Testing by the revised Bloom's taxonomy: linear algebra in physics education

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Abstract:

The aim of the article is to show how can the revised Bloom taxonomy be used to test the effectiveness of linear algebra teaching in university studies of physics, i.e. in education of future physicists and physics teachers. Linear algebra is an integral part of physics as such, and therefore of physics teaching as well. The revised Bloom's taxonomy (RBT), thanks to the principle of gradually increasing increasing difficulty and problem-solving requirements, is very suitable for obtaining information about the extent to which the teaching of linear algebra (and mathematics in general) and the teaching of physics are directly connected in terms of the needs of university physics education. Six levels of the RBT (remember, understand, apply, analyze, evaluate, create) can be tested in three basic stages: knowledge, conceptual knowledge and procedural knowledge (the stage meta-cognition included in general version of RBT is not relevant for our purposes). For students of physics and students of teaching physics, we have newly included, as the fourth stage, "physics application", for testing students ability to use newly acquired mathematical knowledge in describing and solving physical problems. Using a sample test connecting the issues of a typical linear algebra topic, "Eigenvalues, and eigenvectors of linear operators", with the physical issues of "Rigid body rotation", and its evaluation, we demonstrate the effectiveness of the RBT testing method.

Introduction

Teaching mathematics, especially algebra and geometry, mathematical analysis, and more advanced disciplines, is of course a natural part of university studies of physics and teaching physics. However, the problem is often that mathematics is taught separately without any connection to physics, for which it is essential. Unfortunately, the need to connect physics and mathematics directly in teaching is not a priority in general. Nevertheless, studies occasionally appear that deal with the problems of such connections. For the last decade, see for example [1]-[6].

In the study of physics and physics teaching, the interconnection is emphasized especially to linear algebra and geometry as one of fundamental mathematical tools for physics. This can be seen also in recent relevant publications, see e.g. [7]-[11]. Directly related to physics education is testing students' mathematical knowledge and skills in connection with their ability to apply them in solving physics problems. Testing using the revised Bloom's taxonomy (RBT) appears to be particularly suitable for its two-dimensional hierarchical principle of tests formulation and evaluation. Six levels of the RBT (remember, understand, apply, analyze, evaluate, create) can be tested, in general, in four stages (knowledge, conceptual knowledge, procedural knowledge and meta-cognition). For students of physics and students of teaching physics, we have newly replace the stage meta-cognition by the stage "physics application", for testing students ability to use newly acquired mathematical knowledge in describing and solving physical problems. Although we assume that interested readers are familiar with the principles and methods of RBT, we will briefly discuss them in Section 2.

In this contribution we present one of typical sample tests of the fourth (newly added by us) stage "physics applications" (see Sec. 2) within all six levels of RBT, including correct answers and comments relating the mathematical background and the physical essence. The topic of the test is "Eigenvalues and eigenvectors of linear operators" as a mathematical background for solving problems of rotational motion of solid bodies. We present the results of student solutions (the group of students after completing five semesters of study) and their statistical evaluation. We also discuss possible causes of incorrect solutions.

2. The principles of Revised bloom taxonomy

The Bloom taxonomy was first published in 1956 [12]. Bloom and his coauthors showed a hierarchical organization of cognitive processes. They established six gradually increasing levels: Knowledge, Comprehension, Application, Analysis, Synthesis, Evaluation. Today, we use the taxonomy in a revised version [13]. The most important change compared to the original version is the expansion of Bloom one-dimensional version to the two-dimensional one: in addition to the six levels of knowledge, each of them is divided into four degrees (stages):

- A Factual knowledge (basic concepts and terminology)
- B Conceptual knowledge (relationships between concepts)
- C Procedural knowledge (methods, procedures, algorithms)
- D Metacognitive knowledge (self-knowledge, learning strategies)

The original taxonomy included elements of only the first three levels, without specifying them. Another change was the use of verbs in the names of the levels and the order of the highest two levels was reversed. The RBT is schematically depicted on Fig. 1 (where instead of the stage D our new stage F is introduced).

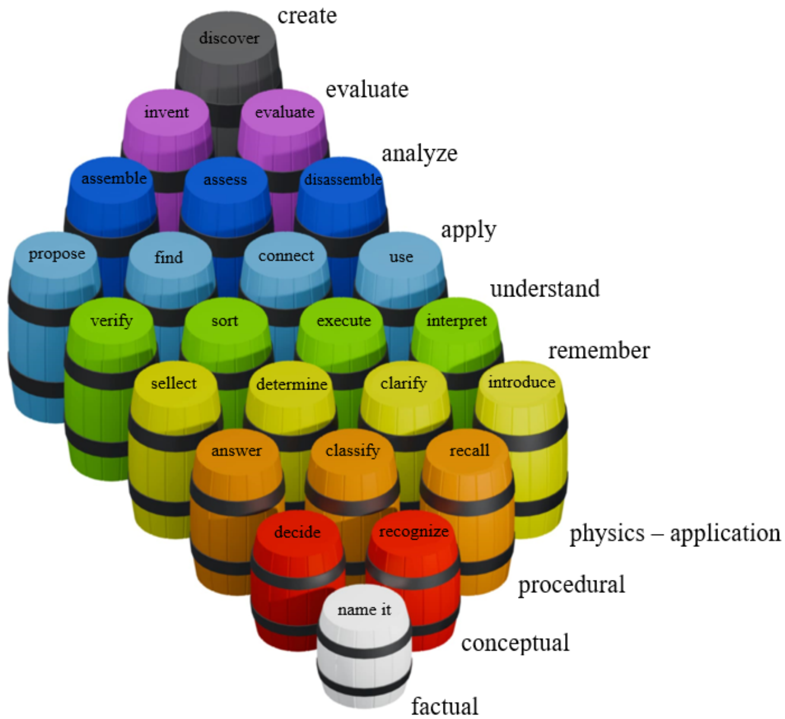

FIG. 1. The "barrel"scheme of the Revised Bloom Taxonomy.

In this spatial image the same height of the "knowledge barrels"corresponds to the same difficulty of the test items belonging to the given cells of the taxonomic table, starting from white (the easiest tasks of type 1A), to black (the most difficult tasks of type 6F). On the top of each barrel wee can see a verb in the imperative, which is supposed to lead to a typical item of this level.

In our work, we did not focus on the last of the four levels (D); the reader can find more information in [14], [15]. For testing students of physics, we have newly introduced the physics-application level of knowledge F. The tasks of the stage F focus on students' abilities to apply mathematical knowledge as an indispensable tool in physics. Using mathematics to describe physical situations and to solve specific problems means a specific knowledge that we require from students of physics and students of teaching physics. We thus emphasize the connection between mathematical and physics courses, which has long been shown to be insufficient in the study of physics.

Let's characterize the individual items of stage F according to the six levels of RBT.

Level 1F – Remember: "Introduce" a mathematical concept to physics. The goal of the tasks is to "recognize"(identify) the appropriate mathematical concepts for physics. It tests the student's ability to "remember"and subsequently "recall"relevant information and connections. The ideal type of questions are closed ones with one or more correct answers, respectively matching or opened tasks.

Level 2F – Understand: "Interpret" a physical situation from a mathematical point of view. We require a deeper understanding of concepts and connections, as well as the ability of students to communicate their meaning. The student's task is not only to choose the right mathematical model, but above all to explain its suitability for a given physical situation. At this level, we usually test with closed questions with multiple correct answers.

Level 3F – Apply: "Use" mathematical tools to solve a physical problem. The tasks tests the ability to use mathematical methods and procedures to solve a known problem, or to implement algorithms for an unknown problem. Typical tasks are the opened problems of the type "determine" or "calculate", where the answer is a number (or several numbers or an expression).

Level 4F – Analyze: "Analyze" mathematical aspects in a given problem of physics. Students ought to be able to see structures, to divide the whole into parts and understand the relationships between them and their relationship to the whole itself, to distinguish, classify, integrate, assign, structure. The most common type of questions are closed ones with multiple correct answers, or assignment questions.

Level 5F – Evaluate: "Evaluate" a physical situation from a mathematical point of view. The tasks test the student judgement based on general criteria. This requires critical thinking when assessing claims or arguments. The ability to evaluate is best demonstrated in oral examinations, or it is appropriate to test it with closed questions with multiple correct answers, or matching.

Level 6F – Create: "Discover" new connection between physics and mathematics. The essence of this level is to present to the student a task that he has not encountered before. For example, if the same task is presented to a first-year student, it may correspond to the type 6F, while a third-year student is tested only at the "apply" level. The tasks require the ability to reorganize elements into a newly created structure. Typical tasks are the opened ones.

3. Physics-application knowledge in the context of RBT

In this chapter, we will focus on the specifics of the knowledge stage newly introduced into teaching on all six knowledge levels, the so-called physical-application knowledge. As mentioned, the basic course of university mathematics for physics studies should be adapted to the aim to implement mathematical concepts, statements and methods into physical theories, of course, provided that the mathematical interpretation is correct. Mathematical courses should not be built "next to" physics courses, but should be in constant parallel connection with them. However, experience shows that the standard way of teaching mathematics using the definition-theorem-proof method without permanent physical demonstrations of the use of mathematical tools does not lead to achieving this goal. Therefore, we have decided to further develop the connection between mathematics and physics teaching in basic mathematics courses. This applies not only to the teaching itself (lecture/exercise), but also, importantly, to testing, for which we use RBT.

To achieve this goal, we primarily focused on the disciplines "Linear algebra and geometry" and "Linear and multilinear algebra". These disciplines are included in the physics study program in the first two semesters and are guaranteed by the Institute of theoretical physics and astrophysics, Faculty of Science, Masaryk University (author's workplace). Tests of the stages 1F-6F are prepared in such a way that they follow the basic topics of both mentioned disciplines as stated in their syllabus (14+14 topics).

4. The sample test

In the physics studies on Faculty of Science, Masaryk University, the courses "Linear algebra and geometry" and "Linear and multilinear algebra" are included in the first two semesters of study. The teaching is carried out in a way that respects the needs of physics courses of the first six semesters (bachelor's degree of study completed by the state final examination).

To give an idea of â€‹â€‹the concept of 1F to 6F type problems, we present a sample test on the topic of "Eigenvalues â€‹â€‹and eigenvectors of linear operators". Let us note that this mathematical background has a direct connection to important topics in physics, such as the rotation of a rigid body (relationship between angular momentum and angular velocity), electricity, magnetism, optics of generally inhomogeneous media (relationship between electric field intensity and polarization of the medium), quantum mechanics (values â€‹â€‹of measured quantities versus eigenvalues â€‹â€‹of the relevant operators, eigenstates of a quantum mechanical systems), and others.

Here we present the test concerning the mentioned topic "Eigenvalues â€‹â€‹and eigenvectors of linear operators" in connection with the physical problems concerning rotational motion of rigid bodies, and with relevant comments on its formulation. First, we would like to make a few notes regarding the specific choice of those sample problems, which fall primarily into the field of Newtonian mechanics.

1) The issue "Eigenvectors and eigenvalues â€‹â€‹of linear operators" is a somewhat more advanced area of â€‹â€‹linear algebra and geometry for students. Although it has a wider and often more natural use in a number of other physical disciplines than just classical mechanics, we choose examples at all levels 1F to 6F of Bloom taxonomy from the field of mechanics, in order to capture the essence and increasing difficulty of individual levels of Bloom taxonomy on issues of a similar type.

2) The tasks involve the concept of a *tangent vector space* $T_O\mathbf{R}^3$ located at point O of Euclidean space $\mathbf{R}^3$ (Fig. 2). This concept is clearly explained to students in both mechanics and linear algebra courses.

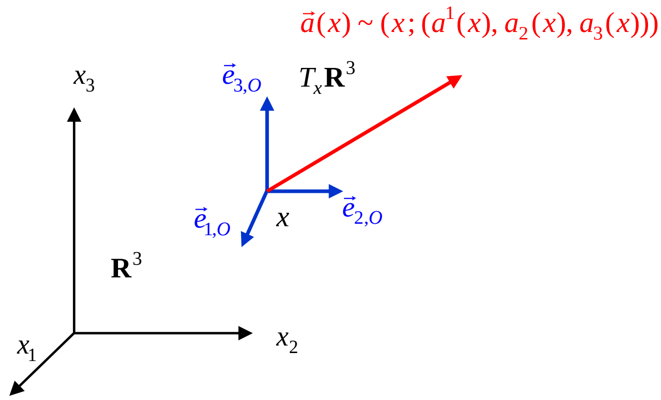

FIG. 2. Tangent (vector) space $T_O\mathbf{R}^3$ to the Euclidean (topological) space $\mathbf{R}^3$.

3) We often choose the location of the tangent vector space $T_O\mathbf{R}^3$ at the center of mass of the body SH, i.e. $O = SH$, so that in tasks of lower difficulty level students do not get entangled in considerations about the location of the reference point for calculating the angular momentum.

4) In introductory physics disciplines in the study of physics programs, especially in mechanics, we are working practically exclusively in Cartesian coordinate systems of the Euclidean space $\mathbf{R}^3$, or in orthonormal bases of the vector space $T_x\mathbf{R}^3$ at a point $x \in \mathbf{R}^3$.

5) The inertia of a body is a symmetric Cartesian tensor – it is defined using components in orthonormal bases (see note 3) above). In mechanics courses, students already work with it practically as a "converter" between angular momentum and angular velocity, and the conditionality of the choice of an orthonormal basis is justified to them. In the teaching of linear algebra, where transitions between bases are discussed simultaneously and with reference to physics, students will become aware of and practice working with the components of the vector product, which is precisely the expression of angular momentum. They will be convinced that the usual notation of the components of the vector product that they used, specifically for

$$\vec{a} \sim (\alpha^1, \alpha^2, \alpha^3) \;\; \vec{b} \sim (\beta^2, \beta^2, \beta^3),$$

$$\vec{a} \times \vec{b} \sim (\alpha^2\beta^3 - \alpha^3\beta^2, \alpha^3\beta^1 - \alpha^1\beta^3, \alpha^1\beta^2 - \alpha^2\beta^1),$$

is correct in right-handed orthonormal bases, but not in general bases. Therefore, in problems of the type mentioned below, orthonormal bases right-handed are automatically taken into account.

6) It should also be noted that the assignment of physics problems is usually more comprehensive than in the case of purely mathematical problems. It is necessary to clearly specify all simplifying assumptions regarding the physics problem.

1F Introduce a mathematical concept into physics

Let $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ be an orthonormal base of the vector space $T_O\mathbf{R}^3$ located at the origin O of the Cartesian coordinate system. The linear relationship between the angular momentum of a rigid body $\vec{L}$ and its angular velocity $\vec{\omega}$ has the form $L_i = J_{i1}\omega_1 + J_{i2}\omega_2 + J_{i3}\omega_3$, where in matrix notation $\vec{L} \sim (L) = (L_1 \;\; L_2 \;\; L_3)$ and $\vec{\omega} \sim (\omega) = (\omega_1 \;\; \omega_2 \;\; \omega_3)$, and $\hat{J} \sim \tilde{J} = (J_{ij})$, $J_{ij} = J_{ji}$, $i = 1, 2, 3$, is the inertia of the body. The matrix relation $(L) = (\omega)\tilde{J}$ can be understood as an expression representing the action of the symmetric linear inertia operator (represented by the matrix $\tilde{J}$) on the angular velocity vector, the image of which is the angular momentum vector.

For the operator $\hat{J}$, it makes sense to discuss the problem of its eigenvalues and eigenvectors. The vector space containing the vectors under consideration is three-dimensional vector space over $\mathbf{R}$. Of the following statements, exactly one is true. Choose them.

a) An eigenvector of the operator $\hat{J}$ is any vector $\vec{\omega}$ for which its image $\vec{L}$ is a numerical multiple of it.

b) The problem of eigenvalues â€‹â€‹and eigenvectors of the operator $\hat{J}$ may not have a solution in some situations.

c) The eigenvalues â€‹â€‹of the operator $\hat{J}$ depend on the chosen basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$.

d) The operator $\hat{J}$ has only real characteristic roots, which are also its eigenvalues.

e) Because the relation $(L) = (\omega)\tilde{J}$ is trivially satisfied for the zero vector $\vec{\omega}$, the zero vector is therefore an eigenvector.

Comments: To solve the problem correctly, the student needs to *know and remember* the definitions of the concepts eigenvector, eigenvalue, characteristic root and also basic information about these concepts related to symmetric linear operators. The physics of the problem is not complicated, it is a common type of linear relationship that the student knows from the study discipline Mechanics.

a) Wrong answer, because the given relation is also satisfied by the zero vector, which is excluded from the definition of an eigenvector.

b) Wrong answer. The problem of eigenvalues â€‹â€‹and eigenvectors of a symmetric linear operator in a vector space over **R** (and even above **C**) always has a solution (the characteristic roots are real and therefore they are also eigenvalues).

c) Wrong answer. The eigenvalues â€‹â€‹of every linear operator are invariant to the choice of basis.

d) Correct answer. A symmetric linear operator in a vector space over **R** (and even over **C**) has only real characteristic roots, which are also its eigenvalues.

e) Wrong answer. The zero vector is not an eigenvector. This is done by the definition.

2F Interpret a physical situation from a mathematical point of view

We work in orthonormal bases. The linear relation between the angular momentum $\vec{L}$ of a rigid body (flywheel) and its angular velocity $\vec{\omega}$ in the vector space $T_O\mathbf{R}^3$, located at the origin O of the Euclidean space $\mathbf{R}^3$, is realized by the symmetric linear operator $\hat{J}$, called the inertia tensor. Let us assume for simplicity that the origin of the coordinate system lies at the center of mass of the body (it represents the zero vector). In the general case, when the body has no symmetry of mass distribution, this operator has three different (positive) eigenvalues ﬃﬁJ_1, J_2, J_3. From the following statements, choose exactly all true ones.

a) If the components of angular momentum $\vec{L} \sim (L_1, L_2, L_3)$ and angular velocity $\vec{\omega} \sim (\omega_1, \omega_2, \omega_3)$ are expressed in any basis of the space $T_O\mathbf{R}^3$, then $L_1 = J_1\omega_1$, $L_2 = J_2\omega_2$,$L_3 = J_3\omega_3$.

b) The operator $\hat{J}$ has (after the addition of the zero vector) just three one-dimensional vector subspaces of eigenvectors, determining the so-called principal axes of the inertia.

c) The operator $\hat{J}$ has exactly three different eigenvectors.

d) It is not clear in advance whether the operator $\hat{J}$ is diagonalizable, i.e. whether there is a basis of the space $T_O\mathbf{R}^3$ in which the operator is represented by a diagonal matrix.

e) Since the operator $\hat{J}$ has a simple spectrum, any vector in the space $T_O\mathbf{R}^3$ is its eigenvector and thus represents the so-called free axis of the flywheel.

f) The eigenvectors of the operator $\hat{J}$ generate the entire vector space $T_O\mathbf{R}^3$.

g) The operator $\hat{J}$ has no eigenvectors because it maps vectors of one type of physical quantity (angular velocity) onto another type of physical quantity (angular momentum).

h) The physical problems of the motion of flywheels are not related to the problem of eigenvalues â€‹â€‹and eigenvectors of operators.

Comments: If the teaching of the topic "Eigenvalues â€‹â€‹and eigenvectors of a linear operator" were conducted without any connection to physics and the test was presented in the discipline "Linear Algebra and Geometry", students would (hopefully) be able to assess the correctness of the answers in which the connection to physics is not explicitly mentioned. A frequent occurrence of answer h) as correct one can also be expected.

a) Wrong answer. In order to correctly assess the correctness of the statement, the student must understand the connections between physical quantities and the concepts of linear algebra and also the fact that the relation $L_i = J_i\omega_i$ holds precisely in the basis associated with the eigenvectors of the operator $\hat{J}$.

b) Correct answer. The student will assess this statement correctly if he/she knows the connection between the principal axes of the inertia and the solution of the problem of the eigenvectors of the operator $\hat{J}$. He/she should also know and understand the fact that to each of the three different eigenvalues â€‹â€‹of the operator in $T_O\mathbf{R}^3$ corresponds just to unique one-dimensional vector subspace of eigenvectors (after adding a zero vector).

c) Wrong answer. The student may perceive the question as purely algebraic. To make the right decision, he must understand the reasons (based on an understanding of the problem of solving systems of linear equations) that an eigenvalue of a linear operator cannot correspond o a single eigenvector, but always to a vector subspace. (This problem is also connected with the understanding of the fact that a homogeneous system of linear equations of reduced rank, which is the system of equations for the components of the eigenvectors after substituting the determined eigenvalue, always has infinitely many solutions.)

d) Wrong answer. To make a decision here, it is sufficient to know the fact that if the operator in three-dimensional vector space has three different eigenvalues â€‹â€‹(a simple spectrum) then it has a diagonal representation. If the student is aware of the connection of the diagonal representation of the operator $\hat{J}$, corresponding relation between the angular momentum and angular velocity of the flywheel, and the connection of this relation with principal axes, then he/she will able to eliminate answer d).

e) Wrong answer. If the student has understood the basic definitions and statements regarding the problem of eigenvalues â€‹â€‹and eigenvectors of a linear operator, he/she knows that every vector (except the zero vector) is an eigenvector of a linear operator if and only if the operator has a single threefold eigenvalue. Alternatively, to make the right decision, it may be enough for him/her to know that the zero vector does not satisfy the definition of an eigenvector. When connecting with physics, it is enough to realize that a general flywheel with asymmetric mass distribution, has exactly three principal axes (each of them is related to a one-dimensional vector space of eigenvectors).

f) Correct answer. The student only needs to realize the meaning of the term "to generate a vector space" and understand it (each vector of the given vector subspace is a linear combination of generators).

g) Wrong answer. It can only be marked as correct by a student who is not aware of any connection of the relationship between angular momentum and angular velocity with the issue of eigenvectors and values â€‹â€‹of linear operators. (This student not understand that the inertia is the physical realization of a linear operator).

h) Wrong answer. Marking it as correct has the same interpretation as in the case of ad g).

3F Use mathematical tools to solve a physics problem

Each component of the angular momentum $\vec{L}$ of a rigid body is, in the simplest physical situation, a linear combination of all components of the angular velocity vector $\vec{\omega}$. This relationship is mediated by the linear operator $\hat{J}$ of the inertia of a body. This operator is symmetric which is symmetric. In a certain orthonormal basis and in agreed units the matrix $\tilde{J} = (J_{ij})$, i, $j = 1, 2, 3$, representing the operator $\hat{J}$ is given, where $J_{11} = J_{22} = J_{33} = 2$, $J_{12} = J_{21} = 0$, $J_{13} = J_{31} = 1$, $J_{23} = J_{32} = 0$.

a) Using the solution of the eigenvalue problem and based on the physical properties of the inertia, decide whether this matrix can represent the operator $\hat{J}$ in the chosen basis.

b) Do the eigenvalues â€‹â€‹of the linear operator $\hat{J}$ depend on the choice of basis?

c) If the given matrix $\tilde{J}$ can represent a linear operator $\hat{J}$, calculate its eigenvectors.

d) Interpret the solution c) from the point of view of physics.

Comments: A relatively simple open task on the direct application of the problem of eigenvalues â€‹â€‹and vectors in a physical situation. (Note that the same problem can be formulated for other linear relations in physics, e.g. for vectors of the electric induction $\vec{D}$ and electric intensity $\vec{E}$ connected by the symmetric linear operator $\hat{\mathcal{E}}$ of dielectric permittivity.)

ad a) The student must assess the given matrix according to the eigenvalues â€‹â€‹of the inertia operator (from the point of view of physical meaning, only positive eigenvalues â€‹â€‹are permissible). In the case of the given matrix, there are three different eigenvalues â€‹â€‹$\mathcal{J}_1 = 1$, $\mathcal{J}_2 = 2$, $\mathcal{J}_3 = 3$. They are therefore permissible and the given matrix can represent the operator $\hat{J}$ in (arbitrary) orthonormal bases.

b) The eigenvalues â€‹â€‹of the operator are independent of the choice of basis.

c) The eigenvectors corresponding to the eigenvalue $\mathcal{J}_1 = 1$ generate the vector subspace

$$\mathcal{L}_1 = [|(1,\, 0,\, -1)|] = \left[\left|\left(\frac{1}{\sqrt{2}},\, 0,\, -\frac{1}{\sqrt{2}}\right)\right|\right]$$

of eigenvectors, the eigenvalue $\mathcal{J}_2 = 2$ corresponds to the vector subspace

$$\mathcal{L}_2 = [|(0,\, 1,\, 0)|],$$

and the eigenvalue $\mathcal{L}_3 = 3$ corresponds to the vector subspace

$$\mathcal{L}_3 = [|(1,\, 0,\, 1)|] = \left[\left|\left(\frac{1}{\sqrt{2}},\, 0,\, \frac{1}{\sqrt{2}}\right)\right|\right].$$

d) From the point of view of the connection of mathematics and physics, the answer to d) (together with the answer to a)) is essential: For example, for any vector $\vec{L} \in \mathcal{L}_3$, i.e. $\vec{L} \sim (\alpha,\, 0,\, \alpha)$, where α is an arbitrary constant, it holds

$$(\alpha \;\; 0 \;\; \alpha) \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix} = (3\alpha \;\; 0 \;\; 3\alpha) \implies \vec{L} = 3\alpha\vec{\omega}.$$

In the case where the angular velocity has the direction of the eigenvector of the operator $\hat{J}$, the angular momentum has the same direction.

4F Analyze mathematical aspects in a physical case

We work in orthonormal bases. The linear relationship between the angular momentum $\vec{L}$ of a rigid body (flywheel) and its angular velocity $\vec{\omega}$ in the vector space $T_O\mathbf{R}^3$ is realized by the linear operator $\hat{J}$, so-called *inertia tensor*. Let us assume that the origin O of the coordinate system lies at the center of mass of the body (the zero vector is located in it). Suppose that the body has symmetrically distributed mass with respect to a certain axis o (rotational symmetry, in the case of a homogeneous body the axis o is the axis of its geometric rotational symmetry), and that it does not have a higher symmetry. The principal axis of the moment of inertia is understood as a straight line passing through the center of mass of the body $O = SH$ such that if the body rotates around it, its angular momentum has the direction of the angular velocity at each moment. From the following statements, select exactly all true ones:

a) The operator $\hat{J}$ has exactly two different eigenvalues ﬃﬃJ_1 and J_2.

b) All vectors directed along the principal axes of the inertia are eigenvectors of the operator $\hat{J}$.

c) The eigenvectors of the operator $\hat{J}$ are exactly all vectors having the direction of the axis of symmetry o.

d) Every vector having the direction of the axis o is an eigenvector of the operator $\hat{J}$.

e) The flywheel has infinitely many principal axes.

f) In an orthonormal basis connected to the principal axes, the matrix representing the operator $\hat{J}$ has a diagonal form, specifically, e.g. $\operatorname{diag} \hat{J} = (J_1, J_2, J_3)$, $J_1 = J_2$.

g) There are special cases when the basis in which the operator $\hat{J}$ would have a diagonal form does not exist.

h) Each eigenvector of the operator $\hat{J}$ determines the direction of a principal axis of the inertia.

Comments: The task supposes understanding the connections between physical concepts and linear algebra concepts, including geometric interpretation. Understanding the fact that the linearity of the relationship between angular momentum and angular velocity mediated by the inertia mathematically implemented by a linear operator (symmetric in orthonormal bases) allows, based on the properties of mathematical objects, to infer (interpret) the properties of the motion of a physical system. In the given task, however, the problem is rather the opposite â€" the student must "to convert" the physical definition of the principal axis of the inertia into the "language" of linear algebra. The gradual steps of the necessary analysis could be as follows:

1) If a body rotates about the principal axis o, i.e. $\vec{\omega} \parallel o$, the relation $\vec{L} = J_o\vec{\omega}$ holds for a certain value J_o.

2) The relationship between the components of the angular momentum $\vec{L}$ and the angular velocity $\vec{\omega}$ in a basis in which the matrix Ëś$\tilde{J}$ representing the operator $\hat{J}$ is diagonal, in matrix notation $\vec{L} \sim (L) = (L_1 \;\; L_2 \;\; L3)$, $\vec{\omega} \sim (\omega_1 \;\; \omega_2 \;\; \omega_3)$, $(L) = (\omega)\tilde{J}$, leads to a special expression of the components of the angular momentum $L_i = J_i\omega_i$.

3) The first two steps are the key to solving the problem, as they lead to the comparison $(L) = (\omega)\tilde{J}$, $(L) = J_0(\omega) \Rightarrow (\omega)(\tilde{J} - J_0E) = (0)$, where E is the unit matrix and (0) is the zero matrix. From this comparison it is clear that each vector ω in the direction of the principal axis is an eigenvector of the operator $\hat{J}$. This transforms the physical problem into the algebraic one.

4) The problem has an important physical aspect resulting from its symmetry (symmetry of the mass distribution with respect to the o-axis, or, in the case of a homogeneous body, geometric symmetry): the o-axis of symmetry is the principal axis. At the same time, $\vec{\omega}$ and $\vec{L}$ are eigenvectors of the operator $\hat{J}$, the value J_0 in the relation $\vec{L} = J_0\vec{\omega}$ is the eigenvalue of the operator. All directions in the plane perpendicular to the o-axis are equivalent from the point of view of the symmetry of the problem.

5) Partial conclusion: The vector space $T_O\mathbf{R}^3$ is a direct sum of two vector subspaces (always after adding a zero vector, which is not an eigenvector by definition): $\mathcal{L}_1$ one-dimensional, generated by the direction vector of the o-axis, $\mathcal{L}_2$ two-dimensional, generated by two arbitrary independent vectors perpendicular to the o-axis.

The previous steps lead to the full connection of the physical problem with the effective tools of linear algebra. The physical problem has been transformed into an essentially routine task by the above analysis. Assuming that the analysis has been carried out in this way, the selection of the correct answers is already very simple. We provide brief comments:

a) Correct answer. The operator has two different eigenvalues, a single one J_1 (corresponding to the vector subspace of eigenvectors $\mathcal{L}_1$) and a twofold one J_2 (corresponding to the vector subspace of eigenvectors $\mathcal{L}_2$).

b) Correct answer. Its marking as correct is a direct consequence of the analysis in the five steps above.

c) Wrong answer. Vectors in the direction of the o-axis are eigenvectors (they form a one-dimensional vector subspace after adding the zero vector). However, there are other eigenvectors. The student who marked the answer as correct may not have realized the importance of the word "exactly".

d) Correct answer. Perhaps somewhat misleading in connection with the answer to c), the student who marked the answer to c) as correct, by means of the formulation of the answer to d) can realize his mistake and return to the answer to c) and correct it. Even such a situation indicates after all, an understanding of the issue.

e) Correct answer. See step 5) above.

f) This correct answer is based on an understanding the connection between the principal axes and eigenvectors of a linear operator $\hat{J}$. 5

g) Trivially wrong answer. Marking it as correct is evidence of a misunderstanding of the issue of eigenvectors and eigenvalues â€‹â€‹of a linear operator as such. However, if the student at least partially understood the connection of the linear relationship between angular momentum and angular velocity with the fact that a flywheel always has principal axes, this may help him/her to eliminate the answer g) and also improve his level of understanding of the issue.

h) Correct answer. It follows quite simply from step 3).

5F Evaluate the physical consequences based on the mathematical properties of concepts

Studying the rotational motion of a rigid body we work in orthonormal bases. Let us assume that the origin of the coordinate system O lies at the center of mass of the body (the zero vector is located in it). The linear relationship between the angular momentum $\vec{L}$ of a rigid body (flywheel) and its angular velocity $\vec{\omega}$ in the vector space $T_O\mathbf{R}^3$ is realized by the linear operator $\hat{J}$, the so-called inertia tensor. In the orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, the angular momentum and angular velocity have the components $\vec{L} \sim (L_1 \quad L_2 \quad L_3)$ and $\vec{\omega} \sim (\omega_1 \quad \omega_2 \quad \omega_3)$ in matrix notation. The operator $\hat{J}$ is represented in the given basis by the symmetric matrix $\hat{J} \sim \tilde{J} = (J_{ij})$, $i, j = 1, 2, 3$. Therefore, $(L) = (\omega)\tilde{J}$. Suppose that in the given specific case the operator $\hat{J}$ has two identical eigenvalues ﬃﬃ$J_1 = J_2$, let us denote them by J_0. We have no information about the third value. Besides, let's assume that in the considered vector space there are non-zero vectors that are not eigenvectors of the operator $\hat{J}$. Based on these facts formulate the physical consequences concerning the motion of the body, especially the principal axes of its inertia. Based on these considerations, choose just all true statements from the following. (Speaking about the principal axes of the inertia, we mean the axes passing through the center of mass of the body.)

a) The assignment is incomplete. Without knowing the third eigenvalue of the operator $\hat{J}$ no conclusions can be made regarding the motion of the body.

b) The body has (at least) infinitely many principal axes of the inertia, such that their direction vectors are all eigenvectors corresponding to the eigenvalue J_0.

c) If the body rotates around an axis whose direction vector is perpendicular to the vector subspace generated by the eigenvectors corresponding to the eigenvalue J_0, its angular momentum will be $\vec{L} = J_0\vec{\omega}$.

d) If the body is homogeneous, then it exhibits rotational symmetry with respect to an axis perpendicular to the plane determined by the eigenvectors corresponding to the eigenvalue J_0.

e) It is not possible to decide whether the third eigenvalue of the operator $\hat{J}$ is the same as or different from the value J_0.

f) The body has spherical symmetry of the mass distribution.

g) The axis (passing through the center of mass of the body) the direction vector of which is perpendicular to the vector subspace generated by the eigenvectors of the operator $\hat{J}$ is the principal axis of the inertia of the body.

h) If a body rotates around an axis passing through its center of mass and having the direction of any eigenvector of the operator $\hat{J}$ corresponding to the eigenvalue J_0, then it holds $\vec{L} = J_0\vec{\omega}$, i.e. the axis of rotation is the principal axis of the inertia of the body.

Comments: The assignment is somewhat longer to achieve the precise formulations, e.g. to distinguish the concepts of "axis"as a straight line, versus "direction vector of the axis", etc. The assignment suggests to the student that physical conclusions result from the connection of physical laws with their mathematical expression/representation, which is the problem of eigenvalues â€‹â€‹and eigenvectors of a symmetric linear operator. Therefore, the student should summarize the basic knowledge from the relevant area of â€‹â€‹linear algebra for the purpose to solve the problem. Let us present them again in successive steps as in the previous problem of level 4F, which was very similar in content, but required the reverse process of reasoning. While task 4F proceeds from physics to mathematics (requiring "to analyze the mathematical aspects of a physical case), task 5F proceeds in the opposite direction (requiring "to formulate the physical consequences based on the mathematical properties of the concepts). So, the task 5F is more difficult. Gradual considerations leading to the solution:

1) A symmetric linear operator is represented in the basis of its eigenvectors by the diagonal matrix with the eigenvalues â€‹â€‹on the diagonal. Such a basis $(\vec{\varepsilon}_1, \vec{\varepsilon}_2, \vec{\varepsilon}_3)$ always exists. In it, the relationship between the components of angular momentum and angular velocity has the form $L_1 = J_1\vec{\omega}_1$, $L_2 = J_2\omega_2$, $L_3 = J_3\omega_3$. (For simplicity, we do not choose a different designation for the components than in the task specification, perhaps there is no risk of collision.)

2) With each eigenvalue â€‹â€‹the vector subspace of eigenvectors is connected (after the addition of the zero vector). The dimension of each such subspace equals to the multiplicity of the corresponding eigenvalue as the characteristic root of the operator. The value $J_0 = J_1 = J_2$ is at least twofold, as specified in the assignment.

3) If the value of J_3 was equal to J_0, the vector space generated by the eigenvectors of the threefold eigenvalue J_0 would be three-dimensional, i.e. it would be the entire space $T_O\mathbf{R}^3$. However, this is in contradiction to the information that in $T_O\mathbf{R}^3$ there are non-zero vectors that are not eigenvectors of the operator. Thus, $J_3 \neq J_0$.

4) The above considerations lead to the following mathematical conclusions: The operator $\hat{J}$ has one twofold eigenvalue J_0, corresponding to the two-dimensional vector subspace of eigenvectors $\mathcal{L}_0$ (after adding the zero vector), and a single eigenvalue $J_3 \neq J_0$, to which belongs (again after adding the zero vector) the one-dimensional vector subspace of eigenvectors $\mathcal{L}_3$ orthogonal to $\mathcal{L}_0$.

5) Physical consequences of considerations 1) to 4): If the direction vector of the rotation axis is the eigenvector corresponding to the eigenvalue J_0, then it holds $\vec{L} = J_0\vec{\omega}$. So the axis is the principal axis of the inertia (this follows from the physical definition of the principal axis). Similar conclusion is valid for the case when the direction vector of the axis is the eigenvector corresponding to the value J_3.

6) Conclusion: In the given case, the body has infinitely many principal axes (going through the center of mass) forming a plane (their orientation is determined by the vector subspace $\mathcal{L}_0$), and one principal axis perpendicular to this plane. This distribution of principal axes must correspond to the symmetry of the mass distribution of the body, which for a homogeneous body is identical to its geometric symmetry.

Deciding on the truth of the offered statements is now easy:

a) Wrong answer. It results from the fact that there are non-zero vectors that are not eigenvectors of the operator $\hat{J}$ that the third eigenvalue J_3 differs from J_0. This is sufficient for the answer, even though J_3 is not given concretely.

b) Correct answer. The word "at least"is important here. There are the axes corresponding to the eigenvalue J_0.

c) Wrong answer. In the described case, $\vec{L} = J_3\vec{\omega}$, $J_3 \neq J_0$.

d) Correct answer. (It would also be correct in the case $J_3 = J_0$.)

e) Wrong answer. The equality $J_3 = J_0$ is excluded by the statement in the assignment about the existence of non-zero vectors that are not eigenvectors of the operator $\hat{J}$.

f) Wrong answer. In the case of spherical symmetry of the mass distribution all lines going through the center of mass would represent equivalent axes of rotation and their direction vectors would be eigenvectors of the operator $\hat{J}$.

g) Correct answer. The direction of the axis corresponds to the orientation of vectors lying in the vector subspace $\mathcal{L}_3$.

h) Correct answer. See arguments above.

6F Discover a new connection of physics with mathematics

The motion of a system of mass points (a body) is described in an inertial frame of reference S, to which an orthonormal basis ($\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ is connected. For the rotational motion of a body, the second momentum theorem holds

$$\frac{\mathrm{d}\vec{L}}{\mathrm{d}t} = \vec{M}_{\mathrm{ext}}$$

where $\vec{L}$ is the angular momentum of the body relative to a pre-selected reference point O (e.g. the origin of the coordinate system) and $\vec{M}_{\text{ext}}$ is the resulting torque of external forces relative to this point. Let us assume that the body is rigid (flywheel). Its rotational motion as a whole is then described by a single kinematic quantity, common to all its particles, the (instantaneous) angular velocity $\vec{\omega}$. The relationship between angular momentum and angular velocity is linear, it is mediated by the linear operator $\hat{J}$, which is represented in the given orthonormal basis by the symmetric matrix $\tilde{J} = (J_{ij}),\ i,\, j = 1, 2, 3$. For components in the matrix notation $\vec{L} \sim (L) = (L_1 \quad L_2 \quad L_3)$, $\vec{\omega} \sim (\omega) = (\omega_1 \quad \omega_2 \quad \omega_3)$, it holds $(L) = (\omega)\tilde{J}$, i.e. $L_i = \sum_{j=1}^{3} J_{ij}\omega_j$ for $i,\, j = 1, 2, 3$. Since the flywheel is rigid, the elements of the inertia operator with respect to an orthonormal basis rigidly connected to the flywheel do not depend on time, i.e., with respect to an observer connected to such a basis, the flywheel does not move. The corresponding reference frame S' is, of course, non-inertial. The flywheel obtains the given angular velocity $\vec{\omega}$ with respect to the frame S (around a certain instantaneous axis going through the center of mass). Assume that in this frame S the body does not perform the translational motion. Moreover, suppose that the axes around which the flywheel can rotate pass through its center of mass. Next, we consider the inertial frame S.

From a mathematical point of view, consider the eigenvalues â€‹â€‹and eigenvectors of the operator $\hat{J}$ and look for the context with physics in following situations.

a) Assume that the resulting torque of external forces acting on the flywheel is zero. What holds for the angular momentum? Is the same valid for angular velocity? If so, is it valid always or only under certain conditions? Is the angular velocity $\vec{\omega}$ parallel to the angular momentum? Always? Or, because of the relation $(L) = (\omega)\tilde{J}$, the relation $\vec{L} \parallel \vec{\omega}$ holds only under specific conditions?

b) What is the connection between the motion of a flywheel governed by the second momentum law and the concept of a fixed axis of rotation? (A fixed axis is a straight line that is immobile with respect to a given inertial frame.) Consider whether, and under what conditions, the axis of rotation can be fixed at zero torque of external forces.

c) Discover the connection of the linear relationship between angular momentum and angular velocity with the problem of eigenvalues â€‹â€‹and eigenvectors of the inertia operator $\hat{J}$.

d) Discover the meaning of the eigenvectors of the operator $\hat{J}$ in terms of the motion of the flywheel and its connection with the symmetry of the flywheel's mass distribution (in the case of a homogeneous body, this is geometric symmetry.)

e) The principal axes of a flywheel are sometimes called free. Can you justify this terminology?

Comments: This is an open task, the solution of which is not easy for first-year students. The task in the assignment is formulated in general terms, does not draw attention to the main axes of the inertia. The note concerning the symmetry of the flywheel may perhaps be helpful for a student with a certain physical intuition. An important aid in the assignment is the explicit reference to the connection of the linear relationship between the angular momentum and the angular velocity through the symmetric operator $\hat{J}$ with the problem of eigenvalues â€‹â€‹and eigenvectors of $\hat{J}$. (On the other hand, even without this reference, the mentioned connection could/should be very logical for a more advanced student: if an operator is suitable for describing a physical relationship, one can certainly expect physical impacts of its mathematical properties. Through this problem, the student can also realize that the constant tensor of inertia, i.e. with respect to the basis associated with the body, is only a property of the distribution of the body's mass, independent of its motion.)

a) This task is both physical (the second momentum theorem) and geometric (the relationship between vectors $\vec{L}$ and $\vec{\omega}$). Solving this problem leads to the understanding of the meaning of the term *fixed axis*. The student will realize that in the case of zero torque the second momentum theorem implies the law of conservation of angular momentum (with respect to the inertial reference frame). The student will be helped by pointing out the fact that the components of the moment of inertia of a rigid body are constant with respect to the basis fixed in the body, but not with respect to the basis connected to the inertial reference frame in general. This leads to the conclusion that even with constant angular momentum, the angular velocity need not be constant. The general linear relationship then leads to the result that the angular velocity is not generally parallel to the angular momentum since $(\omega) = (L)\tilde{J}^{-1}$. The vector $\vec{\omega}$ therefore has a generally different direction at any instant than is the fixed direction determined by the angular momentum. The angular velocity is therefore parallel to the angular momentum only if there exists a value of J_0 such that $\vec{L} = J_0\vec{\omega}$. In such a case, $\vec{\omega}$ will also be constant. (This conclusion turns attention to the connection of physical problem with the problem of eigenvalues and eigenvectors vectors of the operator $\hat{J}$.)

b) The solution to problem a) is related to the concept of a fixed axis. The instantaneous axis has the direction of the instantaneous angular velocity of the flywheel. If it is to be fixed, the direction of the angular velocity must be constant. With zero torque, when the angular momentum is conserved and taking into account the relation $(L) = (\omega)\tilde{J}$, i.e. $(\omega) = (L)\tilde{J}^{-1}$, this is possible only in situations where $\vec{L} = J_0\vec{\omega}$ for a certain value of J_0 (see a)). With zero resultant torque, only the axis in the direction of the conserved angular momentum can be fixed. (If the axis of rotation is to be fixed even in a general situation, the resultant torque cannot be zero.)

c) The connection between the angular momentum and the angular velocity of the flywheel with the problem of eigenvalues â€‹â€‹and eigenvectors of the linear operator $\hat{J}$ is already obvious. If $(L) = (\omega)\tilde{J}$ and $(L) = J_0(\omega)$ hold simultaneously, then by comparison we get

$$(\omega)\tilde{J} = J_0(\omega) \implies (\omega)(\tilde{J} - J_0 E) = (0),$$

where E is the unit matrix and (0) is the zero matrix. The angular velocity vector determining the direction of the principal axis is the eigenvector of the inertia operator corresponding to the eigenvalue J_0. Thus, by solving the problem of eigenvalues â€‹â€‹and eigenvectors of the operator, we obtain the directions of all principal axes.

d) In the case of a certain symmetry of the flywheel's mass distribution, some directions of the principal axes may be equivalent. The basic types of symmetries that make sense to consider in this context are 1) spherical – all directions in $T_O\mathbf{R}^3$ are equivalent, 2) cylindrical (rotational) – all directions lying in a plane perpendicular to the axis of symmetry are equivalent, 3) none of the previous two.

e) Types 1), 2) and 3) in the item d) are related to the number of different eigenvalues â€‹â€‹of the operator $\hat{J}$ as follows: ad 1) a threefold eigenvalue J_0 of the operator $\hat{J}$, ad 2) one twofold eigenvalue, e.g. $J_1 = J_2$ and one different single eigenvalue J_3. In all orthonormal bases connected with its eigenvectors the operator $\hat{J}$ is represented by the diagonal matrix Ëś$\tilde{J}$, with the eigenvalues in the diagonal, i.e. $\tilde{J} = \operatorname{diag}(J_1,\, J_2,\, J_3)$.

   In case 1) $J_1 = J_2 = J_3 = J_0$ the matrix $(\tilde{J} - J_0 E)$ is zero in bases of eigenvectors of $\hat{J}$ and the eigenvectors generate the entire space $T_O\mathbf{R}^3$. All axes passing through the center of mass of the flywheel are principal ones.

   In case 2) it holds $J_1 = J_2 \neq J_3$. Let $J_0 = J_1 = J_2$. The matrix $(\tilde{J} - J_0 E)$ has rank 1. Then, adding the zero vector, it corresponds to the two-dimensional vector subspace $\mathcal{L}_0$, and each axis lying in the plane, whose orientation is determined by this vector subspace, is the principal axis. The matrix $(\tilde{J} - J_3 E)$ has rank 2. Then, again adding the zero vector, it corresponds to the one-dimensional vector subspace $\mathcal{L}_3$ orthogonal (perpendicular) to $\mathcal{L}_0$. The axis of rotation, parallel to this subspace, is the principal axis. It is the axis of rotational symmetry of the flywheel.

   In case 3), when the eigenvalues J_i, $i = 1, 2, 3$, â€‹â€‹of the operator $\hat{J}$ are different, there are the corresponding one-dimensional vector subspaces $\mathcal{L}_i$ of eigenvectors (after adding the zero vector, of course). The flywheel has three different mutually perpendicular principal axes.

   Note: Since the physical meaning of the concept of principal axis is closely related to the problem of eigenvalues â€‹â€‹and eigenvectors of the inertia operator, as was shown solving tasks a) and b), the flywheel cannot have other principal axes than those corresponding to the symmetry of the mass distribution in cases 1), 2) and 3). Of course, the mass distribution can also have symmetries other than 1) and 2) (for example, the mirror symmetry with respect to a plane), but no principal axes will be associated with them.

f) If the body rotates with an angular velocity directed along a principal axis and the resulting torque is zero (idealization), then the angular momentum is constant and it have the same direction as the angular velocity, $\vec{L} = J\vec{\omega}$, where J is the corresponding eigenvalue of the operator $\hat{J}$. The direction of the rotational axis will not change and the flywheel will rotate "freely"around it (in accordance with the "rotational part"of Newton's first law (see e.g. [16]).

5. Test results and their evaluation

The test was solved by 24 students who have successfully completed six semesters of physics studies in the study year 2024/2025 and are prepared for the bachelor's state exam. The test solving time was 50 minutes. The test contains a total of six F tasks with difficulty graded according to revised Bloom's taxonomy, i.e. tasks 1F-6F. These were tasks of the following types (see Sec. 4.):

- one closed task with one correct answer (task 1F),
- three closed tasks with multiple correct answers (tasks 2F, 4F, 5F),
- two open tasks (tasks 3F, 6F).

Methods of evaluation

There are various methods of evaluating tests, differing especially in the case of tasks with multiple correct answers.

- In our test, every task with just one correct answer contains 5 answers. The student receives 1 point if he/she choose the correct answer.
- For evaluating tasks with multiple correct answers we used here the so-called penalty method. Each task of this type in our test contains 8 answers, the number of correct answers can range from zero to eight. The student does not know this number in advance. The student receives one point for each marked correct answer and one point for each unmarked incorrect answer. In the case of marking an incorrect answer and not marking the correct answer, the student always loses a quarter of a point (penalty). The point scale therefore lies in the interval from minus 2 to 8 points. The result (normalized to the interval $x \in [0, 1]$ can be obtained by the transformation $x = \frac{y-a}{b-a}$, where $y \in [a, b]$. (For more details, see e.g. [17].)
- One possible, but overly strict, method of evaluation is the *all-or-nothing method*. When using it, a student receives one point (or the maximum of the used point scale – 8 points in our case) if he marks all correct answers and does not mark any incorrect answers. The student's success in this assessment indicates his/her deep understanding of the issue.

Results and conclusions of their evaluation

- Open tasks (3F, 6F) could not be evaluated. In the case of task 3F, only isolated attempts to include a solution were made, and no student attempted to solve task 6F.
- The closed task 2F was solved by 21 out of 24 students (3 students did not mark any answer). Only one student was successful using the all-or-nothing method.
- The closed task 4F was solved by 18 out of 24 students (8 did not mark any answer). No one of them succeeded using the all-or-nothing method.
- The closed task 5F was solved by 15 students out of 24 (9 did not mark any answer). One student was successful using the evaluation method all-or-nothing method.

The following graphs with comments show the distribution of responses and the total score for each task with the use the penalty method..

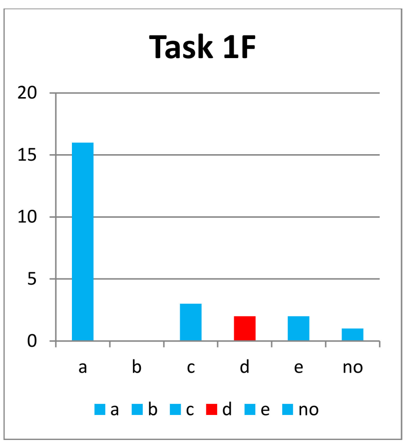

FIG. 3. Task 1F, distribution of answers a)-e). The correct answer is marked by the red colour. **Comment on task 1F:** Only two students marked the correct answer d). The reason for the high frequency of marked incorrect answer a) (76%) can be seen in the fact that students did not think about all the assumptions of the definition. In the case a), they did not take into account that the condition stated in it is also fulfilled by the zero vector, which, however, is not an eigenvector by definition.

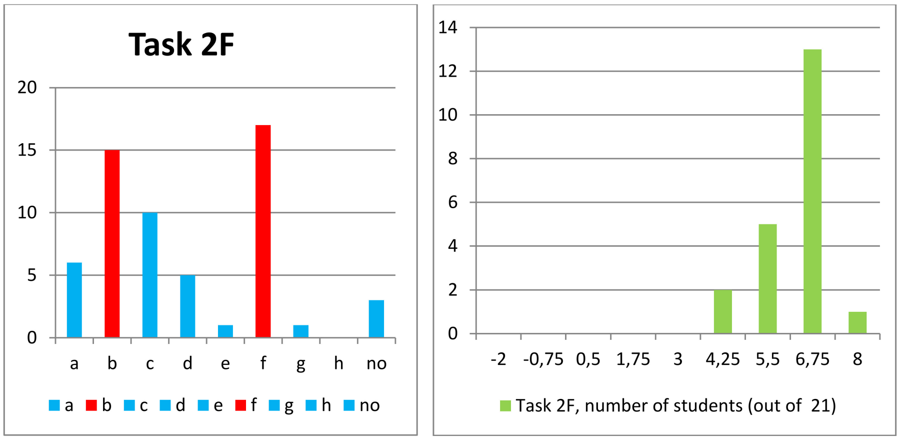

FIG. 4. Task 2F, distribution of answers a)-h) and evaluation of results. The correct answers are marked by the red colour.

Comment on task 2F: The weighted average is $\Phi = 6.3$. The highest frequency corresponds to just two correct answers, b) (71%) and f) (81%). This result indicates an understanding of the issue of eigenvectors of a linear operator: a simple spectrum $\Rightarrow$ independent one-dimensional vector subspaces (after adding the zero vector) corresponding to different eigenvalues $\Rightarrow$ eigenvectors generate the entire space, and the connection with the principal axes of the inertia with the subspaces of eigenvectors. The relatively significant frequency of marking the incorrect statement c) as correct (71%) may not be caused by lack of understanding of the issue of operator eigenvectors and eigenvalues, but by a confusion of the concepts of *vector space* and *vector*.

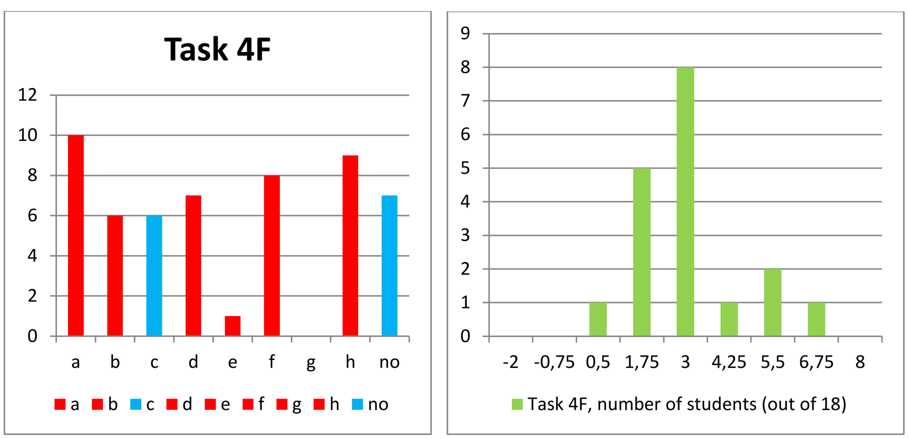

FIG. 5. Task 4F, distribution of answers a)-h) and evaluation of results. The correct answers are marked by the red colour.

Comment on task 4F: The weighted average is $\Phi = 3.1$. The frequency of marking the correct answer a) (55%) indicates (only) an averagely good understanding of the connection between the symmetry of the body mass distribution and the spectrum of the inertia operator. Average understanding of the connection between the eigenvectors of the operator and the principal axes of the inertia is documented by the frequencies of marking the correct answers b) (33%), d) (39%) and f) (44%). Only the answer h) has a somewhat higher frequency (50%), possibly because the connection between the principal axes and the eigenvectors of the inertia operator is explicitly stated in it. The fact that no one considered the incorrect answer g) to be correct can be attributed to physical experience. Students know that there are always principal axes of inertia in a body and can relate their existence to the diagonal form of the operator. We did not expect the relatively high frequency (33%) of answer c) (33%), possibly due to the fact that students did not pay attention to the word "just".

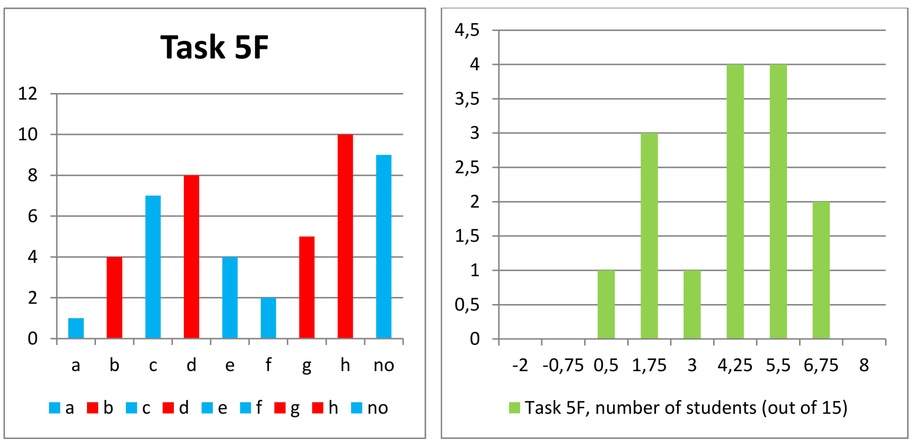

FIG. 6. Task 5F, distribution of answers a)-h) and evaluation of results. The correct answers are marked by the red colour.

Comment on task 5F: The weighted average is $\Phi = 4.1$. Considering that 9 out of 24 students did not attempt to solve the problem at all, more general conclusions can be drawn based on the evaluation of fifteen tests. The correct answers d) (53%) and h) (67%) were generally successful, which demonstrated a relatively good, but rather intuitive, understanding of the connection between the eigenvectors of the inertia operator and the principal axes. The relatively high frequency of marking the incorrect answer c) as correct (47%) may be due to the fact that the students did not realize the significance of the eigenvalue J_0 of the inertia operator specified in the task assignment.

Comment on tasks 3F and 6F: The failure of students (of 6th semester of their study) to solve these open tasks may seem somewhat alarming. It can be seen mainly in two reasons: 1) insufficient practice of knowledge acquired during studies and forgetting (linear algebra is taught in the first two semesters), and 2) especially insufficient linking of mathematics (in our case, linear algebra) and physics (in our case, mechanics) teaching.

Conclusion resulting from the test evaluation: Even though the group of students who solved the test tasks is not too large for detailed statistical analysis, certain relevant conclusions can still be drawn. The students' success is quite satisfactory where they can apply essentially separate simple knowledge and skills from linear algebra, or experience from teaching physics. However, the desired connection of the two disciplines has not been proven to a sufficient extent. This conclusion supports the view that the teaching of linear algebra (and mathematics in general) and the teaching of physics subjects are not consistently and purposefully connected. It would be considered satisfactory (of course, in addition to the ability to solve open problems independently) if the majority of students were successful in solving closed problems with multiple correct answers when assessed using the "all or nothing"method. Only in this way would the desired connection between mathematics and physics teaching and a deeper understanding of physical laws based on a solid mathematical foundation be demonstrated.

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