# Integral on $\mathbb{R}^n$ and differential manifolds

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# $k$-tensors on $V_n$, vector structure on sets of $k$-tensors

**DEFINITION:** $k$-tensor (covariant $k$-order tensor)

Let $V_n$ be an $n$-dimensional vector space over $\mathbb{R}$. A mapping

$$u : V_n \times \cdots \times V_n \ni (a_1, \cdots, a_k) \longrightarrow \mathbb{R}$$

is called $k$-tensor on $V_n$ if it is linear in every of its vector arguments,

$$u(a_1, \ldots, \alpha a_j + \beta b_j, \ldots, a_n) =$$

$$= \alpha u(a_1, \ldots, a_j, \ldots, a_n) + \beta u(a_1, \ldots, b_j, \ldots, a_n)$$

for every $j = 1, \ldots, k$, $a_1, \ldots, a_j, b_j, \ldots, a_n \in V_n$, $\alpha, \beta \in \mathbb{R}$. Denote as $V_n^*$ the set of all 1-tensors on $V_n$ and $\mathcal{T}_k(V_n)$ the set of all $k$-tensors on $V_n$.

We introduce a vector structure on sets of k-tensors by defining operations the sum of two $k$-tensors and the product of a $k$-tensor and a real number (scalar).

**PROPOSITION:** vector operations on sets of tensors

Let $u, v \in \mathcal{T}_k(V_n)$, $\alpha \in \mathbb{R}$. Then mappings

$$\text{(i)} \quad w : V_n \times \cdots \times V_n \ni (a_1, \dots, a_k) \longrightarrow w(a_1, \dots, a_k) \in \mathbb{R},$$

$$\text{(ii)} \quad z : V_n \times \cdots \times V_n \ni (a_1, \dots, a_k) \longrightarrow z(a_1, \dots, a_k) \in \mathbb{R},$$

$$w(a_1, \dots, a_k) = u(a_1, \dots, a_k) + v(a_1, \dots, a_k),$$

$$z(a_1, \dots, a_k) = \alpha u(a_1, \dots, a_k), \quad \forall a_1, \dots, a_n,$$

are $k$-tensors. Denote $w = u + v$, $z = \alpha u$.

(Proof of the proposition: exercise.)

**THEOREM:**

The set of all k-tensors together with operations "$+$" and multiplication by scalars is a vector space over $\mathbb{R}$ of dimension $n^k$.

(Proof: exercise, except for the assertion concerning dimension.)

**EXAMPLE:** dual space, dual base

$(e_1, \ldots, e_n)$ ... base in $V_n$. Define $e^1, \ldots, e^n \in V_n^*$: $e^i(e_j) = \delta^i_j$, $i, j = 1, \ldots, n$. The family $(e^1, \ldots, e^n)$ is a base in $V_n^*$, so $\dim V_n^* = n$.

(Proof: exercise.)

**TERMINOLOGY:** $(e^1, \ldots, e^n)$ ... dual base induced by $(e_1, \ldots, e_n)$, $V_n^*$ ... dual space to $V_n$

## .EXERCISE:

1) Let $T$ be the transition matrix from the base $(e_1, \dots, e_n)$ to the base $(\bar{e}_1, \dots, \bar{e}_n)$ in $V_n$. Derive the transformation relations for components of 1-tensors in corresponding induced bases.
2) Let $(e_1, \dots, e_n)$ be a base in $V_n$, $(u_1, \dots, u_n)$, $(v_1, \dots, v_n)$ components of 1-tensors $u$, $v$ in the induced dual base. Derive the relations for components of 1-tensors $w = u + v$, $z = \alpha u$.
3) Let $T$ be the transition matrix from the base $(e_1, \dots, e_n)$ to the base $(\bar{e}_1, \dots, \bar{e}_n)$ in $V_n$. Derive the transformation relations between the corresponding dual bases.
4) Prove all assertions denoted as exercises in the previous text.

NOTE: Proofs concerning bases in vector spaces have two steps: a) proof of linear independence and b) proof of completeness.

# Tensor and wedge product of tensors

**PROPOSITION:** tensor product (Proof: exercise.)

Let $u \in \mathcal{T}_k(V_n)$, $v \in \mathcal{T}_l(V_n)$. The mapping

$$w : V_n \times \cdots \times V_n \ni (a_1, \ldots, a_{k+l}) \longrightarrow w(a_1, \ldots, a_{k+l}) \in \mathbb{R},$$

$$w(a_1, \ldots, a_{k+l}) = u(a_1, \ldots, a_k) v(a_{k+1}, \ldots, a_{k+l}), \quad \forall a_i \in V_n,$$

is the $(k+l)$-tensor, $w = u \otimes v$ ... tensor product of $u$ and $v$.

**PROPOSITION:** properties of tensor product (Proof: exercise.)

$u, u_1, u_2 \in \mathcal{T}_k(V_n)$, $v, u_1, u_2 \in \mathcal{T}_l(V_n)$, $w \in \mathcal{T}_m(V_n)$, $\alpha, \beta \in \mathbb{R}$

1) $u \otimes (\alpha v_1 + \beta v_2) = \alpha(u \otimes v_1) + \beta(u \otimes v_2)$
2) $(\alpha u_1 + \beta u_2) \otimes v = \alpha(u_1 \otimes v) + \beta(u_2 \otimes v)$
3) $(u \otimes v) \otimes w = u \otimes (v \otimes w)$

WARNING: No commutativity.

**PROPOSITION:** induced bases in $\mathcal{T}_k(V_n)$

Let $(e_1, \dots, e_n)$ be a base in $V_n$. Then $(e^{i_1} \otimes \dots \otimes e^{i_k})$, $1 \leq i_1, \dots, i_k \leq n$, is a base in $\mathcal{T}_k(V_n)$.

**TERMINOLOGY:** $(e^{i_1} \otimes \dots \otimes e^{i_k})$, $1 \leq i_1, \dots, i_k \leq n$, ... induced base by $(e_1, \dots, e_n)$

**EXAMPLES:** induced bases for $n = 2$, $k = 2$, $k = 3$ (Einstein summation)

$$(e^1 \otimes e^1, e^1 \otimes e^2, e^2 \otimes e^1, e^2 \otimes e^2), \quad u = u_{ij} e^i \otimes e^j, \quad u_{ij} = u(e_i, e_j)$$

$$(e^1 \otimes e^1 \otimes e^1, e^1 \otimes e^1 \otimes e^2, e^1 \otimes e^2 \otimes e^1, e^1 \otimes e^2 \otimes e^2,$$

$$e^2 \otimes e^1 \otimes e^1, e^2 \otimes e^1 \otimes e^2, e^2 \otimes e^2 \otimes e^1, e^2 \otimes e^2 \otimes e^2),$$

$$u = u_{ijl} e^i \otimes e^j \otimes e^l, \quad u_{ijl} = u(e_i, e_j, e_l)$$

**DEFINITION:** completely antisymmetric tensors

A tensor $\eta \in \mathcal{T}_k(V_n)$ is called completely antisymmetric $\eta(a_1, \dots, a_i, \dots, a_j, \dots, a_k) = \eta(a_1, \dots, a_j, \dots, a_i, \dots, a_k)$ for arbitrary vector arguments and arbitrary argument positions.

**THEOREM:**

The set $\Lambda_k(V_n)$, $k \geq 2$, of all completely antisymmetric $k$-tensors is a vector subspace of $\mathcal{T}_k(V_n)$, of dimension $\binom{n}{k}$. (For $k = 1$ denote $\Lambda_1(V_n) = \mathcal{T}_1(V_n)$.)

**PROPOSITION:** (Proof: exercise.)

The mapping (alternation) $\mathrm{Alt} : \mathcal{T}_k(V_n) \ni \eta \longrightarrow \mathrm{Alt}\, \eta \in \Lambda_k(V_n)$,

$$\mathrm{Alt}\, \eta(a_1, \dots, a_k) = \frac{1}{k!} \sum_{\sigma \in P_k} \mathrm{sgn}\, \sigma \cdot \eta(a_{\sigma(1)}, \dots, a_{\sigma(k)})$$

for arbitrary $a_1, \dots, a_k$, is the completely antisymmetric $k$-tensor.

**PROPOSITION:** properties of alternation

$$\eta \in \Lambda_k(V_n) \Longrightarrow \mathrm{Alt}\,\eta = \eta, \quad u \in \mathcal{T}_k(V_n) \Longrightarrow \mathrm{Alt}\,(\mathrm{Alt}\,u = \mathrm{Alt}\,U)$$

**DEFINITION:** wedge product

The mapping

$$\wedge : \Lambda_k(V_n)\times\Lambda_l(V_n) \ni (\omega, \eta) \longrightarrow \omega\wedge\eta = \frac{(k+l)!}{k!l!}\mathrm{Alt}\,\omega\otimes\eta \in \Lambda_{k+l}(V_n),$$

is called a wedge product.

**EXERCISE:**

Prove that $\omega \wedge \eta$ is completely antisymmetric.

**PROPOSITION:** properties of the wedge product

Let $\omega_1, \omega_2 \omega \in \Lambda_k(V_n)$, $\eta_1, \eta_2 \eta \in \Lambda_l(V_n)$, $\chi \in \Lambda_m(V_n)$, $\alpha, \beta \in \mathbb{R}$.
Then

1) $(\alpha\omega) \wedge \eta = \omega \wedge (\alpha\eta) = \alpha(\omega \wedge \eta)$
2) $\omega \wedge (\alpha\eta_1 + \beta\eta_2) = \alpha(\omega \wedge \eta_1) + \beta(\omega \wedge \eta_2)$
3) $(\alpha\omega_1 + \beta\omega_2) \wedge \eta = \alpha(\omega_1 \wedge \eta) + \beta(\omega \wedge \eta_2)$
4) $(\omega \wedge \eta) \wedge \chi = \omega \wedge (\eta \wedge \chi)$
5) $\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$

**EXERCISE:** Prove all properties of the wedge product. Use the definition.

**PROPOSITION:** induced bases in $\Lambda_k(V_n)$

$(e^{i_1} \wedge \ldots \wedge e^{i_k})$, $1 \leq i_1 < \cdots < i_k \leq n$

$$\eta = \sum_{i_1 < \cdots < i_k} \tilde{\eta}_{i_1 \ldots i_k} e^{i_1} \wedge \ldots \wedge e^{i_k} =$$
$$= \eta_{j_1 \ldots j_k} e^{j_1} \wedge \ldots \wedge e^{j_k}, \; j_s \in \{1, \ldots, n\}$$

**EXAMPLE:** induced bases for $k = 2$ and $n = 2$

$$\eta = \tilde{\eta}_{12} e^1 \wedge e^2 = (\eta_{12} e^1 \wedge e^2 + \eta_{21} e^2 \wedge e^1)$$
$$\tilde{\eta}_{12} = \eta_{12} - \eta_{21}, \quad \text{put} \quad \eta_{21} = -\eta_{12} \quad (\text{antisymmetrization})$$

**DEFINITION:** contraction of an antisymmetric tensor by a vector

$$i_\xi : \Lambda_k(V_n) \ni \eta \longrightarrow i_\xi \eta \in \Lambda_{k-1}(V_n)$$
$$i_\xi \eta(a_1, \ldots, a_{k-1}) = \eta(\xi, a_1, \ldots, a_{k-1})$$

**DEFINITION:** volume element

$V_n$ ... a vector space with a scalar product and orientation $\mu$
volume element ... a form $\omega_0 \in \Lambda_n(V_n)$ such that
$\omega_0(e_1, \dots, e_n) = 1$ for every orthonormal base $(e_1, \dots, e_n)$
belonging to $\mu$

**PROPOSITION:**

For a vector space with a given scalar product and orientation there exists the unique volume element.

It holds $\omega_0 = e^1 \wedge \dots \wedge e^n$. (Proof: exercise.)

**EXAMPLE:**

for $\xi_1, \dots, \xi_n$, where $\xi_i = \xi_i^j e_j$:
$\omega_0(\xi_1 m \dots, \xi_n) = \det(\xi_i^j)$. (Proof: exercise.)

## .EXERCISE:

1) The operation of tensor product is not commutative. Explain.
2) Explain the relation for the dimension of $\Lambda_k(V_n)$.
3) Prove all previously mentioned assertions concerning tensor and wedge products.
4) Express components of $u \otimes v$ and $\omega \wedge \eta$ via components of $u$, $v$ and $\omega$, $\eta$, respectively.
5) For a general completely antisymmetric $k$-tensor $\eta$ find the relation between components $\tilde{\eta}_{i_1 \dots i_k}$ and $\eta_{j_1 \dots j_k}$ after the antisymmetrization procedure.
6) Derive transformation relations for components of completely antisymmetric $k$-tensors in various induced bases with help of the transition matrix $T$ between initial bases in $V_n$.

# Vector and tensor fields on $\mathbb{R}^n$, differential $k$-forms

We the following definition we introduce bounded vectors and tensors in $\mathbb{R}^n$, and then vector and tensor fields.

**DEFINITION:** tangent space, vector fields, tensor fields

Tangent space $T_x\mathbb{R}^n$ at $x \in \mathbb{R}^n$ is $n$-dimensional real vector space bounded at $x$. Elements of $T_x\mathbb{R}^n$ are pairs of $n$-tuples $\xi(x) = (x^i, \xi^i)$, $i \in \{1, \dots, n\}$.

Algebraic operations: only for vectors bounded at the same point.

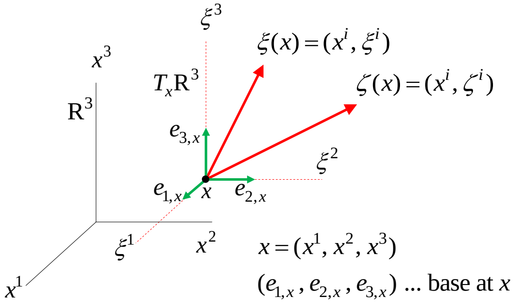
Vector field (continuous, differentiable, smooth, ...) ... a mapping (continuous, differentiable, smooth, ...)

$$\xi : \mathbb{R}^n \longrightarrow \xi(x) \in T_x\mathbb{R}^n$$

Tensor field ... analogously ... a mapping

$$\tau : \mathbb{R}^n \longrightarrow \tau(x) \in \mathcal{T}_k(T_x\mathbb{R}^n)$$

## .**EXAMPLE:** threedimensional situation




$$v(x) = \alpha\xi(x) + \beta\zeta(x), \quad v(x) = (x^i, v^i), \quad v^i = \alpha\xi^i + \beta\zeta^i$$

**NOTATION:**

$$T\mathbb{R}^n = \bigcup_{x\in\mathbb{R}^n} T_x\mathbb{R}^n,\ T_x^*\mathbb{R}^n = \Lambda_1(T_x\mathbb{R}^n),\ \Lambda_k\mathbb{R}^n = \bigcup_{x\in\mathbb{R}^n} \Lambda_k(T_x\mathbb{R}^n)$$

**DEFINITION:** differential $k$-form

differential $k$-form, $k \geq 1$ ... completely antisymmetric differentiable tensor field, i.e. differentiable (up to a given order) mapping $\omega : \mathbb{R}^n \ni x \longrightarrow \omega(x) \in \Lambda_k(T_x\mathbb{R}^n) \subset \Lambda_k\mathbb{R}^n$
differential 0-form ... a function on $\mathbb{R}^n$

**EXAMPLE:** standard bases in $T_x\mathbb{R}^n$, $T_x^*\mathbb{R}^n$

$$e_{i,x} = (x^j, \delta_i^j), \quad e_x^i = (x^j, \delta_j^i), \quad e_{i,x} = \frac{\partial}{\partial x^i}, \quad \xi(x) = \xi^{(}x)\frac{\partial}{\partial x^i}$$

**EXERCISE:** Explain the motivation for the notation $e_{i,x} = \partial/\partial x^i$. (Use the concept of the derivation of a function along a vector.)

# Exterior derivative and pullback of forms

**EXAMPLE:** motivation to exterior derivative ... function

The derivative of a function $f(x)$ at $x \in \mathbb{R}^n$ along a vector $\xi(x) \in T_x\mathbb{R}^n$: $\partial_{\xi(x)} f(x) = \partial_i \xi^i$, $\partial_i = \partial/\partial x^i$, can be interpreted as the value of the 1-form $e^i_x$ at $x$ evaluated on the vector $\xi(x)$.

**DEFINITION:** exterior derivative of a 0-form

exterior derivative of a 0-form ... mapping

$$\mathrm{d} : \Lambda_0\mathbb{R}^n \ni f \longrightarrow \mathrm{d}f \in \Lambda_1\mathbb{R}^n, \quad \mathrm{d}f(x)(\xi(x)) = \partial_i f(x)\xi^i(x)$$

**EXAMPLE:** exterior derivative of coordinate functions

$$x^j : \mathbb{R}^n \ni x \longrightarrow x^j(x) = x^j \in \mathbb{R}, \ \mathrm{d}x^j(x)(e_{i,x}) = \delta^j_i \dots e^j_x \equiv \mathrm{d}x^j$$

$$\eta = \sum_{j_1 < \dots < j_k} \tilde{\eta}_{j_1 \dots j_k} \mathrm{d}x^{j_1} \wedge \dots \wedge \mathrm{d}x^{j_k} = \eta_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k}$$

components $\eta_{i_1 \dots i_k}$ are supposed to be antisymmetrized.

**DEFINITION:** exterior derivative of a $k$-form, $k \geq 1$

exterior derivative . . . a mapping

$$\mathrm{d} : \Lambda_k\mathbb{R}^n \ni \eta \longrightarrow \mathrm{d}\eta \in \Lambda_{k+1}\mathbb{R}^n, \quad \mathrm{d}\eta(x) = \mathrm{d}\eta_{i_1\dots i_k}\wedge\mathrm{d}x^{i_1}\wedge\dots\wedge\mathrm{d}x^{i_k}$$

**PROPOSITION:** properties of exterior derivative (Proof: exercise)
$\alpha, \beta \in \mathbb{R}$, $\eta_1, \eta_2 \in \Lambda_k\mathbb{R}^n$, $\omega \in \Lambda_k\mathbb{R}^n$, $\eta \in \Lambda_l\mathbb{R}^n$ . . . arbitrary,

1) $\mathrm{d}(\alpha\eta_1 + \beta\eta_2) = \alpha\,\mathrm{d}\eta_1 + \beta\,\mathrm{d}\eta_2$,
2) $\mathrm{d}(\omega \wedge \eta) = \mathrm{d}\omega \wedge \eta + (-1)^k\omega \wedge \mathrm{d}\eta$
3) $\mathrm{d}^2 \equiv \mathrm{d} \circ \mathrm{d} = 0$, i.e. $\mathrm{d}(\mathrm{d}\omega) = 0$

**DEFINITION:** closed and exact forms

closed form . . . a form $\omega$ such that $\mathrm{d}\omega = 0$, exact form on $U \subset \mathbb{R}^n$ . . . a form $\omega \in \Lambda_k\mathbb{R}^n$ such that there exists $\eta \in \Lambda_{k-1}\mathbb{R}^n$, $\omega = \mathrm{d}\eta$

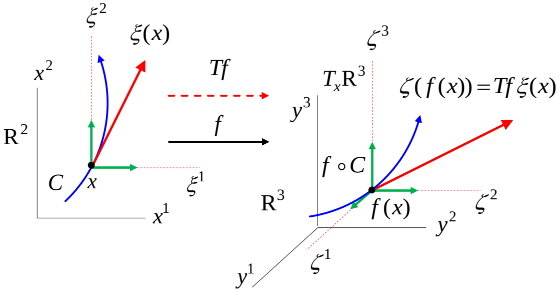
NOTE: closedness $\Longrightarrow$ exactness, not vice versa in general (depends on the set $U$) ... discuss

**DEFINITION:** tangent mapping to $f : \mathbb{R}^n \to \mathbb{R}^m$

$$Tf : T\mathbb{R}^n \ni \xi(x) \longrightarrow \zeta(f(x)) = Tf\xi(x) \in T\mathbb{R}^m$$

$$\zeta(f(x)) = \left(\frac{\partial y^\alpha f(x)}{\partial x^i}\right) \frac{\partial}{\partial y^\alpha}\bigg|_{y=f(x)}, \quad i \in \{1, \dots, n\},\ \alpha \in \{1, \dots, m\}$$

**EXAMPLE:** $n = 2,\ m = 3$



.**EXAMPLE:** matrix expression of tangent mapping

$$\zeta(f(x)) = (y^\alpha f(x), \zeta^\alpha f(x)), \quad (\zeta^1 \ \dots \ \zeta^m)|_{f(x)} = Df(x) \cdot (\xi^1 \ \dots \ \xi^n)|_x$$

$Df(x) \dots$ Jacobi matrix of the mapping $f$ ($Tf$ is $\mathbb{R}$-linear)

**DEFINITION:** pullback of forms

pullback of forms by $f : \mathbb{R}^n \to \mathbb{R}^m$ ... for $k \geq 1$ the mapping

$$f^* : \Lambda_k \mathbb{R}^n \ni \omega \longrightarrow \eta = f^* \omega \in \Lambda_k \mathbb{R}^m, \quad f^* F = F \circ f \text{ for } k = 0$$

$$\eta(\xi_1, \dots, \xi_k)|_x = \omega(\zeta_1, \dots, \zeta_k)|_{f(x)}, \ \ \zeta_j(f(x)) = Tf \xi_j(x)$$

**EXAMPLE:** pullback of coordinate forms

$$f^* \, \mathrm{d}y^\alpha(x)(\xi(x)) = \mathrm{d}y^\alpha(f(x))(Tf\xi(x)) = \zeta^\alpha(f(x)) =$$

$$= \frac{\partial y^\alpha f}{\partial x^i} \xi^i \bigg|_x, \ \xi(x) = \mathrm{d}x^i(\xi(x)) \implies f^* \, \mathrm{d}y^\alpha = \frac{\partial y^\alpha}{\partial x^i} \, \mathrm{d}x^i$$

**PROPOSITION:** properties of pullback (Proof: exercise.)

$\alpha, \beta \in \mathbb{R}$, $\omega_1, \omega_2, \omega \in \Lambda_k\mathbb{R}^m$, $\eta \in \Lambda_l\mathbb{R}^m$ ... arbitrary

1) $f^*(\alpha\omega_1 + \beta\omega_2) = \alpha f^*\omega_1 + \beta f^*\omega_2$
2) $f^*\,\mathrm{d}\omega = \mathrm{d}f^*\omega$
3) $f^*(\omega \wedge \eta) = f^*\omega \wedge f^*\eta$

**EXAMPLE:** Property 2) for coordinate functions

$$f^*y^\alpha(x) = y^\alpha f(x), \;\; \mathrm{d}(f^*y^\alpha) = \frac{\partial y^\alpha f}{\partial x^i}\,\mathrm{d}x^i = f^*\,\mathrm{d}y^\alpha$$

**EXAMPLE:** pullback – general expression in coordinates

$$\omega = \omega_{\alpha_1\dots\alpha_k}\,\mathrm{d}y^{\alpha_1}\wedge\ldots\wedge\mathrm{d}y^{\alpha_k}, \quad f^*\omega = (\omega_{\alpha_1\dots\alpha_k}\circ f)f^*\mathrm{d}y^{\alpha_1}\wedge\ldots\wedge f^*\mathrm{d}y^{\alpha_k} =$$

$$= (\omega_{\alpha_1\dots\alpha_k}\circ f)\frac{\partial y^{\alpha_1}}{\partial x^{i_1}}\cdots\frac{\partial y^{\alpha_k}}{\partial x^{i_k}}\,\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_k}$$

## EXERCISE:

1) Prove properties of the exterior derivative operator and properties od pullback. For the proof of relation $\mathrm{d}^2 = 0$ use the antisymmetry of the wedge product and symmetry of second order partial derivatives.
2) $F$ ... a vector field, $f$,$\Phi$ ... functions on $\mathbb{R}^3$. Denote
$\omega_F^{(1)} = F_1 \,\mathrm{d}x^1 + F_2 \,\mathrm{d}x^2 + F_3 \,\mathrm{d}x^3$
$\omega_F^{(2)} = F_1 \,\mathrm{d}x^2 \wedge \mathrm{d}x^3 + F_2 \,\mathrm{d}x^3 \wedge \mathrm{d}x^1 + F_3 \,\mathrm{d}x^1 \wedge \mathrm{d}x^2$
$\omega_\Phi^{(3)} = \Phi \,\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3$
Prove relations: $\mathrm{d}f = \omega^{(1)}_{\mathrm{grad}\, f}$, $\mathrm{d}\omega_F^{(1)} = \omega^{(2)}_{\mathrm{rot}\, F}$, $\mathrm{d}\omega_F^{(2)} = \omega^{(3)}_{\mathrm{div}\, F}$
3) Using 2) prove that $\mathrm{div}\,\mathrm{rot}\, F = 0$, $\mathrm{div}\,\mathrm{grad}\, f = \Delta f$, $\mathrm{rot}\,\mathrm{grad}\, f = 0$.
4) If $\xi(x)$ is the vector tangent to the curve $\mathcal{C}$ at $x$, prove that $\zeta(f(x)) = Tf\xi(x)$ is tangent to the curve $f \circ \mathcal{C}$ at $f(x)$.

**DEFINITION:** one-parameter group of a vector field

one-parameter group of a vector field $\xi(x)$ ... the family of mappings

1) $\alpha_u : W \ni x \longrightarrow \alpha_u(x) \in \alpha_u(W) \subset \mathbb{R}^n$, $U \subset \mathbb{R}^n \ldots$ open set, $u \in (-\varepsilon, \varepsilon)$
2) $\alpha_{u+v} = \alpha_{v+u} = \alpha_v \circ \alpha_u$, $\alpha_0 = \mathrm{id}_W$, i.e. $\alpha_u^{-1} = \alpha_{-u}$
3) $\alpha : (-\varepsilon, \varepsilon) \times W \ni (u, x) \longrightarrow \alpha(u, x) = \alpha_u(x) \in \alpha_u(W)$
   $\alpha \ldots$ differentiable
4) $\xi^i(x) = \left.\frac{\mathrm{d}x^i\alpha_u(x)}{\mathrm{d}u}\right|_{u=0}$

**EXAMPLE:** integral lines in a plane $(n = 2,\ u \in \mathbb{R})$

$\xi(x) = (1,\ 3x^1)$, $x^1\alpha_u(x) = u + x^1$, $x^2\alpha_u(x) = \frac{3}{2}u^2 + 3ux^1 + x^2$

We know how to describe changes of a function $f$ on $\mathbb{R}^n$ along a given vector field $\xi$: by the derivative of $f$ along $\xi$, $\partial_\xi f = \frac{\partial f}{\partial x^i}\xi^i$. It is in fact the Lie derivative of $f$ with respect to $\xi$. Generalization of this concept to differential forms:

**DEFINITION:** Lie derivative of a differential $k$-form

$$\partial_\xi \omega = \left.\frac{\mathrm{d}\alpha_u^*\omega}{\mathrm{d}u}\right|_{u=0} = \lim_{u\to 0} \frac{\alpha_u^*\omega - \omega}{u}$$

**EXAMPLE:** Lie derivative of a 1-form

$$\omega = \omega_i \,\mathrm{d}x_i, \quad \xi = \xi^j \frac{\partial}{\partial x^j}, \quad \partial_\xi \omega = \left( \omega_j \frac{\partial \xi^j}{\partial x^i} + \xi^j \frac{\partial \omega_j}{\partial x^i} \right) \mathrm{d}x^i$$

**PROPOSITION:** some properties of Lie derivatives

$\omega_1,\, \omega_2,\, \omega \in \Lambda_k\mathbb{R}^n,\ \eta \in \Lambda_l\mathbb{R}^n,\ \alpha,\, \beta \in \mathbb{R}$

1) $\partial_\xi(\omega \wedge \eta) = \partial_\xi\omega \wedge \eta + \omega \wedge \partial_\xi\eta$
2) $\partial_\xi(\alpha\omega_1 + \beta\omega_2) = \alpha\partial_\xi\omega_1 + \beta\partial_\xi\omega_2$
3) $\partial_\xi\omega = i_\xi\,\mathrm{d}\omega + \mathrm{d}i_\xi\omega$
4) $\partial + \xi i_\xi\omega = i_\xi\partial_\xi\omega$
5) $\partial_{f\xi} = f\partial_\xi\omega + \mathrm{d}f \wedge i_\xi\omega$

**EXERCISE:**

1) Derive the expression for components of the Lie derivative of a 1-form $\omega$ with respect to a vector field $\xi$.
2) Derive the above properties of Lie derivatives from the definition in general, or at least for 1-forms – use calculation in components.

# Parametrized pieces and singular cubes in $\mathbb{R}^n$

Concept of integration: we need to define some subsets of $\mathbb{R}^m$ as appropriate domains of integration, while differential forms will be integrated objects (instead of functions)

**DEFINITION:** *n*-dimensional parametrized pieces and singular cubes in $\mathbb{R}^m$

*n*-dimensional parametrized piece of a surface in $\mathbb{R}^m$, resp. *n*-dimensional singular cube in $\mathbb{R}^m$ ... a differentiable mapping

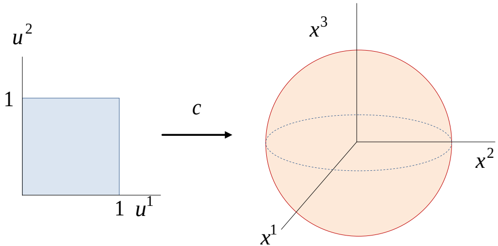
$$c : [0, 1]^n \ni u = (u^1, \dots, u^n) \longrightarrow c(u) = (x^1 c(u), \dots, x^m c(u)) \in \mathbb{R}^m,$$

such that $c$ is one-to-one or obeys the condition $\operatorname{rank} Dc(u) = n$ on $[0, 1]^n$ for parametrized pieces or for singular cubes, respectively

**EXERCISE:** Specify the difference between definitions of an *n*-dimensional parametrized piece of a surface and an *n*-dimensional singular cube in $\mathbb{R}^m$.

## .EXAMPLE:

$$c : [0, 1]^2 \ni u = (u^1, u^2) \longrightarrow c(u) \in \mathbb{R}^3$$



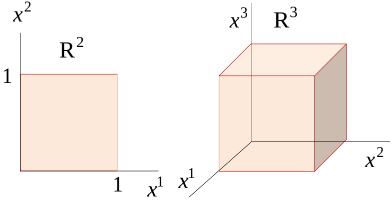

$$x^1 c(u^1, u^2) = \sin(2\pi u^1)\cos(2\pi u^2)$$

$$x^2 c(u^1, u^2) = \sin(2\pi u^1)\sin(2\pi u^2),\ x^3 c(u^1, u^2) = \cos(2\pi u^1)$$

Is this the parametrized piece or the singular cube? What dimension?

A question is: how to describe the boundary of a parametrized piece or a singular cube using the mapping $c$. Is it possible to find a description including orientation?

**EXAMPLE:** standard $n$-dimensional cubes in $\mathbb{R}^n$



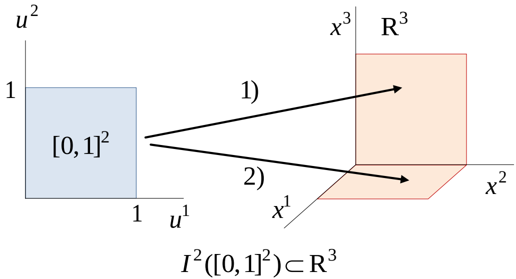

$I^2([0,1]^2) \subset \mathrm{R}^2$ $\quad$ $I^3([0,1]^3) \subset \mathrm{R}^3$

$$x^i I^n(u^1, \ldots, u^n) = u^i, \;\; i \in \{1, \ldots, n\}$$

$$I^2 : [0,1]^2 \ni (u^1, u^2) \longrightarrow (x^1 I^2(u^1, u^2), x^2 I^2(u^1, u^2)) \in \mathbb{R}^2$$
$$I^3 : [0,1]^3 \ni (u^1, u^2, u^3) \longrightarrow (x^i I^3(u^1, u^2, u^3)) \in \mathbb{R}^3, \;\; i \in \{1, 2, 3\}$$

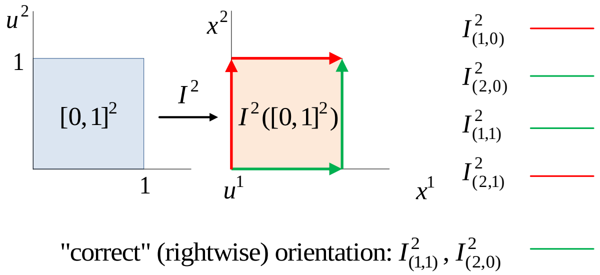
.**EXAMPLE:** standard $n$-dimensional cubes in $\mathbb{R}^m$, $m > n$

on $(m-n)$-coordinate positions are fixed values $\beta = 0$ or $\beta = 1$




1) $I^2 : [0,1]^2 \ni (u^1, u^2) \longrightarrow (\beta, u^1, u^2) \in \mathbb{R}^3$
2) $I^3 : [0,1]^3 \ni (u^1, u^2, u^3) \longrightarrow (u^1, u^2, \beta) \in \mathbb{R}^3$
3) $I^3 : [0,1]^3 \ni (u^1, u^2, u^3) \longrightarrow (u^1, \beta, u^2) \in \mathbb{R}^3$

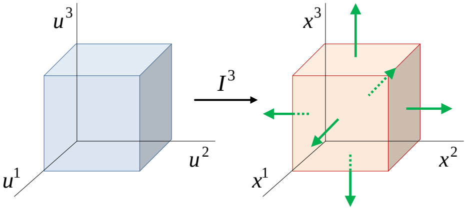
## .EXAMPLE: walls of standard cubes




$$I^2_{(1,0)} : [0,1] \ni t^1 \longrightarrow (x^1 I^2_{(1,0)}(t^1), x^2 I^2_{(1,0)}(t^1)) = (0, t^1) \in \mathbb{R}^2$$
$$I^2_{(1,1)} : [0,1] \ni t^1 \longrightarrow (x^1 I^2_{(1,1)}(t^1), x^2 I^2_{(1,1)}(t^1)) = (1, t^1) \in \mathbb{R}^2$$
$$I^2_{(2,0)} : [0,1] \ni t^1 \longrightarrow (x^1 I^2_{(2,0)}(t^1), x^2 I^2_{(2,0)}(t^1)) = (t^1, 0) \in \mathbb{R}^2$$
$$I^2_{(2,1)} : [0,1] \ni t^1 \longrightarrow (x^1 I^2_{(2,1)}(t^1), x^2 I^2_{(1,0)}(t^1)) = (t^1, 1) \in \mathbb{R}^2$$

## .EXERCISE: walls of standard cubes

Arrows ... "correct"(ext normal) orientation of the cube boundary. Parametrize walls and compare with the correct orientation.




## DEFINITION: walls of standard $n$-dimensional cubes

$(i, \alpha)$-wall of a standard $n$-dimensional cube in $\mathbb{R}^n$, $\alpha = 0$ or 1
$I^n_{i,\alpha} : [0, 1]^{n-1} \ni (t^1, \ldots, t^n) \to (t^1, \ldots, t^{i-1}, \alpha, t^i, \ldots, t^{n-1}) \in \mathbb{R}^n$
for $I^n$ in $\mathbb{R}^m$, $m > n$, positions occupied in $I^n$ by $\beta = 0$ or $\beta = 1$ remain unchanged

Following definitions look rather formally – their meaning will be completely clear later, in relation to the concept of integral.

**DEFINITION:** boundary of an $n$-dimensional singular cube

boundary of a standard $n$-dimensional singular cube in $\mathbb{R}^m$

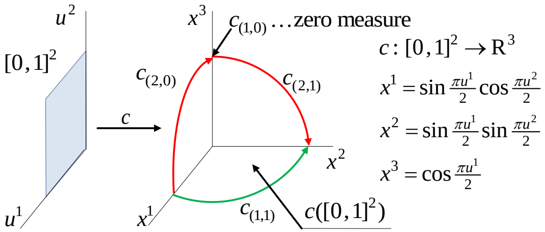
$$\partial I^n = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} I^n_{(i,\alpha)}$$

boundary of an $n$-dimensional singular cube $c$ in $\mathbb{R}^m$

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)}$$

$$c_{(i,\alpha)} = c \circ I^n_{(i,\alpha)} \ldots (i,\alpha) \, \text{wall of the cube} \, c$$

## .EXAMPLE: boundary of a 2-dim parametrized piece in $\mathbb{R}^3$




$$c_{(1,0)}(t^1) = (0,\ 0,\ 1),\ c_{(1,1)}(t^1) = \left(\cos\frac{\pi t^1}{2},\ \sin\frac{\pi t^1}{2},\ 0\right)$$
$$c_{(2,0)}(t^1) = \left(\sin\frac{\pi t^1}{2},\ 0,\ \cos\frac{\pi t^1}{2}\right),\ c_{(2,1)}(t^1) = \left(0,\ \sin\frac{\pi t^1}{2},\ \cos\frac{\pi t^1}{2}\right)$$

**EXERCISE:** Discuss orientation of walls with respect to $i$ and $\alpha$.

# Integral of differential forms

Now we introduce the concept of integral:

- integrated objects ... differential $n$-forms
- integration domains ... $n$-dimensional parametrized pieces or singular cubes in $\mathbb{R}^m$, $m \geq n$

**DEFINITION:** integral of forms

$c : [0, 1]^n \ni u = (u^i)_{i=1,\ldots,n} \longrightarrow c(u) = (x^\alpha c(u^i))_{\alpha=1,\ldots,m} \in \mathbb{R}^m$
$n$-dimensional parametrized piece or singular cube in $\mathbb{R}^m$, $m \geq n$,
$\omega$ ... an $n$-form defined on an open set $A \subset \mathbb{R}^m$, $c([0, 1]^n) \subset A$

integral of $\omega$ on $c$

$$\int_c \omega = \int_{[0,1]^n} c^*\omega = \int_{[0,1]^n} f \, (\mathrm{d}u^1 \ldots \mathrm{d}u^n) \text{ (Riemann integral)}$$

$c^*\omega = f(u^1, \ldots, u^n) \, \mathrm{d}u^1 \wedge \ldots \wedge \mathrm{d}u^n$ ($f$ unique component of $c^*\omega$)

Explanation of the formal expression for the boundary $\partial c$ of $c$

**DEFINITION:** integral on chains

$c_s$, $s \in \{1, \dots, p\}$, ... $n$-dimensional parametrized pieces od singular cubes in $\mathbb{R}^m$

$\omega$ ... an $n$-form defined on an open set $A \subset \mathbb{R}^m$, $c_s([0, 1]^n) \subset A$

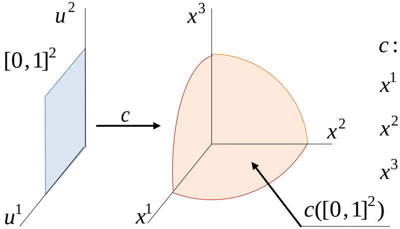
a formal notation $\Gamma = k_1 c_1 + \cdots + k_p c_p$ is understood in the sense of the integral on an $n$-dimensional singular chain,

$$\int_\Gamma \omega = k_1 \int_{c_1} \omega + \cdots + k_p \int_{c_p} \omega$$

The expressions for $\partial c$ and $\partial \Gamma$ are understood in this sense as well.

**EXERCISE:** Prove that $\partial(\partial c) = 0$ and analogously $\partial(\partial \Gamma) = 0$.

## .EXAMPLE: integral of a 2-form on a 2-dimensional domain




$c : [0,1]^2 \to R^3$

$x^1 = \sin \frac{\pi u^1}{2} \cos \frac{\pi u^2}{2}$

$x^2 = \sin \frac{\pi u^1}{2} \sin \frac{\pi u^2}{2}$

$x^3 = \cos \frac{\pi u^1}{2}$

$c \ldots$ parametrized piece of a unit sphere

$$\omega = x^3 \, dx^1 \wedge dx^2, \quad c^*\omega = (c \circ x^3) \, c^* \, dx^1 \wedge c^* \, dx^2$$

.**EXAMPLE:** (continued)

$$\begin{aligned} c^* \, \mathrm{d}x^1 &= \frac{\partial x^1}{\partial u^1} \mathrm{d}u^1 + \frac{\partial x^1}{\partial u^2} \mathrm{d}u^2 = \\ &= \frac{\pi}{2} \cos \frac{\pi u^1}{2} \cos \frac{\pi u^2}{2} \mathrm{d}u^1 + \frac{\pi}{2} \sin \frac{\pi u^1}{2} \sin \frac{\pi u^2}{2} \mathrm{d}u^2 \\ c^* \, \mathrm{d}x^2 &= \frac{\partial x^2}{\partial u^1} \mathrm{d}u^1 + \frac{\partial x^2}{\partial u^2} \mathrm{d}u^2 = \\ &= \frac{\pi}{2} \cos \frac{\pi u^1}{2} \sin \frac{\pi u^2}{2} \mathrm{d}u^1 + \frac{\pi}{2} \sin \frac{\pi u^1}{2} \cos \frac{\pi u^2}{2} \mathrm{d}u^2 \end{aligned}$$

$$\begin{aligned} c^* \, \mathrm{d}x^1 \wedge c^* \, \mathrm{d}x^2 &= \frac{\pi^2}{4} \sin \frac{\pi u^1}{2} \cos \frac{\pi u^1}{2} \mathrm{d}u^1 \wedge \mathrm{d}u^2 \\ (c \circ x^3) \, c^* \, \mathrm{d}x^1 \wedge c^* \, \mathrm{d}x^2 &= \frac{\pi^3}{8} \sin^2 \frac{\pi u^1}{2} \cos^2 \frac{\pi u^1}{2} \mathrm{d}u^1 \wedge \mathrm{d}u^2 \\ \int_c \omega &= \int_{[0,1]^3} \frac{\pi^2}{4} \sin^2 \frac{\pi u^1}{2} \cos^2 \frac{\pi u^1}{2} (\mathrm{d}u^1 \, \mathrm{d}u^2) = \frac{\pi}{6} \end{aligned}$$

# Stokes theorem and its classical versions

**THEOREM:** Stokes theorem

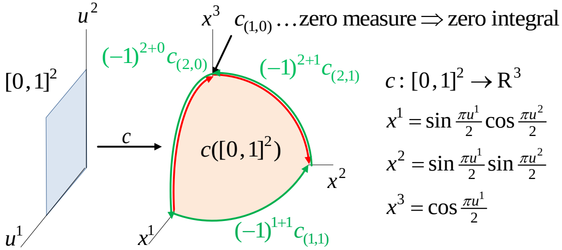
$\Gamma$ ... $n$-dimensional singular chain in $\mathbb{R}^m$, $m \geq n$
$\omega$ ... an $(n-1)$-form on an open set in $\mathbb{R}^m$ containing $\Gamma$. It holds

$$\int_{\partial\Gamma} \omega = \int_{\Gamma} \mathrm{d}\omega$$

Steps of the proof:

1) for $\Gamma = I^n$ in $\mathbb{R}^n$ ...
$\omega = f(x^1, \dots, x^n)\, \mathrm{d}x^1 \wedge \mathrm{d}x^{i-1} \wedge \mathrm{d}x^{i+1} \wedge \dots \wedge \mathrm{d}x^n$
2) for $\Gamma = c \circ I^n$, $c:\ I^n \ni (x^j)_{j=1,\dots,n} \to (y^\alpha c(x^j))_{\alpha=1,\dots,m} \in \mathbb{R}^m$

**EXAMPLE:** $c$ ... parametrized piece of unit sphere in $\mathbb{R}^3$




$$c : [0,1]^2 \to \mathrm{R}^3$$
$$x^1 = \sin \tfrac{\pi u^1}{2} \cos \tfrac{\pi u^2}{2}$$
$$x^2 = \sin \tfrac{\pi u^1}{2} \sin \tfrac{\pi u^2}{2}$$
$$x^3 = \cos \tfrac{\pi u^1}{2}$$

$$\partial c = (-1)^{1+0} c_{(1,0)} + (-1)^{1+1} c_{(1,1)} + (-1)^{2+0} c_{(2,0)} + (-1)^{2+1} c_{(2,1)}$$

$$\omega = x^1\,\mathrm{d}x^1 + x^2\,\mathrm{d}x^2 + x^3\,\mathrm{d}x^3, \quad \mathrm{d}\omega = 0 \;\Rightarrow\; \int_c \mathrm{d}\omega = 0$$

$$\int_{\partial c} = \int_{c_{(1,1)}} \omega + \int_{c_{(1,1)}} \omega - \int_{c_{(2,1)}} \omega = \ldots$$ (integral on $c_{1,0}$ is zero)

**EXERCISE:** complete the calculation

**THEOREM:** $n = m = 2$ Green theorem

$\omega = P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$

$P(x, y)$, $Q(x, y)$ ... (differentiable) functions

$$\int_{\partial c} P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = \int_{c} \left( \frac{\partial Q(x,y)}{\partial x} - \frac{\partial P(x,y)}{\partial y} \right) \mathrm{d}x \wedge \mathrm{d}y$$

**THEOREM:** $n = 2$, $m = 3$ classical Stokes theorem

$\omega = \omega_F^{(1)} = F_1(x, y, z)\,\mathrm{d}x + F_2(x, y, z)\,\mathrm{d}y + F_3(x, y, z)\,\mathrm{d}z$

$$\int_{\partial c} F_1(x, y, z)\,\mathrm{d}x + F_2(x, y, z)\,\mathrm{d}y + F_3(x, y, z)\,\mathrm{d}z =$$

$$= \int_{c} \mathrm{rot}_x\,\vec{F}\,\mathrm{d}y \wedge \mathrm{d}z + \mathrm{rot}_y\,\vec{F}\,\mathrm{d}z \wedge \mathrm{d}x + \mathrm{rot}_z\,\vec{F}\,\mathrm{d}x \wedge \mathrm{d}y$$

**THEOREM:** $n = 3$, $m = 3$ Gauss-Ostrogradsky theorem

$$\omega = \omega_F^{(2)} =$$

$$F_1(x, y, z)\,\mathrm{d}y \wedge \mathrm{d}z + F_2(x, y, z)\,\mathrm{d}z \wedge \mathrm{d}x + F_3(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y$$

$$\int_{\partial c} F_1(x, y, z)\,\mathrm{d}y \wedge \mathrm{d}z + F_2(x, y, z)\,\mathrm{d}z \wedge \mathrm{d}x + F_3(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y =$$

$$= \int_c \operatorname{div} \vec{F}(x, y, z)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

## EXERCISE:

Discuss classical theorems (Green, classical Stokes and Gauss-Ostragradsky) with respect to the general Stokes theorem.

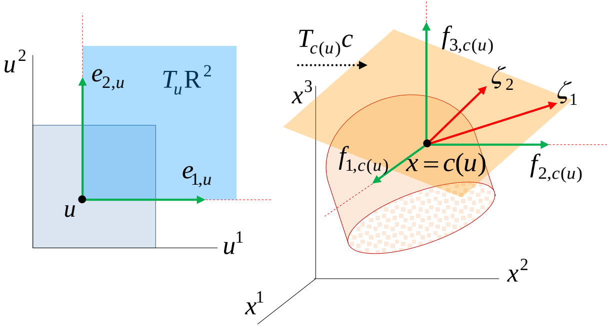
**DEFINITION:** volume element on $c$

Assumptions:

1) $u \in [0, 1]^n \subset \mathbb{R}^n$, $(e_{1,u}, \dots, e_{n,u})$ ... standard base in $T_u\mathbb{R}^n$
2) $c$ ... $n$-dim parametrized piece or singular cube in $\mathbb{R}^m$, $m \geq n$
3) $T_x\mathbb{R}^m$, $x \in \mathbb{R}^m$, is supposed with the standard scalar product $(\xi, \zeta) = \xi^1\zeta^1 + \dots + \xi^m\zeta^m$, standard orientation of $T_x\mathbb{R}^m$ is given by a standard base $(f_{1,x}, \dots, f_{m,x})$
4) $T_{c(u)}c = [|\zeta_1, \dots, \zeta_n|]$, $\zeta_i = Tc\, e_i$, ... tangent space to $c$ in $c(u)$, $x \in [0, 1]^n$ ... generated by vectors $\zeta_1 = T_u c$
5) scalar product on $T_{c(u)}c$ ... restriction of $(\xi, \zeta)$ to $T_{c(u)}c$
6) orientation in $T_{c(u)}c$ compatible with $c$ ... $\mu = [\zeta, \dots, \zeta_n]$

volume element on $c$ ... the $n$-form $\omega$ such that $\omega(c(u))$ is the volume element in $T_{c(u)}c$

## .**EXAMPLE** structures for volume element for $n = 2$, $m = 3$




## EXERCISE:

For $c : [0, 1]^2 \ni u = (u^1, u^2) \to c(u) = (x^1 c(u), x^2 c(u), x^3 c(u)) \in \mathbb{R}^3$
$x^1 = \sin \frac{\pi u^1}{2} \cos \frac{\pi u^2}{2}$, $x^2 = \sin \frac{\pi u^1}{2} \sin \frac{\pi u^2}{2}$, $x^1 = \cos \frac{\pi u^1}{2}$ calculate vectors
$\zeta_1 = Tc\, e_{1,u}$, $\zeta_2 = Tc\, e_{2,u}$.

**PROPOSITION:** calculation of the volume element on $c$

$\omega_0 \dots$ volume element on an $n$-dimensional $c$ in $\mathbb{R}^m$, $\zeta_= \zeta_i(c(u)) = Tc\, e_{i,u}$, $i \in \{1, \dots, n\}$, then

$$\omega_0(c(u))(\zeta_1, \dots, \zeta_n) = \sqrt{\det(\zeta_i, \zeta_j)}$$

**EXERCISE:** Prove the above proposition using following notes:

- $\omega_0(\zeta_1, \dots, \zeta_2) = \det A$, $A = (\bar{\zeta}_i^j)_{i,j=1,\dots,n}$, $\zeta_i = \bar{\zeta}_i^j \epsilon_j$, $(\epsilon_1, \dots, \epsilon_n) \dots$ orthonormal base in $T_{c(u)}c$ belonging to the orientation $\mu$
- $\det^2 A = \det A \cdot \det A^T = \det G$, $G = (g_{ij})$, $g_{ij} = (\zeta_i, \zeta_j)$

NOTE: Recall that the scalar product is invariant with respect to the choice of a base, thus the calculation can be made with help of components of vectors $\zeta_1, \dots, \zeta_n$ in the base $(f_1, \dots, f_m)$, $\zeta_i = \zeta_i^\alpha f_\alpha$, $\alpha \in \{1, \dots, m\}$.

With the volume element we can introduce a specific type of integral on $c$, useful e.g. for practical calculations of geometric and physical characteristics of real bodies

**DEFINITION:** first-type integral

first-type integral of a function $f : A \ni x \longrightarrow \mathbb{R}$, $A \subset \mathbb{R}^m$ ... an open set containing $c([0, 1])$ is the integral of the form $f\omega_0$ on $c$

$$I = \int_c f\omega_0 = \int_{[0,1]^n} (f \circ c)\, c^*\omega_0$$

**PROPOSITION:**

$$\int_c f\omega_0 = \int_{[0,1]^n} (f \circ c)\sqrt{\det G}\, (\mathrm{d}u^1 \ldots \mathrm{d}u^n)$$

NOTE: It is the direct consequence of the expression for $\omega_0(\zeta_1, \ldots, \zeta_n)$.

# Differential manifolds

**DEFINITION:** topological manifold

$n$-dimensional topological manifold ... topological space $(X, \tau_X)$

1) Hausdorff
2) there exists a countable base of $\tau_X$ ($2^{\text{nd}}$ axiom of countability)
3) locally isomorphic with $\mathbb{R}^n$

**DEFINITION:** maps on a topological manifold

1) $U \subset X$ ... open set, $\varphi: U \ni x \longrightarrow \varphi(x) \in \mathbb{R}^n$, $\varphi(U) \in \tau_X$
2) $\varphi$ is a homeomorphism

$(U, \varphi)$ ... local map on $X$, $(X, \varphi)$ ... global map (if exists)

NOTE: $n$-dimensional topological manifold is "locally" $\mathbb{R}^n$

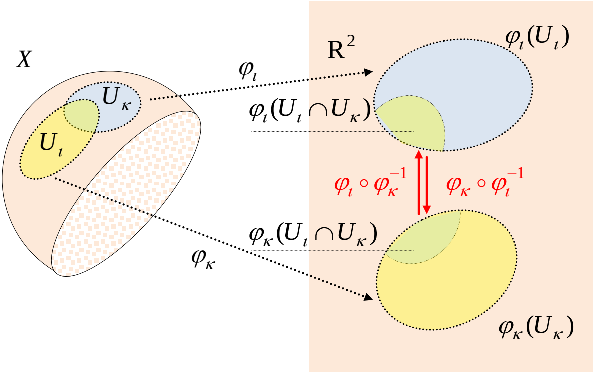
**DEFINITION:** $\mathbb{R}^n$-atlas on an $n$-dim topological manifold $X$

$\mathbb{R}^n$ atlas of class $C^r$, or smooth (class $C^\infty$) on $X$ ... family of maps $\mathcal{A} = \{(U_\iota, \varphi_\iota) \,|\, U_\iota \in \tau_X, \, \iota \in I\}$, $I$ ... a set of indices:

1) $\bigcup_{\iota \in I} U_\iota = X$
2) every $\varphi_\iota$ is a homeomorphism of $U_\iota$ on the open set $\varphi(U_\iota) \subset \mathbb{R}^n$
3) for every pair $\iota, \kappa \in I$ the maps
   $\varphi_\iota \circ \varphi_\kappa^{-1} : \varphi_\kappa(U_\iota \cap U_\kappa) \longrightarrow \varphi_\iota(U_\iota \cap U_\kappa)$
   $\varphi_\kappa \circ \varphi_\iota^{-1} : \varphi_\iota(U_\iota \cap U_\kappa) \longrightarrow \varphi_\kappa(U_\iota \cap U_\kappa)$
   are one to one $C^r$-differentiable, or diffeomorphisms (of class $C^\infty$)

NOTE: $\mathbb{R}^n$ is understood as the $n$-dimensional Euclidean space, i.e. the set of all real $n$-tuples $(\alpha^1, \ldots, \alpha^n)$ with the Euclidean (natural) topology.

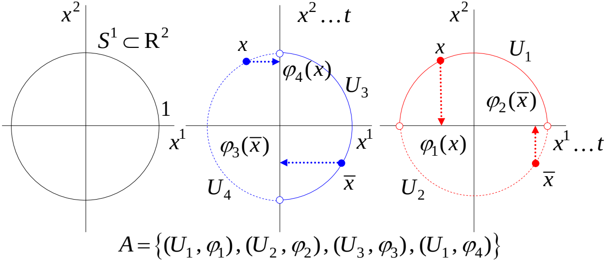
## .**EXAMPLE:** $\mathbb{R}^2$-atlas on 2-dimensional $X$ – overlapping maps




$$\varphi_\iota \circ \varphi_\kappa^{-1} : \varphi_\kappa(U_\iota \cap U_\kappa) \ni (x^i \varphi_\kappa(x^j \varphi_\iota))_{i=1,2} \longrightarrow (x^i \varphi_\kappa(x^j \varphi_\kappa))_{i=1,2} \in \varphi_\kappa(U_\iota \cap U_\kappa)$$

$$\varphi_\kappa \circ \varphi_\iota^{-1} : \varphi_\iota(U_\iota \cap U_\kappa) \ni (x^i \varphi_\iota(x^j \varphi_\kappa))_{i=1,2} \longrightarrow (x^i \varphi_\kappa(x^j \varphi_\iota)_{i=1,2}) \in \varphi_\kappa(U_\iota \cap U_\kappa)$$

## .**EXAMPLE:** $\mathbb{R}^1$-atlas on a unit circle $S^1 \subset \mathbb{R}^2$




Mappings $\varphi_i$ are cartesian projections, $t_i = \varphi_i(x)$, for example for $x = (x^1, x^2) \in U_1 \cap U_3$, i.e. $x^1, x^2 > 0$, it holds

$$\varphi_3 \circ \varphi_1^{-1} : \; U_1 \cap U_2 \ni x \longrightarrow t_3 = \sqrt{1 - t_1^2}$$

**EXERCISE:** Express all $\varphi_j \circ \varphi_i^{-1}$. Show that $\mathcal{A}$ is an atlas on $S^1$.

**DEFINITION:** maximal $\mathbb{R}^n$-atlas on $X$, differential structure

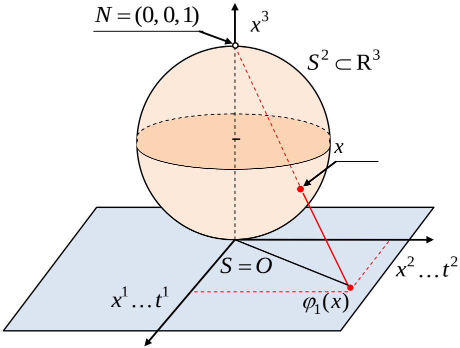
1) $(U, \varphi)$ ... map compatible with $\mathcal{A}$, if the family $\bar{\mathcal{A}} = \{(U_\iota, \varphi_\iota), (U, \varphi)\}$ is again an atlas
2) atlases $\mathcal{A}$, $\bar{\mathcal{A}}$ ... equivalent, if every map of $\mathcal{A}$ is compatible with $\bar{\mathcal{A}}$ and vice versa
3) equivalence class $[\mathcal{A}]$ ... differential structure on $X$

The following theorem is important for the concept of integration on manifolds:

**THEOREM:**

On every differentiable manifold it exists a countable atlas.

## .EXAMPLE: stereographic projection "SP"




$$A=\{(U,\varphi),(\bar{U},\bar{\varphi})\}$$

$$U=S^2\setminus\{N\},\bar{U}=S^2\setminus\{S\}$$

$$\bar{\varphi}\circ\varphi^{-1}:(t^1,t^2)\to(\bar{t}^1,\bar{t}^2)=$$
$$\left(\frac{t^1}{(t^1)^2+(t^2)^2},\frac{t^2}{(t^1)^2+(t^2)^2}\right)$$

$$\varphi\circ\bar{\varphi}^{-1}:(\bar{t}^1,\bar{t}^2)\to(t^1,t^2)$$

exercise

## EXERCISE:

1) For SP of $S^2 \subset \mathbb{R}^3$ derive relations $\varphi:\ (t^1,\ t^2) \to (x^1,\ x^2,\ x^3)$ and $\varphi:\ (\bar{t}^1,\ \bar{t}^2) \to (x^1,\ x^2,\ x^3)$.
2) With SP construct the atlas on $S^1 \subset \mathbb{R}^2$ and show that it is compatible with that based on cartesian projections.

The following proposition is important for practical description of $n$-dimensional manifolds in $\mathbb{R}^m$, $m > n$

**PROPOSITION:** Let $X \subset \mathbb{R}^m$ be an $n$-dimensional manifold. Then for every point $x \in X$ it exists a map $(W, \psi)$, $\psi(x) = (y^1\psi(x), \dots, y^m\psi(x))$ such that $(U, \varphi)$, $U = W \cap X$, $\varphi = \psi|_U$ is a map on $X$ and

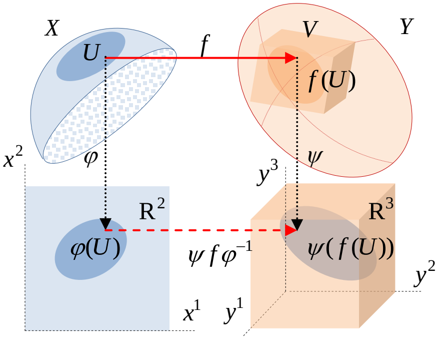
$$y^{n+1}\psi(x) = \cdots = y^m\psi(x) = 0$$

**EXAMPLES:**

1) $S^1 \subset \mathbb{R}^2$ (unit circle in $\mathbb{R}^2$) ... 1-dimensional manifold in $\mathbb{R}^2$, $W = \mathbb{R}^2$, $\psi = (y^1, y^2)$, $y^1 = \vartheta$, $y^2 = r - 1$, where $\vartheta$, $r$ are polar coordinates; for $x \in S^1$ is $y^2 = 0$
2) $S^2 \subset \mathbb{R}^3$ (unit sphere in $\mathbb{R}^3$) ... 2-dimensional manifold in $\mathbb{R}^3$; exercise: describe $S^2$ with the use of the above proposition

## **DEFINITION:** differentiable mappings of manifolds

$(X, \tau_X)$, $(Y, \tau_Y)$ ... diff. manifolds $\dim X = n$, $\dim Y = m$
$f : W \to Y$ ... a mapping on an open set $W \subset X$
$(U, \varphi)$, $x \in U$, $(V, \psi)$, $f(x) \in f(U) \subset V$ ... maps on $X$ and $Y$




$f$ ... differentiable, if $\psi \circ f \circ \varphi^{-1}$ is differentiable

# Differential manifolds with boundaries

**EXAMPLES:** manifolds without a boundary

differential manifolds $S^1 \subset \mathbb{R}^2$, $S^2 \subset \mathbb{R}^3$ ... examples of manifolds without boundary

**EXERCISE:** motivation: $S^1 \subset \mathbb{R}^2$, $S^2 \subset \mathbb{R}^3$ can be described also as singular cubes; prove that $\partial S^1 = 0$, $\partial S^2 = 0$

What means manifold with a boundary?

**DEFINITION:** $n$-dimensional manifold with a boundary in $\mathbb{R}^m$

$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \,|\, x^n \geq 0\}$ with induced Euclidean topology
$X \subset \mathbb{R}^m$, $X \neq \emptyset$, $m > n$ $X$ ... a manifold with boundary if there exists a maximal $\mathbb{R}^n_+$ atlas on $X$

NOTE: axioms of maximal $\mathbb{R}^n_+$ are formally the same as for a maximal $\mathbb{R}^n$-atlas, taking into account $\mathbb{R}^n_+$ instead of $\mathbb{R}^n$

**DEFINITION:** boundary of a manifold $X$

interior point $x \in X$ ... there exists a map $(U, \varphi)$ such that $x \in U$, $\varphi(U) \subset \mathbb{R}^n_+$ is open in $\mathbb{R}^n$, i.e. $x^n \varphi(x) > 0$

boundary point ... a point $x \in X$ which is not interior, i.e. there exists a map $(U, \varphi)$ such that $x \in U$, $x^n \varphi(x) = 0$

boundary of $X$ ... the set $\partial X$ of all boundary points

manifold without a boundary ... $\partial X = \emptyset$

**PROPOSITION:**

the boundary of $n$-dimensional manifold with boundary is $(n-1)$-dimensional manifold (without boundary)

**EXAMPLE:** $X = \{(x^1, x^2) \in \mathbb{R}^2 \,|\, (x_1)^2 + (x_2)^2 < 1\}$, $\partial X = \emptyset$,
$X = \{(x^1, x^2) \in \mathbb{R}^2 \,|\, (x_1)^2 + (x_2)^2 \leq 1\}$, $\partial X = S^1$

**DEFINITION:** tangent spaces to a manifold $X$

$X$ ... $n$-dimensional manifold with boundary in $\mathbb{R}^m$, define

1) curve in $X$ with the origin at $x \in$
   $\mathcal{C} : [0, \varepsilon) \to \mathbb{R}^m$, $\mathcal{C}([0, \varepsilon)) \subset X$
2) tangent vector to $X$ at $x$
   $\xi \in T_x\mathbb{R}^m$, if it is tangent to a curve in $X$ with the origin at $x$
3) tangent space to $X$ at $x$
   the set $T_xX$ of all tangent vectors to $X$

**PROPOSITION:**

tangent space $T_xX \subset T_x\mathbb{R}^m$ is a vector subspace of $T_x\mathbb{R}^m$

NOTE: Above properties hold for all points of $X$, including boundary ones.

$\mathcal{C} : [0, \varepsilon) \to \mathcal{C}(t) \in \mathbb{R}^m$, $\mathcal{C}(0) = x$, ($\mathcal{C}$ differentiable) ... a curve in $\mathbb{R}^m$ with origin at $x$; tangent vectors to $\mathbb{R}^m$ at $x = (y^1, \dots, y^m)$:

$$\xi(x) = \mathrm{const}\, \dot{\mathcal{C}}(0) = \mathrm{const}\, \left( x, \left( \frac{\mathrm{d}y^1\mathcal{C}(t)}{\mathrm{d}t}, \dots, \frac{\mathrm{d}y^m\mathcal{C}(t)}{\mathrm{d}t} \right)\Bigg|_{t=0^+} \right)$$

**PROPOSITION:** coordinate expressions of tangent vectors to $X$ for $x \in X$, $(U, \varphi)$, $\varphi = (x^1, \dots, x^n)$ ... a map on $X$, $x \in U$:

$$\xi(x) = \mathrm{const}\, \dot{\mathcal{C}}(0) = \mathrm{const}\, \left( x, \left( \frac{\mathrm{d}x^1\varphi\mathcal{C}(t)}{\mathrm{d}t}, \dots, \frac{\mathrm{d}x^n\varphi\mathcal{C}(t)}{\mathrm{d}t} \right)\Bigg|_{t=0^+} \right)$$

**EXERCISE:** Prove that $(e_{1,x}, \dots, e_{n,x})$, $e_{i,x} = \dot{\varphi}_{i,x}(0)$, $\varphi_{i,x}$ ... coordinate curves $[0, \varepsilon) \ni t \to \varphi_{i,x}(t) = \varphi^{-1}(x^1, \dots, x^i + t, \dots x^n) \in U$ is a base in $T_xX$. Discuss if components of a vector depend on a concrete map $(U, \varphi)$, $x \in U$.

# Differential forms on manifolds, decomposition of unity

Having defined tangent spaces to a manifold $X$ we have automatically defined dual spaces $T_x^*X$ and spaces of tensors, especially $\Lambda_k(T_xX)$ by the analogous way as for $k$-forms on Euclidean spaces.

NOTE: bases and dual bases

$(e_{1,x}, \ldots, e_{n,x}) = \left(\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^1}\right)_x$, $(e_x^1, \ldots, e_x^n) = (\mathrm{d}x^1(x), \ldots, \mathrm{d}x^n(x))$

$\mathrm{d}x^i(x)(e_{j,x}) = \left.\frac{\mathrm{d}x^i\varphi_{j,x}}{\mathrm{d}t}\right|_{t=0} = \delta^i_j$

**DEFINITION:** differential $k$-forms on $X$

mappings $\omega : X \supset W \in x \longrightarrow \omega(x) \in \Lambda_k(T_xX) \subset \Lambda_k X$

$$\omega(x) = \omega_{i_1 \ldots i_k}\,\mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_k}, \quad i_1, \ldots, i_k \in \{1, \ldots, n\}$$

**DEFINITION:** orientable manifolds

volume element on $n$-dimensional $X$ ... an (at least continuous) $n$-form $\omega$ on $X$, $\omega(x) \neq 0$ for every $x \in X$

orientable manifold $X$ ... there exists the volume element on $X$

$(U, \varphi)$, $\varphi = (x^1, \ldots, x^n)$, $x \in X$ ... $\omega(x) = f(x)\,\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^n$

**PROPOSITION:**

1) $n$-dimensional manifold $X$ is orientable iff there exist a countable atlas $\mathcal{A} = \{(U_\iota, \varphi)\iota \,|\, \iota \in I\}$ such that for every $\iota, \kappa \in I$ it holds $\det D(\varphi_\iota \circ \varphi_\kappa^{-1}) > 0$ on $\varphi_\kappa(U_\iota \cap U_\kappa)$
2) the boundary of an orientable manifold with boundary is orientable

NOTE:The expression of a volume element $\omega$ for every $(U_\iota, \varphi_\iota)$ is unique, $\omega = f_\iota\,\mathrm{d}x_\iota^1 \wedge \ldots \wedge \mathrm{d}x_\iota^n$, $f_\iota > 0$ or $f_\iota < 0$; $(U_\iota, \varphi_\iota)$ ... positive, if $f_\iota > 0$.

**PROPOSITION:** transformation of coordinates

$(U_\iota, \varphi_\iota)$, $(U_\kappa, \varphi_\kappa)$, $U_\iota \cap U_\kappa \neq \emptyset$, on $U_\iota \cap U_\kappa$ it holds

$$\omega = f_\iota \, \mathrm{d}x_\iota^1 \wedge \ldots \wedge \mathrm{d}x_\iota^n = \left( f_\kappa \cdot \det D(\varphi_\kappa \circ \varphi_\iota^{-1}) \right) \mathrm{d}x_\kappa^1 \wedge \ldots \wedge \mathrm{d}x_\kappa^n$$

**DEFINITION:** oriented manifolds

$X$ ... $n$-dimensional orientable manifold

orientation of $X$ ... a mapping $\mu : X \ni x \longrightarrow \mu(x)$, where $\mu(x)$ is an orientation of $T_x X$

induced orientation by the volume element $\omega$ ... for every $x \in X$ and all bases $(e_{1,x}, \ldots, e_{n,x}) \in \mu(x)$ is $\omega(x)(e_{1,x}, \ldots, e_{n,x}) > 0$

a manifold $X$ with orientation ... oriented manifold

**EXAMPLE:** $S^2 \subset \mathbb{R}^3$ ... orientable (e.g. with help of a continuous external normal), Mobius strip ... not orientable

The concept of integration of forms on differentiable manifold is analogous as that using parametrized pieces or singular cubes $c : [0, 1]^n \to \mathbb{R}^m$, $m \geq n$ and chains $\Gamma = k_1 c_1 + \cdots + k_p c_p$ in $\mathbb{R}^m$. However, it is more general: while integrated forms can be defined on the whole manifold (as e.g. volume element) there need not exist a global map (coordinate system) on $X$. The coordinate expression of differential forms (integrated object) are different in various coordinates (transformable, of course, each to other on intersections of various maps). We solve this problem with help of so called decomposition of unity.

**PROPOSITION:** decomposition of unity

$X$ ... an $n$-dimensional differential manifold with boundary
$\mathcal{A} = \{(U_\iota, \varphi_\iota \,|\, \iota \in I)\}$ ... an atlas on $X$

Then there exists a family $(\chi_\iota)$, $\chi_\iota : X \to \mathbb{R}$, $\iota \in I$, of $C^\infty$-differentiable functions (decomposition of unity on $X$ associated with $\mathcal{A}$) with properties:

1) for every $\iota \in I$ and every $x \in X$ it holds $0 \leq \chi_\iota \leq 1$
2) for every $x \in X$ there exist an open set $W \subset X$ such that only a finite number of functions belonging to $\{\chi_\iota \,|\, \iota \in I =\}$ is nonzero on $W$
3) for very $x \in X$ only a finite number of numbers $\chi_\iota(x)$, $\iota \in I$, is nonzero; moreover it holds $\sum_{\iota \in I} \chi_\iota(x) = 1$
4) for every $\iota$ in holds $\operatorname{supp} \chi_\iota \subset U_\iota$ (support of the function $\chi_\iota$)

.**EXAMPLE:** an example of decomposition of unity on $\mathbb{R}$

$X = \mathbb{R}$, $\mathcal{A} = \{(U_j, \varphi_j), (V_j, \psi_j) \,|\, j \in \mathbf{N} \cup \{0\}\}$

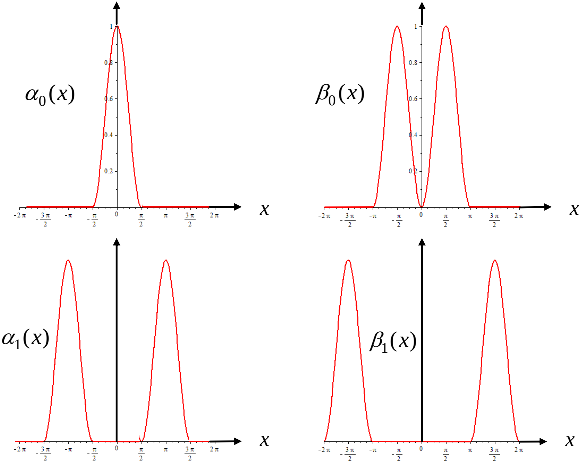
$U_j = \left(-(2j+1)\frac{\pi}{2}, -(2j-1)\frac{\pi}{2}\right) \cup \left((2j-1)\frac{\pi}{2}, (2j+1)\frac{\pi}{2}\right)$

$V_j = (-(j+1)\pi, -j\pi) \cup (j\pi, (j+1)\pi)$

$\alpha_j(x) = \cos^2 x$, $x \in U_j$, $\alpha = 0$, $x \in \mathbb{R} \setminus U_j$

$\beta_j(x) = \sin^2 x$, $x \in V_j$, $\beta_j(x) = 0$, $x \in \mathbb{R} \setminus V_j$

## .**EXAMPLE:** decomposition of unity on $\mathbb{R}$ (continued)



# Integral of forms on manifolds, Stokes theorem

**DEFINITION:** integral

Assumptions

1) $X$ ... $n$-dimensional oriented manifold with boundary
2) $\mathcal{A} = \{(U_i, \varphi_i) \,|\, i \in \mathbf{N}\}$ ... a countable atlas on $X$ with positive maps, such that all sets $\bar{U}_j$ are compact
3) $\bar{\mathcal{A}} = \{(V_j, \psi_j) \,|\, j \in \mathbf{N}\}$ ... another atlas with the same properties as $\mathcal{A}$
4) $(\chi_i)$, $(\upsilon_j)$ ... decompositions of unity associated with $\mathcal{A}$, $\bar{\mathcal{A}}$
5) $\omega$ ... an arbitrary $n$-form on $X$,
   $\omega = f_i \, \mathrm{d}x_i^1 \wedge \ldots \wedge \mathrm{d}x_i^n = g_j \, \mathrm{d}y_j^1 \wedge \ldots \wedge \mathrm{d}y_j^n$

**DEFINITION:** integral – continued

Denote integrals (explain why they exist)

$$I_i = \int\limits_{\varphi_i(U_i)} (\chi_i f_i) \circ \varphi_i^{-1} (\mathrm{d}x_i^1 \dots \mathrm{d}x_i^n)$$

$$J_j = \int\limits_{\psi_j(V_j)} (\upsilon_j g_j) \circ \psi_j^{-1} (\mathrm{d}y_j^1 \dots \mathrm{d}y_j^n)$$

**PROPOSITION:** important for introducing integral

absolute convergence of $\sum_i I_i \Longrightarrow$ absolute convergence of $\sum_j J_j$ and $\sum_i I_i = \sum_j J_j$

**DEFINITION:** integral – continued

for absolutely convergent $\sum_i I_i$ ... form $\omega$ is integrable on $X$

$$\int_X \omega = \sum_i \int\limits_{\varphi_i(U_i)} (\chi_i f_i) \circ \varphi_i^{-1} (\mathrm{d}x_i^1 \ldots \mathrm{d}x_i^n) \quad \text{integral of } \omega \text{ on } X$$

**PROPOSITION:** properties of integral

1) linearity: $\int\limits_X (\alpha\omega_1 + \beta\omega_2) = \alpha \int\limits_X \omega_1 + \beta \int\limits_X \omega_2$
2) for $X = X_1 \cup X_2$, $X_1 \cap X_2 = \emptyset$ ... $\int\limits_X \omega = \int\limits_{X_1} \omega + \int\limits_{X_2} \omega$
3) for $\varphi_i(U_i \cap (X \setminus Y)) \subset \mathbb{R}^n_+$ zero measure set
$I_X = \int\limits_X \omega$ exists iff $I_Y = \int\limits_Y \omega$ exists, and then $I_X = I_Y$

**THEOREM:** Stokes theorem

Assumptions

- $X$ ... $n$-dimensional compact oriented manifold with boundary
- $\omega$ ... differential $(n-1)$ form on $X$

Then

$$\int_X \mathrm{d}\omega = \int_{\partial X} \omega, \text{for } \partial X = \emptyset \ldots \int_X \mathrm{d}\omega = 0$$

NOTE: The compactness of the manifold $X$ cannot be weakened in general. On the other hand, Stokes theorem is valid even for non-compact manifolds if $\omega$ has compact support.

**DEFINITION:** volume of a manifold and integral of $f : X \to \mathbb{R}$

Assumptions

1) $X$ ... $n$-dimensional oriented manifold in $\mathbb{R}^m$ with boundary, $m \geq n$, orientation $\mu$
2) for every $x \in X$ ... $(U, \varphi)$, $\varphi = (x^i)$, $x \in U$,
   $\xi(x), \zeta(x) \in T_x X$, $g_x(\xi, \zeta) = (\xi(x), \zeta(x))$, i.e. $g_x \in \mathcal{T}_2(T_x X)$
   mapping $g : (x \to g_x)$, $g = g_{ij} \, \mathrm{d}x^i \otimes \mathrm{d}x^j$, $g_{ij} = (e_{i,x}, e_{i,x})$, ...
   Riemann metric on $X$,
3) $\omega(x) = \sqrt{\det(g_{ij})} \, \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n$ ... volume element defined by Riemann metric

$$v(X) = \int_X \omega \text{ volume of } X \text{ (if it exists)} \quad I(f) = \int_X f \, \omega$$

NOTE: The volume of a manifold exists for every compact manifold.

## .EXERCISE

1) Volume (length) of a 1-dimensional manifold: for $X$ ... a singular cube $c : [0, 1] \ni t \to c(t) \in \mathbb{R}^m$ prove that

$$v(X) = \int_0^1 \sqrt{\left(\frac{\mathrm{d}x^1 c(t)}{\mathrm{d}}\right)^2 + \cdots + \left(\frac{\mathrm{d}x^n c(t)}{\mathrm{d}}\right)^2}$$

and verify for $S^1 \subset \mathbb{R}^2$.

2) Volume of a 2-dimensional manifold: for $X$ ... a singular cube $c : [0, 1]^2 \ni (u^1, u^2) \to c(u^1, u^2) \in \mathbb{R}^3$ prove that

$$v(X) = \int_{[0,1]^1} \sqrt{g_{11}g_{22} - g_{12}^2}\,(\mathrm{d}u^1\,\mathrm{d}u^2) \quad \text{where}$$

$$g_{ij} = \frac{\partial x^1 c}{\mathrm{d}u^i}\frac{\partial x^1 c}{\mathrm{d}u^j} + \frac{\partial x^2 c}{\mathrm{d}u^i}\frac{\partial x^2 c}{\mathrm{d}u^j} + \frac{\partial x^3 c}{\mathrm{d}u^i}\frac{\partial x^3 c}{\mathrm{d}u^j}$$

and verify for $S^2 \subset \mathbb{R}^3$.

## .**EXERCISE:** continued

3) Calculate the inertia $J_3$ of the 3-dimensional unit sphere in $\mathbb{R}^3$, $S^3 = \{(x^1, x^2, x^3) \in \mathbb{R}^3 \,|\, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$ with respect to $x^3$-axis, where

$$J_3 = \int_{S^3} f_3(x^1, x^2, x^3)\, \omega \quad f_3(x^1, x^2, x^3) = (x^1)^2 + (x^2)^2$$

4) Calculate the position $(x_1^0, x_2^0, x_3^0)$ of center of mass of the cone

$$X : \{(x^1, x^2, x^3) \in \mathbb{R}^3 \,|\, (x^3)^2 - (x^2)^3 - (x^1)2 = 0,\, 0 \leq x^3 \leq 1\}$$

$$x_0^i = \frac{1}{v(X)} \int x^i\, \omega$$

NOTE:In all exercises $\omega$ is the volume element defined by Riemann metric.

# Books

[1] Spivak, M.: *Calculus on manifolds: A modern approach to classical theorems of advanced calculus*, 27. ed., Perseus Books Publishing, L. L. C., Massachusetts 1998.

[2] Krupka, D., Musilová, J.: *Integral calculus on Euclidean spaces and differential manifolds.* (In Czech.) SPN Praha, 1982.