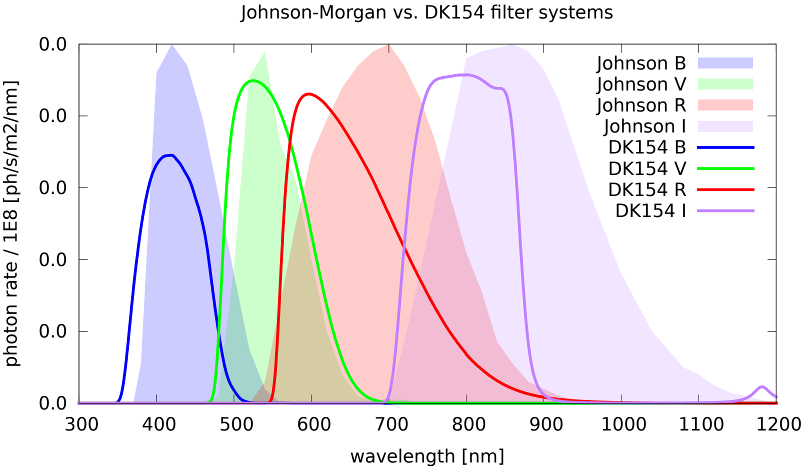
Mean difference 0.042. The absolute calibration is limited to a few percent!

† units in [photons/s/m^2]

Part II

Color transformations

Johnson-Morgan and DK154 filter systems



The basics of approximations

- Right approximation function selection
- Criterion of a good approximation

Distance Of Functions

The distance of functions is defined as the functional ($\mathcal{C}(.) \to \mathbb{R}$):

$$S[f|g] = \int w(x)\|f(x) - g(x|a)\|\,\mathrm{d}x$$

where $\|.\|$ is a measure.
For filters, $w(x) = \phi_\nu(\nu)q(\nu)$:

$$S = \int_{-\infty}^{\infty} \phi_\nu(\nu)q(\nu)\|f(\nu) - f'(\nu)\|\,\mathrm{d}\nu$$

General approximation of functions:

- we choose "suitable" functions from a space (set) of functions $\mathcal{C}$
- we try choose of parameters ()
- we choose a measure

Design of approximation of filters

Scaling

Norm factor (f' is instrumental, f standard)

$$f'(\nu) = r \cdot f(\nu)$$

$$S(r) = \int\limits_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) [f'(\nu) - r f(\nu)]^2 \, \mathrm{d}\nu$$

Solution for $\delta S / \delta r = 0$:

$$\frac{\mathrm{d}S}{\mathrm{d}r} = -2 \int\limits_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) [f'(\nu) - r f(\nu)] f(\nu) \, \mathrm{d}\nu$$

so

$$\int\limits_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f'(\nu) f(\nu) \, \mathrm{d}\nu = r \int\limits_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f^2(\nu) \, \mathrm{d}\nu \qquad (5)$$

Design of approximation of filters

Two filter Multi-Linear Approximation

Norm factor (f' is instrumental, f standard)

$$f'_B(\nu) = c_{BB} \cdot f_B(\nu) + c_{BV} \cdot f_V(\nu) f'_V(\nu) = c_{VB} \cdot f_B(\nu) + c_{VV} \cdot f_V(\nu)$$

$$S(A,B,C,D) = \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) [f'_B(\nu) - A f_B(\nu) - B f_V(\nu)]^2 \,\mathrm{d}\nu + \int_{-\infty}^{\infty} \phi_\nu(\nu) q($$

The solution is a set of equations

$$\frac{\partial S}{\partial A} = 0$$

$$\int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f'_B f_B \,\mathrm{d}\nu = c_{BB} \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f_B^2 \,\mathrm{d}\nu + c_{BV} \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f_B f_B$$

$$\int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f'_B f_V \,\mathrm{d}\nu = c_{BB} \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f_B f_V \,\mathrm{d}\nu + c_{BV} \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f_V^2$$

...

Design of approximation of filters

Multi-Linear approximation

Norm factor (f' is instrumental, f standard)

$$f'_B(\nu) = A \cdot f_B(\nu) + B \cdot f_V(\nu) f'_V(\nu) = C \cdot f_B(\nu) + D \cdot f_V(\nu)$$

$$f'_i = \sum_{ij} c_{ij} f_j, \qquad i,j = B,V$$

$$S(c_{ij}) = \sum_j \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) \sum_i [f'_j(\nu) - c_{ij} f_i(\nu)]^2 \,\mathrm{d}\nu$$

The solution

$$\frac{\partial S}{\partial c_{ij}} = 0$$

is a set of equations

$$\int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f'_j f_i \,\mathrm{d}\nu = \sum_l c_{ij} \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f_i f_l \,\mathrm{d}\nu i,j = B,V,R,I$$

Interpretation:

- diagonal elements are proportional common filter

Filter Approximation of DK 154

From DK154 to Johnson(-Morgan)

Vega

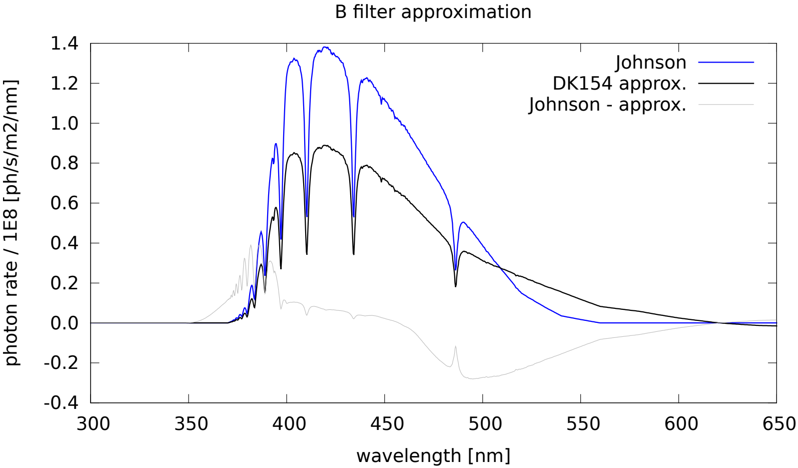
$$\begin{pmatrix} n_B \\ n_V \\ n_R \\ n_I \end{pmatrix} = \begin{pmatrix} 1.5271 & -0.2223 & 0.0825 & -0.0673 \\ 0.1125 & 0.8968 & -0.1655 & 0.0126 \\ 0.0030 & -0.1854 & 1.4262 & -0.6064 \\ 0.0004 & -0.0533 & 0.1591 & 1.0928 \end{pmatrix} \begin{pmatrix} c_B \\ c_V \\ c_R \\ c_I \end{pmatrix}$$

Sun

$$\begin{pmatrix} n_B \\ n_V \\ n_R \\ n_I \end{pmatrix} = \begin{pmatrix} 1.6163 & -0.2064 & 0.0653 & -0.0497 \\ 0.1877 & 0.8932 & -0.0149 & 0.0106 \\ 0.0065 & -0.2764 & 1.5378 & -0.6388 \\ 0.0011 & -0.0859 & 0.2086 & 1.0971 \end{pmatrix} \begin{pmatrix} c_B \\ c_V \\ c_R \\ c_I \end{pmatrix}$$

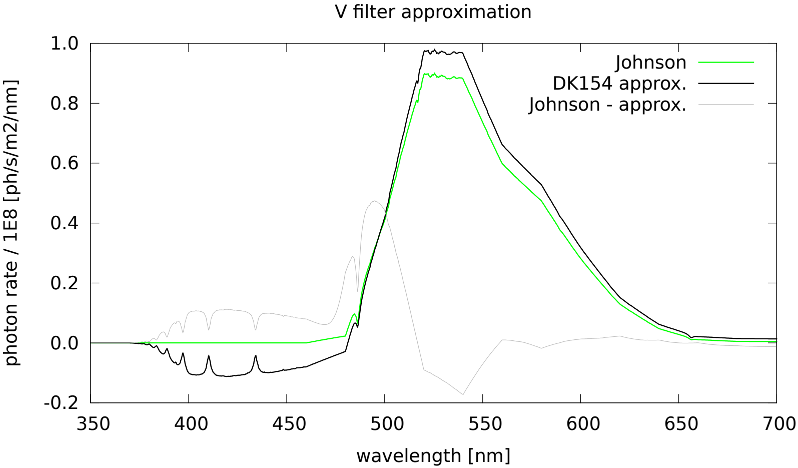
Approximation of B filter

Vega and DK154 instrumental filter



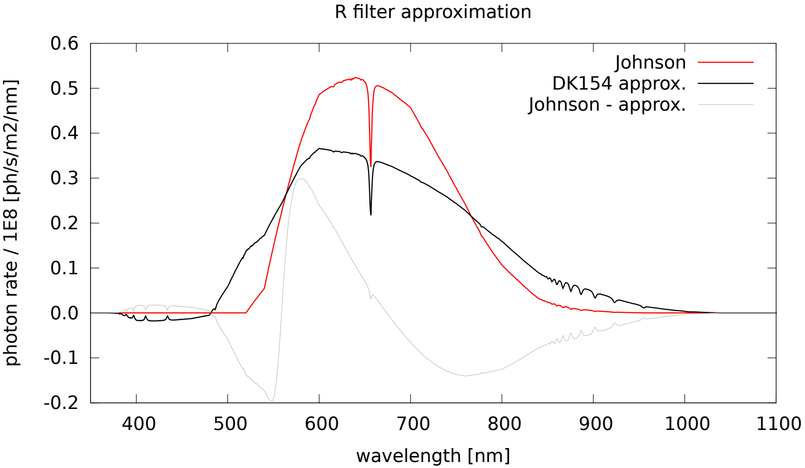
Approximation of V filter

Vega and DK154 instrumental filter



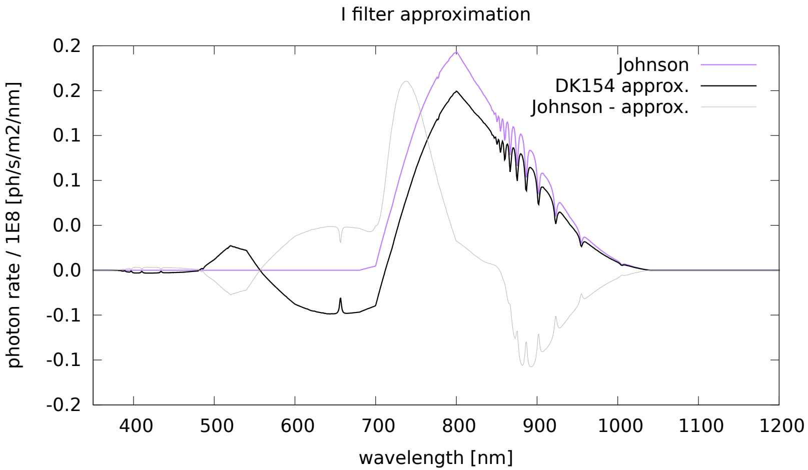
Approximation of R filter

Vega and DK154 instrumental filter



Approximation of I filter

Vega and DK154 instrumental filter



Single Filter Approximation of DK 154

Determination of r from (5):

$$\int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f'(\nu) f(\nu)\, \mathrm{d}\nu = r \int_{-\infty}^{\infty} \phi_\nu(\nu) q(\nu) f^2(\nu)\, \mathrm{d}\nu$$

filter	Vega	Sun
B	2.3546	2.2873
V	1.6307	1.6421
R	1.1455	1.1581
I	0.76748	0.74333

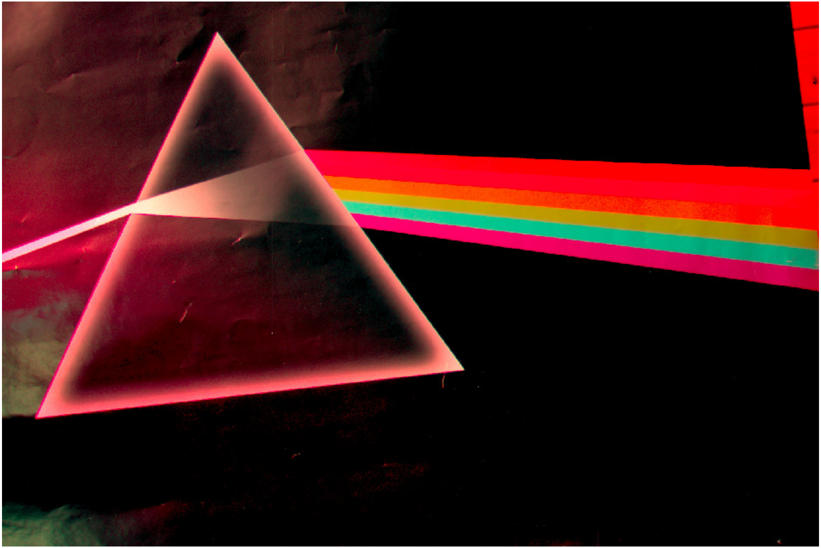
Natural colors

"reconstruction of natural colors (by human being perception) from astronomical filters"

- SBIG ST-8 with BVRI filter set
- MonteBoo dome, solar light
- best check of the approximations !

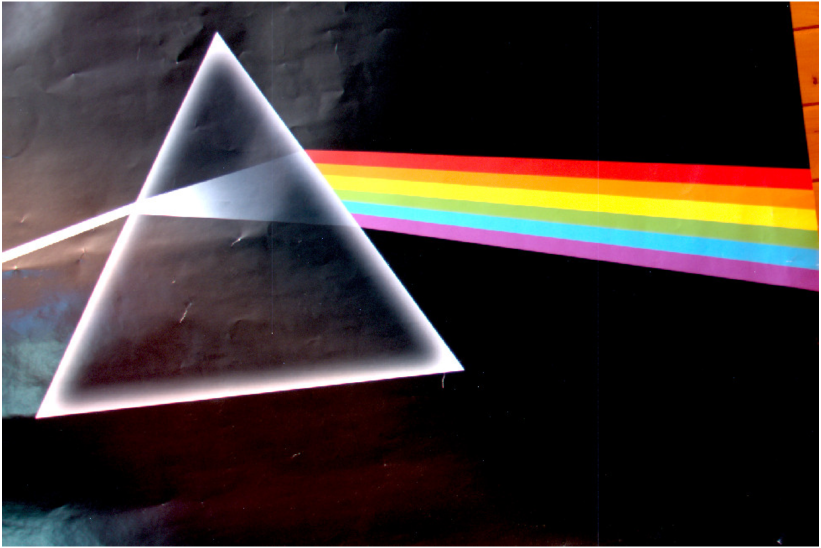
Poorly Reconstructed colors

Instrumental MonteBoo BVR to RGB



Natural colors

Instrumental MonteBoo BVR to Johnson-Morgan



Canon EOS30D

Check colors

SMC

DK154

Part III

Poisson's Nature of Photons

Poisson distribution

Let's, expected amount of photons is n, probability observing of events λT is

$$P_n(\lambda T) = \frac{(\lambda T)^n \mathrm{e}^{-\lambda T}}{n!}, \quad (n = 0, 1 \ldots) \tag{6}$$

- λ is event rate, λT is number of occurred events per time period
- comparisons counts of particles, not normalized fluxes
- for independently occurred events

Mean:

$$\bar{\lambda} = \lambda \tag{7}$$

Variance:

$$\sigma^2 = \lambda \tag{8}$$

Median ν is ($\nu \neq \bar{c}$!)

$$\lambda - \ln 2 < \nu < \lambda + \frac{1}{3} \tag{9}$$

Principle of maximum likelihood

- probability distribution of every single data point x_i is *a priory* $p(x_i|\theta)$
- like composing of probabilities $p = p_1 \cdot p_2 \dots p_N$, join distribution is $p(x_1, x_2 \dots x_N|\theta) = p(x_1|\theta) \cdot p(x_2|\theta) \dots p(x_N|\theta) \equiv L$
- parameter θ is determined for maximum of $p(x_1, x_2 \dots x_N|\theta)$

$$L = \prod_{i=1}^{N} p(x_i|\theta) \tag{10}$$

Common method to get maximum is use of derivation $\ln L$

$$\frac{\partial \ln L}{\partial \theta} = \frac{\partial}{\partial \theta} \sum_{i=1}^{N} \ln p(x_i|\theta) = 0$$

Determination of response — beginning

Use of maximum likelihood for calibration sources $i = 1, 2 \ldots N$ with expected number of photons n_i and observed per 1 s period c_i:

$$\lambda_i(c_i|r) = rc_i, \quad (r > 1)$$

$$L = \prod_{i=1}^{N} p_{n_i}(\lambda_i|r) = \prod_{i=1}^{N} \frac{\lambda_i^{n_i} \mathrm{e}^{-\lambda_i}}{n_i!}.$$

Localization of maximum ($\max L = \max \ln L$):

$$\ln L = \sum_{i=1}^{N} (n_i \ln \lambda_i - \lambda_i) - \sum_{i=1}^{N} n_i!$$

$$\frac{\mathrm{d} \ln L}{\mathrm{d} r} = \sum_{i=1}^{N} \left(\frac{n_i}{\lambda_i} - 1 \right) \frac{\mathrm{d}\lambda_i}{\mathrm{d} r} = 0 \tag{11}$$

Determination of response — result

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}r} = c_i$$

and its derivation

$$\sum_{i=1}^{N} \left(\frac{n_i}{r c_i} - 1 \right) c_i = 0$$

with some algebra

$$\sum_{i=1}^{N} \left(\frac{n_i}{r} - c_i \right) = 0$$

and finally

$$r = \frac{\sum\limits_{i=1}^{N} n_i}{\sum\limits_{i=1}^{N} c_i} \tag{12}$$

as expected and equivalent to $r = \bar{n}/\bar{c}$.

Determination of multi-response — beginning

Use maximum likelihood for calibration sources $i = 1, 2 \ldots N$ with expected number of photons n_{ik} in a filter set $F_1, F_2, \ldots F_K$ and observed counts c_{ik} per 1 s period:

$$\lambda_{ik}(c_{ik}|r_{kj}) = \sum_{j=1}^{K} r_{kj} c_{ij}, \quad (k = 1, 2, \ldots K, i = 1, 2, \ldots N)$$

and also $\forall r_{kj} > 1$.

$$L = \prod_{\substack{i=1\\k=1}}^{N,K} P_{n_{ik}}(\lambda_{ik}) = \prod_{\substack{i=1\\k=1}}^{N,K} \frac{\lambda_{ik}^{n_{ik}} \mathrm{e}^{-\lambda_{ik}}}{n_{ik}!}.$$

$$\ln L = \sum_{\substack{i=1\\k=1}}^{N,K} (n_{ik} \ln \lambda_{ik} - \lambda_{ik}) - \sum_{\substack{i=1\\k=1}}^{N} n_{ik}!$$

$$\frac{\partial \ln L}{\partial r_{ik}} = \sum_{\substack{i=1\\k=1}}^{N,K} \left(\frac{n_{ik}}{\lambda_{ik}} - 1 \right) \frac{\partial \lambda_{ik}}{\partial r_{ik}} = 0$$

Determination of parameters for multi-filter — result

$$\frac{\partial \lambda_{ik}}{\partial r_{ik}} = c_{ik}$$

and its derivation

$$\sum_{i=1}^{N} \left(\frac{n_i}{\sum_j r_{jk} c_i} - 1 \right) c_{ik} = 0$$

and finally

$$\sum_{i=1}^{N} \frac{n_{ik} c_{ik} - c_{ik} \sum_j r_{jk} c_{ij}}{\sum_j r_{jk} c_{ij}} = 0 \qquad (13)$$

we have got a non-linear system of equations for r_{jk}.

Normal and Poisson distributions connection

For Poisson distribution, formula like (11) is minimized

$$\ln L = \sum \left(\frac{n_i - \lambda_i}{\lambda_i} \right) \frac{\partial \lambda_i}{\partial r} = 0 \qquad (14)$$

For Normal distribution, χ^2 (least-squares with weights) is used

$$\chi^2 = \sum \left(\frac{n_i - \lambda_i}{\sigma} \right)^2$$

Applying property (7) and (8) of Poisson systems $\sigma^2 = \lambda$ gives

$$\chi^2 = \sum \frac{(n_i - \lambda_i)^2}{\lambda_i}$$

and asymptotically[3] for

$$\frac{\partial \chi^2}{\partial r} = \sum \frac{\lambda_i^2 - n_i^2}{\lambda_i^2} \frac{\partial \lambda_i}{\partial r} \overset{n_i \to \lambda_i}{=} \sum \left(\frac{\lambda_i - n_i}{\lambda_i} \right) \frac{\partial \lambda_i}{\partial r} = 0 \qquad (15)$$

Poisson distribution suggests the same minimization way !

[3] simple, but not correct way is to minimize $(\lambda_i - n_i)^2 / n_i$

Part IV

Robust Statistics

Robust Methods

Outliers didn't matter[4]

Huber[4]:

> *"robustness signifies insensitivity to small deviations from assumptions"*

- insensitive to outliers (by unexpected errors, apparatus defects, cosmics, . . .)
- equivalent to least square for well noised data (the same dispersion)
- ideal for machine processing

[4] Huber[4]

Merged Distributions

Tail of Outliers[5] for 100 thousands data points

$$F(x) = (1-\epsilon)\Phi\left(\frac{x-\mu}{\sigma}\right) + \epsilon\Phi\left(\frac{x-\mu}{3\sigma}\right),$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}dt$$

ϵ	s	d	σ
0.0	0.998	0.796	1.008
0.01	1.040	0.819	1.026
0.05	1.179	0.875	1.067
0.1	1.361	0.967	1.146
0.2	1.604	1.112	1.296
1.0	2.994	2.389	3.023

Measure of scatter:

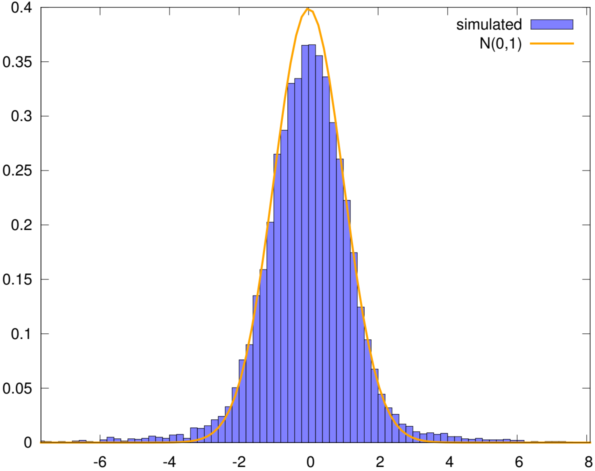
$$d = \frac{1}{N}\sum |x_i - \bar{x}|,$$

$$s = \sqrt{\frac{1}{N}\sum (x_i - \bar{x})^2}$$

[5] Example by Tukey(1960), Hubber(1980)

Two Normal distributions

$\epsilon = 0.1$



General Principles of Robust Statistics

Determine a parameter [6] by maximum likelihood:

$$L = \prod_{i=1}^{N} p(x_i|\tilde{x}) = \prod_{i=1}^{N} f(x_i - \tilde{x}),$$

assumption $p(x|\tilde{x}) = f(x - \tilde{x})$ and substitution $\rho(x) = -\ln p(x)$

$$\ln L = -\sum_{i=1}^{N} \rho(x_i - \tilde{x})$$

A standard way to look for minimum, $\psi(x) \equiv \rho'(x)$

$$\frac{\mathrm{d} \ln L}{\mathrm{d}\tilde{x}} = 0, \quad \sum_{i=1}^{N} \psi(x_i - \tilde{x}) = 0$$

[6] tilde $\tilde{x}$ is a robust estimator with contrast to the least square's $\bar{x}$

Least Square Method — I.

Derivation

The distribution $p(x_i|\bar{x})$ is Normal (Gaussian):

$$p(x_i|\bar{x}) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_i-\bar{x})^2/2\sigma^2}$$

Likelihood:

$$L = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_i-\bar{x})^2/2\sigma^2}$$

To get an analytic solution, we introduces

$$-\ln L = \sum_{i=1}^{N} \frac{(x_i-\bar{x})^2}{2\sigma^2} + N \ln \sqrt{2\pi}\sigma$$

which we identified as the sum of squares S (second term is an additive constant):

$$S \equiv \sum_{i=1}^{N} \frac{(x_i-\bar{x})^2}{2\sigma^2}$$

Least Square Method — II.

Arithmetical Mean

The minimum is located as:

$$-\frac{\partial \ln L}{\partial \bar{x}} = -\sum_{i=1}^{N} \frac{x_i - \bar{x}}{2\sigma^2} =$$

so

$$\sum_{i=1}^{N} x_i = \sum_{i=1}^{N} \bar{x}$$

and by this way

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

where we used identity

$$\sum_{i=1}^{N} \bar{x} = \bar{x} \sum_{i=1}^{N} 1 = \bar{x} N$$

Mean Absolute Deviation

Laplace

The distribution $p(x_i|\nu)$ is Laplace's:

$$p(x_i|\nu) = \mathrm{e}^{-|x_i-\nu|}$$

Likelihood:

$$L = \prod_{i=1}^{N} \mathrm{e}^{-|x_i-\nu|}$$

and its logarithm:

$$-\ln L = \sum_{i=1}^{N} |x_i - \nu|$$

$$-\frac{\partial \ln L}{\partial \nu} = \sum_{i=1}^{N} \mathrm{sgn} x_i - \nu = 0$$

Numerical solution only.

General Distribution

A general (robust) distribution will

$$p(x_i|\tilde{x}) = \mathrm{e}^{(}\varrho(x_i - \tilde{x}))$$

Maximum likelihood

$$L = \prod_{i=1}^{N} \mathrm{e}^{-\varrho(x_i - \tilde{x})}$$

$$-\frac{\partial \ln L}{\partial \tilde{x}} = \sum_{i=1}^{N} \varrho'(x_i - \tilde{x})$$

with common designation $\psi = \varrho'$ is

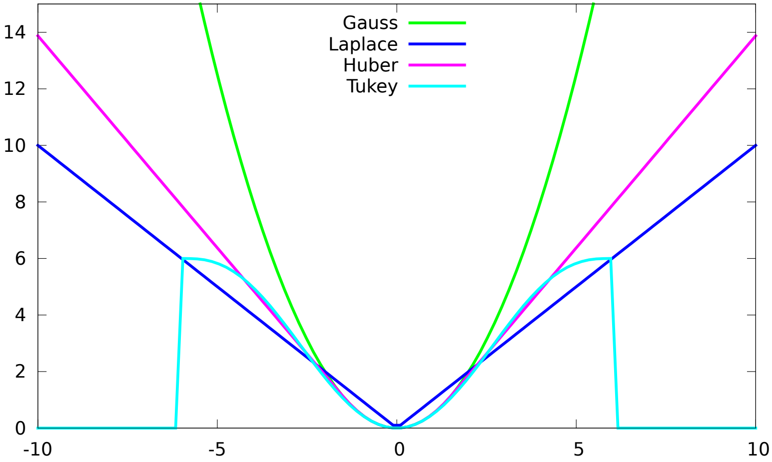
$$\sum_{i=1}^{N} \psi(x_i - \tilde{x}) = 0$$

The equation can be solved numerically.

Remarkable Distributions

	$\rho = -\ln p$	$\psi = \rho'$
Gauss	$x^2/2$	x
Laplace	$\lvert x\rvert$	$\mathrm{sgn}(x)$
Huber	$\begin{cases} -ax - a^2/2, & x < -a \\ x^2/2, & -a < x < a \\ ax - a^2/2, & x > a \end{cases}$	$\begin{cases} -a \\ x \\ a \end{cases}$
Tukey	$\begin{cases} x^6/6c^4 - (x^2/2)(1 - x^2/c^2), \\ 0 \end{cases}$	$\begin{cases} x(1 - x^2/c^2)^2 \\ 0 \end{cases}$

Graphs of Remarkable Distributions



The Algorithm for Robust Mean

1. Initial estimation by median: $\tilde{x}_0 = \nu = \mathrm{med}(x_i)$
2. Scatter estimation (median of absolute deviations — MAD) by median or by simplex method
$$s = \mathrm{med}(|x_i - \tilde{x}_0|)$$
3. Robust estimator
$$\sum_i \psi\left(\frac{x_i - \tilde{x}}{s}\right) = 0.$$
by Newton's method, Levendberg-Marquart (Minpack)
4. Approximation of deviations on minimum (robust analogy of RMS):
$$\sigma^2 = \frac{N}{N-1} s^2 \frac{(1/N)\sum_i \psi^2[(x_i - \tilde{x})/s]}{\left\{(1/N)\sum_i \psi'[(x_i - \tilde{x})/s]\right\}^2}$$

Robust Photometry

Maximum likelihood for Poisson's:

$$\chi^2 = \sum_{i=1}^{N} \left(\frac{n_i - \lambda_i(c_i|r)}{\lambda_i(c_i|r)} \right)^2$$

may be[7] asymptotically (for > 20) replaced by

$$\chi^2 \to R = \sum_{i=1}^{N} \varrho \left(\frac{n_i - \lambda_i(c_i|r)}{\lambda_i(c_i|r)} \right)$$

[7] Important! There is no proof for Poisson distribution.

Single Band Calibration

$$R = \sum_{i=1}^{N} \varrho\left(\frac{n_i - \lambda_i}{\lambda_i}\right)$$

where

$$\lambda_i(c_i|r) = rc_i, \quad \frac{\partial \lambda_i}{\partial r} = c_i, \quad \varrho' = \psi$$

$$\frac{\partial R}{\partial r} = -\sum_{i=1}^{N} \psi\left(\frac{n_i - \lambda_i}{\lambda_i}\right)\left(\frac{n_i}{\lambda_i^2}\right)\frac{\partial \lambda_i}{\partial r}$$

where the last term after ψ can be reduced onto $n_i/rc_i \to 1$ and asymptotically in minimum to one.
To improve precision, $c_i \to c_i'$ can be computed from known color transformation matrix:

$$c'_{ik} = \sum_j t_{jk} c_{ij}$$

Color Transformation Determination

$$R = \sum_{i=1}^{N} \varrho \left(\frac{n_i - \lambda_i}{\lambda_i} \right)$$

where

$$\lambda_{ik}(c_{ik}|r_{jk}) = \sum_j r_{jk} c_{ij}, \quad \frac{\partial \lambda_{ik}}{\partial r_{jk}} = c_{ik}$$

$$\frac{\partial R}{\partial r_{jk}} = -\sum_{i=1}^{N} \psi \left(\frac{n_i - \lambda_i}{\lambda_i} \right) \left(\frac{n_i}{\lambda_i^2} \right) c_{jk},$$

$$k = 1, \ldots K, j = 1, \ldots J$$

Color transformation

Determination by using (**??**) should get a matrix:

$$\begin{pmatrix} n_B \\ n_V \\ n_R \\ n_I \end{pmatrix} \overset{?}{=} \begin{pmatrix} 3.1344 & 0.0930 & 0.0108 & -0.0220 \\ -0.3033 & 1.4834 & -0.0882 & 0.0054 \\ -0.7558 & 0.4635 & 1.3393 & 0.1252 \\ -2.0691 & 1.6536 & -1.0988 & 3.0274 \end{pmatrix} \begin{pmatrix} c_B \\ c_V \\ c_R \\ c_I \end{pmatrix}$$

What's going on?

- Off-diagonal elements are too large (means overlays!)
- Test data gives correct values for small noise, fails for large.
- Residuals looks sufficiently: small and correct.
- Failed due to various disturbances (noise, rounding errors, ...).

Regularization

To get disturbances-free solution, we introduces additional condition:

$$\sum_{\substack{i,j \\ |i-j|>1}} r_{ik} \to 0$$

(minimizing of sum of off-tridiagonal elements)

$$R = \sum_{i=1}^{N} \varrho\left(\frac{n_i - \lambda_i}{\lambda_i}\right) + \lambda \left(\sum_{\substack{jk \\ |j-k|>1}} r_{jk} - 1 \right)$$

where λ is Lagrange's multiplicator.

Regularized Color Transformation

Field of T Phe

Calibration on Stars:

$$\begin{pmatrix} n_B \\ n_V \\ n_R \\ n_I \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} 2.0043 & 0.0499 & -0.0000 & 0.0000 \\ 0.0592 & 0.9392 & -0.0159 & 0.0000 \\ 0.0000 & 0.0439 & 1.1834 & -0.0213 \\ 0.0000 & 0.0000 & -0.0376 & 1.8829 \end{pmatrix} \begin{pmatrix} c_B \\ c_V \\ c_R \\ c_I \end{pmatrix}$$

Filters (Vega):

$$\begin{pmatrix} n_B \\ n_V \\ n_R \\ n_I \end{pmatrix} = \begin{pmatrix} 1.5271 & -0.2223 & 0.0825 & -0.0673 \\ 0.1125 & 0.8968 & -0.1655 & 0.0126 \\ 0.0030 & -0.1854 & 1.4262 & -0.6064 \\ 0.0004 & -0.0533 & 0.1591 & 1.0928 \end{pmatrix} \begin{pmatrix} c_B \\ c_V \\ c_R \\ c_I \end{pmatrix}$$

Advises for Color Transformation

- Regularization is absolutely necessary.
- Blurred regularization term can be proposed.
- Classical photometrics (Hardie) recommends only diagonal and upper diagonal (unstable).

Part V

Modeling Extinction

The Extinction

Light passing a medium lost its energy (or photons are scattered and absorbed) as

$$\frac{\mathrm{d}F_\nu}{\mathrm{d}x} = \kappa(\nu)F_\nu$$

Its solution:

$$F_\nu = F_0 \mathrm{e}^{-\kappa(\nu)x}$$

$F = I$ for plane wave.
Typical dependencies of $\kappa \sim 1/\lambda, 1/\lambda^4$

$$\kappa(\nu) \sim \nu, \sim \nu^4$$

Monochromatic Extinction

Flux for an object in the filter:

$$F_V = \int F(\nu) f_V(\nu) \mathrm{e}^{-\kappa(\nu)x} \,\mathrm{d}\nu \approx F_{V_0} f_V \Delta\nu_V \mathrm{e}^{-\kappa(\nu_V)x}$$

Photon flux for a plane wave in a filter

$$n_V = n_{V_0} \frac{f_V \Delta\nu_V}{h\nu_V} \mathrm{e}^{-\kappa_V x}$$

with substitution $c_{V_0} = n_{V_0}(f_V \Delta\nu_V / h\nu_V)$ that leads to a simple dependence of observed counts c on its path

$$c_V = c_{V_0} \mathrm{e}^{-\kappa_V x}$$

DK154 extinction

Without color correction

- observer Selected Areas (SA) by P.Škoda.
- available 3-5 stars per SA
- Johnson-Morgan photometry system, Landolt (1992)
- aperture photometry, radius 3-FWHM (= 7 arcsec)

Attenuation is modeled as

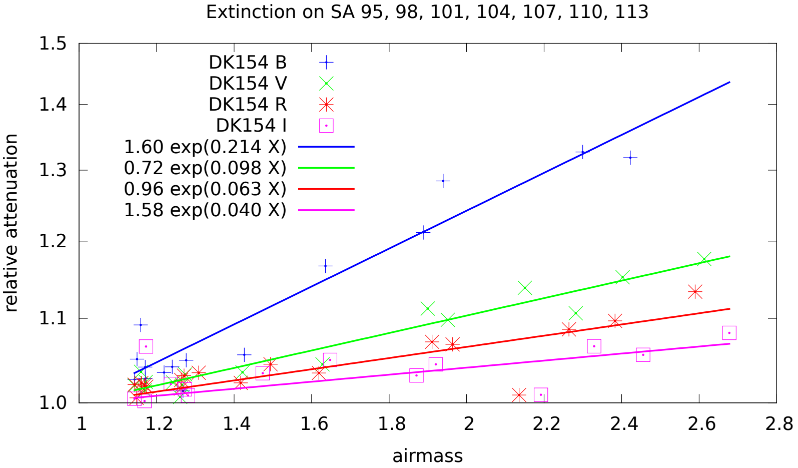
$$r(X) = A e^{KX} (\sim \frac{n_V}{c_V})$$

filter	A [ct/ph]	K [ph/airmas]
B	1.600 ± 0.042	0.214 ± 0.016
V	0.722 ± 0.008	0.098 ± 0.007
R	0.958 ± 0.016	0.063 ± 0.010
I	1.582 ± 0.031	0.040 ± 0.011

A is extraterrestrial value, K is the extinction

Graph of DK154 extinction

Without color correction



DK154 extinction

Color correction

Attenuation is modeled as

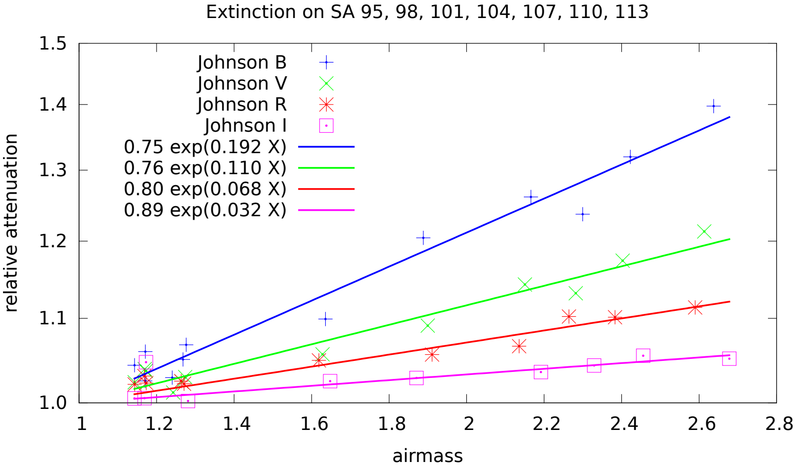
$$r(X) = A\mathrm{e}^{KX}(\sim \frac{n_V}{c_V})$$

filter	A [ct/ph]	K [ph/airmas]
B	0.753 ± 0.016	0.192 ± 0.011
V	0.764 ± 0.010	0.110 ± 0.007
R	0.799 ± 0.011	0.068 ± 0.008
I	0.888 ± 0.014	0.032 ± 0.008

A is extraterrestrial value, K is the extinction

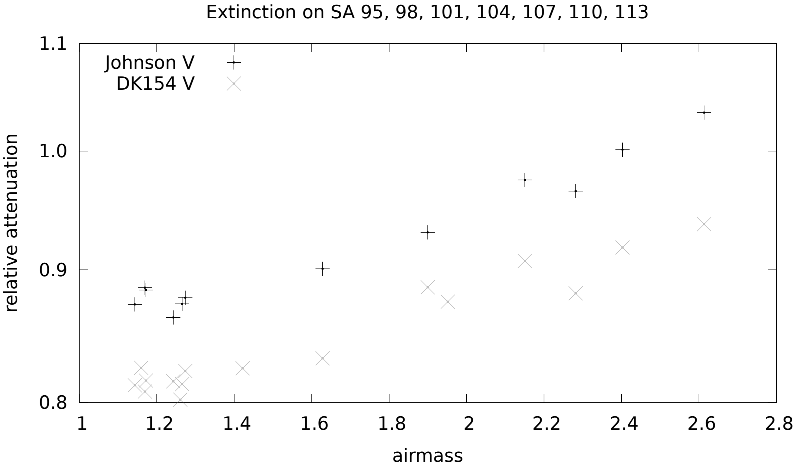
Graph of DK154 extinction

Color correction



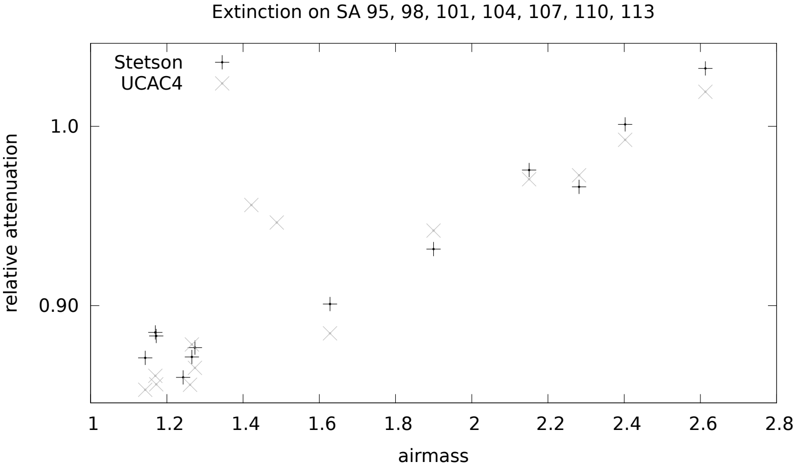
Graph of DK154 extinction

Comparison with and without color correction



Graph of DK154 extinction

Comparison Stetson vs. UCAC4



Color Extinction and Photon Calibration

$$\kappa(\nu - \nu_V) = \kappa(\nu_V) + \left.\frac{\mathrm{d}\kappa}{\mathrm{d}\nu}\right|_V (\nu - \nu_V) + \cdots \approx \kappa(\nu_V) + \kappa'_V \Delta\nu$$

so

$$c_V = c_0 \mathrm{e}^{-(\kappa_V - \kappa'_V \Delta\nu) X}$$

therefore differently coloured object will different-falling exponential.
Fluency on color transformation[8]

$$r_{ik} = r_{ik}(n, c) \cdot r_{ik}(\kappa)$$

$$n_k \approx \sum \mathrm{e}^{\kappa X} r_{ik} c_k \tag{16}$$

[8] No data — no love.

Part VI

Hell Of Magnitudes

Magnitudes

Magnitudes (Nobody expects the Spanish Inquisition![9])

$$m - m_0 = 2.5 \log_{10} \frac{F}{F_0}$$

- Defined by Pogson in mid 19 century to formalize ancient magnitudes of Hipparcos.
- Logarithm of flux ratio.
- Designed as an analogy of psycho-physiological law for sound (obsoleted at late 1920s by Wright, Guild: perception $\sim F^{1/3}$)
- Used exclusively by optical astronomers
- Chief confusing framework in astronomy (!).

[9] The Spanish Inquisition seeds "violence, terror an torture" like magnitudes. See the sketch of Monty Python [1] for details.

Magnitude — Flux and Photons Connection

Magnitudes in filter

$$m - m_0 = -2.5 \log_{10} \frac{F_i}{F_{0i}} = -2.5 \log_{10} \frac{n_i}{n_{0i}}$$

- ??

Magnitude — Calibration in Magnitudes

From (**??**), we know

$$n \approx rc$$

and

$$F = \int_\Omega nh\nu_{\mathrm{eff}}\,\mathrm{d}\Omega$$

and therefore

$$F_{\mathrm{reference}} = nh\nu_{\mathrm{eff}}\Delta\Omega$$

$$F_{\mathrm{instrumental}} = ch\nu_{\mathrm{eff}}\Delta\Omega$$

$$m_{\mathrm{instrumental}} - m_{\mathrm{reference}} = 2.5\log_{10} r$$

Basic rule: difference of magnitudes is logarithm of ratio of fluxes.
Work for both energy and photon fluxes.

Magnitude — Color index

Defined as

$$m_k - m_l = -2.5 \log_{10} \frac{F_k}{F_l} = -2.5 \log_{10} \frac{n_k}{n_l} \frac{\Delta\nu_k}{\Delta\nu_l} \frac{\nu_l}{\nu_k}$$

or alternatively

$$m_k - m_l = -2.5 \log_{10} \frac{n_k}{n_l}$$

Magnitude — Color Transformation

From (**??**), we know

$$n_k = \sum_j r_{jk} c_k, \quad (k = U, B, ..)$$

and

$$m_k - m_{k-1} = -2.5 \log_{10} \frac{n_k}{n_{k-1}} = -2.5 \log_{10} \frac{r_k c_k}{n_{k-1}}$$

Magnitude — Atmospheric Extinction

From (16), we know

$$n_k \approx \sum e^{\kappa X} r_{ik} c_k$$

when we define extinction coefficient $\frac{2.5}{\ln 10}\kappa \equiv k$

$$m - m0 = -2.5 \log_{10} \frac{rc}{n} + k\kappa X$$

Magnitude's Hell

- We cannot use robust methods (because distribution is non-normal).
- The use of $\sigma^2 = \bar{n}$ is obscured.
- The question of superior: Can we use magnitudes in our case?

Part VII

Munipack

Key Features

* power of combinations

Routines Overview

Aperture Photometry

Input: Output:

Color Transformation

Photometry System Identification

The photometry system is identified by its name and a set of filters.
Representation by a structure:
Source: The Asiago Database on Photometric Systems [3]

Photometry Calibration

Splited on two phase process:

- color calibration
- ratios calibration

Color calibration

Properties (for 4 filters):

- fitting many parameters: 17 (!)
- needs many stars (at least 5 in every filter)
- sensitive on statistical errors

```
$ munipack phfotran -c [cat] --label [filters] b.fits v.fit
```

Uses formula..

Ratio calibration

Properties:

- fitting one parameter per filter
- single star is sufficient
- for precise calibration uses r_{ik} matrix

```
$ munipack phcal -c [cat] --label [filters] b.fits,b_cal.fi
```

Uses formula..

Part VIII

Conclusions

Conclusions

Key points:

- photon nature of modern detectors
- robust statistical methods
- Poisson statistic
- Regularization

References

- http://en.wikipedia.org/wiki/The_Spanish_Inquisition_(Monty_Python)
- http://www.stsci.edu/institute/software_hardware/stsdas/synphot
- http://ulisse.pd.astro.it/Astro/ADPS/
- P. J. Huber: Robust Statistics, Willey (2004)