

Variationality of physics theories

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Abstract:

It turns out that key physical theories, usually presented through equations of motion, are variational, i.e. resulting from certain variational principle. For solving the corresponding inverse problem (proving the variational nature of the equations of motion and finding the corresponding Lagrangian structure), the variational sequence on fibred manifolds and their finite prolongations represents an effective geometric tool. Our contribution is focused mainly on physical applications of the mathematical theory well developed in mathematical literature. It presents an overview of the inverse problem solution by the variational sequence for fundamental theories: classical mechanics, electrodynamics, wave theory, quantum mechanics.

Keywords: fibred manifolds, variational sequences, inverse problem, variational physics theories

1. Introduction.

It is well-known that the solution of many key problems of physics is based on an appropriate variational principle. Research publications are still currently devoted to variational theories and especially their generalization to various scientific fields, see e.g. [1, 2]. A strong geometric background for the formulation of variational theories are fibred manifolds and their jet prolongations, given in many various publications (for our purposes see e.g. [3, 4, 8, 9], in more detail with examples and problems e.g. [10]). The solution of a variational problem in physics means to obtain equations of motion of a physical system as equations for stationary points of the appropriate functional (action function).

Physical theories are usually formulated on the basis of experimental experience directly in the form of equations of motion of the studied physical system, whether in integral or differential form. The task of so called inverse problem of calculus of variations is to find the criteria for determining whether a given set of equations of motion is variational, i.e. whether it results from solving a certain variational problem. (The reader can find the well written overview of various approaches and sources up to year 2010 in [11].) This problem is effectively solved with the use of variational sequence on finite jet prolongations of fibred manifolds. The idea of the finite variational sequence was first formulated by Krupka [12] on the basis of variational bicomplex theory developed by various authors (let us cite the primary sources for all of them, [13, 14, 15, 16]). For practical calculations concerning variational nature of physics theories the

representation of the finite order variational sequence by forms is useful. Such a representation was looked for by various authors primarily for mechanics. The general solution for field theory using formally the Euler operator [14] (constructed for infinite order variation bicomplex automatically ensuring the globality) was in full given in [17] for finite order case (there the globality had to be proven). The results were then applied for solving various problems, see e.g. [18].

We suppose that the interested reader is familiar with basic properties of geometrical objects on fibred manifolds and their jet prolongations – vector fields, differential forms and their key properties related to the fibred structure of the underlying manifold. Thus we only summarize these concepts as prerequisites in Sec. 2.

In Sec. 3 the concept of dynamical forms representing generally equations of motion is presented, as well as the concept of a variational dynamical form.

Sec. 4 presents the variational sequence on finite jet prolongations of fibred manifolds and its representation by differential forms constructed with help of so called Euler operator. In the same section the meaning of such representation in calculus of variation is explained, including the inverse problem.

In Sec. 5 the general solution of the inverse problem obtained with help of just mentioned geometrical tools is then applied to equations of motion of key physics theories (some aspects of classical mechanics, wave mechanics, electrodynamics, quantum mechanics) proving that these theories are variational, including the construction of the relevant minimal Lagrangians.

(Naturally, variational theories are related to the issues of symmetries of the Lagrangian and equations of motion and the corresponding conservation laws. We do not deal with them here due to the length of the paper, but we refer the reader to [21]-[25].)

Let us explain the nature and purpose of our contribution in more detail. Variational problems appear in geometry, physics, engineering and other fields, even in economics [26]. Their solutions have specific forms within these fields. However, the variational problem as such has an essentially uniform mathematical background. It is based on the above mentioned abstract general concepts: fibred manifolds and their jet prolongations and structures defined on these underlying spaces. Mathematics formulate the geometric background as unified for all variational theories whose equations of motion are based on the variational principle, i.e. they are Euler-Lagrange equations for stationary points of a functional formulated on principles of the theory in question.

Physical theories in research and especially in physics teaching are mostly based directly on equations of motion (given by laws of motion derived from experiment) and work in time-space coordinates. The starting point is not always a variational functional, even if the theory itself is variational in principle. (For classical mechanics Newton's second law for conservative forces, for classical electrodynamics Maxwell's equations as a consequence of Coulomb, Faraday and Ampere laws, the non-existence of a magnetic monopole and the charge continuity equation.) It is therefore not usual for the formulations of physical theories, even well-known ones, to be based on an abstract and coordinate-

independent mathematical basis. However, such abstract and coordinate free background discloses the fundamental structure of the relevant physical theory. In particular, it enables the solution of the mentioned inverse variational problem. Even though the mathematical background for variational physics is known, and the key classical physical theories are individually developed by common procedures, we find in physics research and teaching the connection between these two perspectives rather exceptionally.

In short: Mathematics searches for and constructs suitable structures as a background for physical (and other) theories, but does not solve the corresponding physical problems to the end. Physics proceeds within the framework of a given theory in solving practical problems and do not search for an abstract basis.

The goal and supposedly our own benefit of the contribution is to show and emphasize the natural context of mathematical and physical approaches on the examples of the direct connection of general abstract mathematical tools and key physical theories. Using the finite order variational sequence as the mathematical background we show the effective solution of the inverse variational problem in general. (For specific theory under consideration it is, of course, important to assign the corresponding fibred manifold and relevant structures.) We show the application of the variational sequence in connection with physics using above mentioned variational physical theories as examples.

The combination of a general unified mathematical background and physics can be used not only in developing new theories based on the variational principle, but also as a didactic tool, especially in advanced physics teaching. Our contribution is thus not purely research, but it shows the inherence of abstract mathematics in physical research. It is a certain overview of the use of current abstract results within the specific mathematical area for physical research and advanced teaching.

Summarized: Both, the physical theories themselves in the field of physics, and the abstract mathematical background in the field of mathematics, are known individually. We see our original contribution in their natural and logical connection, especially in physics, teaching physics and other relevant applications.

2. Geometric background, notations and chart expressions.

In this section, we will summarize in the form of prerequisites and references to sources the basic concepts and structures on fibred manifolds, essential for the interpretation of the variational sequence and for the practical solution of the inverse problem in concrete physical situations. (There are several scientific schools dealing with geometric analysis on fibred manifolds. In this overview section we rely mainly on the Prof. Krupka's school.)

Underlying spaces and mappings: Fibred manifold (Y, π, X) with base X , total space Y and the projection π , $\dim X = n \geq 1$, $\dim Y = m + n$, sections γ of π and their r -jets $J_x^r \gamma$ in $x \in X$, r -jet prolongation $(J^r Y, \pi_r, X)$

of (Y, π, X) , local isomorphisms (α, α_0) of (Y, π, X) and their prolongations $(J^r \alpha, \alpha_0)$, [3, 4, 8, 9].

Charts: Fibred chart on (Y, π, X) , associated chart on X and associated fibred charts on $(J^r Y, \pi_r, X)$ are (V, ψ) , $(\pi(V), \varphi)$, $(\pi_r^{-1}(V), \psi_r)$, where $V \subset Y$ is an open subset and

$$\psi = (x^i, y^\sigma), \quad \varphi = (x^i), \quad \psi_r = (x^i, y^\sigma, y_j^\sigma, \dots, y_{j_1, \dots, j_r}^\sigma), \quad (1)$$

$1 \leq i \leq n, 1 \leq \sigma \leq m, 1 \leq j_s \leq n, 1 \leq s \leq r, y_{j_1 \dots j_s}^\sigma$ are fully symmetric in indices $j_1, \dots, j_s, \dim J^r Y = n + m \sum_{s=0}^r \binom{n+s-1}{s}$. In mechanics $n = 1$ and

$$\psi = (t, q^\sigma), \quad \varphi = (t), \quad \psi_r = (t, q^\sigma, \dot{q}^\sigma, \ddot{q}^\sigma, \dots, q_r^\sigma). \quad (2)$$

Vector fields, projectable, vertical: Consider vector fields ξ_0 on X , ξ on Y and ξ_s on $J^s Y$, respectively, $1 \leq s \leq r$, (see e.g. [3, 4, 20, 10]),

$$\begin{aligned} \xi_0 &= \xi^i(x^j) \frac{\partial}{\partial x^i}, \quad \xi = \xi^i(x^j, y^\nu) \frac{\partial}{\partial x^j} + \Xi^\sigma(x^j, y^\nu) \frac{\partial}{\partial y^\sigma}, \dots \\ \xi_s &= \xi^i \frac{\partial}{\partial x^i} + \Xi^\sigma \frac{\partial}{\partial y^\sigma} + \dots + \Xi_{J_s}^\sigma \frac{\partial}{\partial y_{J_s}^\sigma}, \quad J_s = (j_1 \dots j_s), \quad 1 \leq s \leq r, \end{aligned} \quad (3)$$

all components of ξ_s are functions on $J^s Y$, $s = |J_s|$ is the length of the multiindex J_s .

π_r -projectable vector field on $J^r Y$: (ξ, ξ_0) , where $T\pi_r \circ \xi = \xi_0 \circ \pi_r$, ξ_0 on X is unique,

$\pi_{r,s}$ -projectable vector field on $J^r Y$: (ξ, ξ_s) , $T\pi_{r,s} \circ \xi = \xi_s \circ \pi_{r,s}$, ξ_s on $J^s Y$ is unique.

$$\xi_0 = \xi_0^i(x^j) \frac{\partial}{\partial x^i}, \quad \xi_s = \xi^i \frac{\partial}{\partial x^i} + \sum_{p=0}^s \Xi_{J_p}^\sigma \frac{\partial}{\partial y_{J_p}^\sigma},$$

all components of ξ_s depend only on $(x^j, y^\nu, y_\ell^\nu, \dots, y_{\ell_1 \dots \ell_s}^\nu)$.

π_r -vertical vector field on $J^r Y$: $T\pi_r \circ \xi = 0 \Rightarrow \xi^i = 0, 1 \leq i \leq n$.

$\pi_{r,s}$ -vertical vector field on $J^r Y$: $T\pi_{r,s} \circ \xi = 0 \Rightarrow \xi^i, \Xi^\sigma, \Xi_\ell^\sigma, \dots, \Xi_{L_s}^\sigma = 0, 1 \leq i \leq n, 1 \leq \sigma \leq m, L_s = (\ell_1, \dots, \ell_s)$.

Prolongations of vector fields: Let ξ be a π -projectable vector field on Y , $(\alpha_u, \alpha_{u0}), \alpha_u : W \rightarrow Y, \alpha_0 : \pi(W) \rightarrow X$ for $u \in (-\varepsilon, \varepsilon)$, its one parametric group of local isomorphisms, $W \subset Y$ is open set,

r -jet prolongation of ξ : The vector field $J^r \xi, 1 \leq r$, on $J^r Y$ having the one-parameter group $(J^r \alpha_u, \alpha_{u0})$. The recursive formula:

$$\begin{aligned} \xi &= \xi^i(x^j) \frac{\partial}{\partial x^i} + \Xi^\sigma(x^j, y^\nu) \frac{\partial}{\partial y^\sigma}, \\ J^r \xi &= \xi^i(x^j) \frac{\partial}{\partial x^i} + \Xi^\sigma(x^j, y^\nu) \frac{\partial}{\partial y^\sigma} + \sum_{s=1}^r \Xi_{J_s \dots j_s}^\sigma(x^j, y^\nu, \dots, y_{\ell_1 \dots \ell_s}^\nu) \frac{\partial}{\partial y_{J_s \dots j_s}^\sigma}, \\ \Xi_{J_s \dots j_s}^\sigma &= d_{J_s} \Xi_{j_1 \dots j_{s-1}}^\sigma - y_{j_1 \dots j_{s-1} \ell}^\sigma \frac{\partial \xi^\ell}{\partial x_{j_s}}, \quad 1 \leq s \leq r. \end{aligned} \quad (4)$$

Differential forms. Antisymmetric covariant k -th order tensor fields on X, Y and $J^s Y$, respectively, $1 \leq s \leq r$, operations with forms (algebraic operations, contraction, pullback, exterior derivative, total derivative, Lie derivative, see e.g. [3, 4, 5, 10, 20, 28]),

$$\omega : J^r Y \supset W_r \ni y \longrightarrow \omega(y) \in \Lambda_k(T_y J^r Y) \subset \Lambda_k(T J^r Y), \quad 1 \leq k \leq \dim J^r Y,$$

$W_r \subset J^r Y$ is an open set. A function f on W_r is considered as a 0-form.

$$\omega = \sum_{p=0}^k \sum_{|J_q|=0}^r A_{\sigma_1 \dots \sigma_p}^{J_1 \dots J_p} \omega_{i_{p+1} \dots i_k} dy_{J_1}^{\sigma_1} \wedge \dots \wedge dy_{J_p}^{\sigma_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_k}, \quad (5)$$

the coefficients antisymmetric in $i_{p+1}, \dots, i_k$ and in multiindices $J_1 \dots J_p$.

Horizontal forms on $J^r Y$: π_r -horizontal k -form: $i_\xi \omega = 0$ for every π_r -vertical vector field ξ on $J^r Y$.

$$\omega = \omega_{i_1 \dots i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}, \quad 1 \leq k \leq n.$$

$\pi_{r,s}$ -horizontal k -form: $i_\xi \omega = 0$ for every $\pi_{r,s}$ -vertical vector field on $J^r Y$. In charts see (5), where $0 \leq |J_1|, \dots, |J_p| \leq s$.

Contact forms on $J^r Y$: $J^r \gamma^* \omega = 0$ for every section γ of π . Contact 1-form is called 1-contact, horizontal form is called 0-contact.

1-contact k -form, $k > 1$: $i_\xi \omega$ is 0-contact, for every π_r -vertical ξ .

q -contact k -form, $2 \leq q \leq k$, $k > 2$: $i_\xi \omega$ is $(q-1)$ -contact for every π_r -vertical ξ .

$\mathcal{I}_c = \mathcal{I}_c(J^r Y) = \{\omega \in \Lambda_k(T J^r Y), k \geq 1 \mid \omega \text{ contact}\}$. Properties:

$$\omega \in \mathcal{I}_c(J^r Y) \Rightarrow \omega \wedge \eta \in \mathcal{I}_c(J^r Y) \forall \eta, \quad d\omega \in \mathcal{I}_c(J^r Y), \quad \mathcal{I}_c(J^r Y) \text{ is an ideal.}$$

Bases adapted to contact structure on $J^r Y$ and $J^{r+1} Y$:

$$(x^i, \omega^\sigma, \dots, \omega_{j_1 \dots j_{r-1}}^\sigma, dy_{j_1 \dots j_r}^\sigma), \quad (x^i, \omega^\sigma, \dots, \omega_{j_1 \dots j_{r-1}}^\sigma, \omega_{j_1 \dots j_r}^\sigma). \quad (6)$$

$$\omega^\sigma = dy^\sigma - y_i^\sigma dx^i, \quad \omega_i^\sigma = dy_i^\sigma - y_{ij}^\sigma dx^j, \dots, \omega_{j_1 \dots j_s}^\sigma = dy_{j_1 \dots j_s}^\sigma - y_{j_1 \dots j_s i}^\sigma dx^i. \quad (7)$$

Decomposition of forms, horizontalization, contactization: $\omega \in \Lambda_k(T J^r Y)$

$$\pi_{r+1,r}^* \omega = h\omega + p_1 \omega + \dots + p_k \omega, \quad \text{where } p_q \omega \text{ is } q\text{-contact, } h\omega = p_0 \omega. \quad (8)$$

Horizontalization: $h : \Lambda_k(T J^r Y) \ni \omega \rightarrow h\omega \in \Lambda_{k,X}(T J^{r+1} Y)$,

q -contactization: $p_q : \Lambda_k(T J^r Y) \ni \omega \rightarrow p_q \omega \in \Lambda_k(T J^{r+1} Y)$,

contactization: $p : \Lambda_k(T J^r Y) \ni \omega \rightarrow p\omega = \pi_{r+1,r}^* \omega - h\omega$.

Properties: 1) $h(\omega \wedge \eta) = h\omega \wedge h\eta$, 2) $h\partial_{J^r \xi} \omega = \partial_{J^{r+1} \xi} h\omega$, 3) $\omega \in \mathcal{I}_c \Rightarrow \partial_{J^r \xi} \omega \in \mathcal{I}_c$, 4) $\omega \in \mathcal{I}_c$, ζ on $J^r Y$ and $\partial_\zeta \omega \in \mathcal{I}_c$, then $\zeta = J^r \xi$ for certain ξ on Y .

3. Dynamical forms, equations of motion and variationality.

In general, we find trajectories of physical systems as solutions of equations of motion. Trajectories are represented by sections of a fibred manifold (Y, π, X) , where X is the domain of (time-space) variables, a fibre over $x \in X$ is the configuration space. Equations of motion are given by vanishing of *dynamical form* E on $J^s Y$ along relevant prolongations of trajectories $J^s \gamma$. This form is $\pi_{s,0}$ -horizontal and 1-contact and in physics it is affine in variables $y_{j_1 \dots j_s}^\sigma$:

$$E = E_\sigma \omega^\sigma \wedge \omega_0 = (A_\sigma + B_{\sigma\nu}^{j_1 \dots j_s} y_{j_1 \dots j_s}^\nu) \omega^\sigma \wedge \omega_0, \quad E \in \Lambda_{n+1}(TJ^s Y), \quad (9)$$

where $\omega_0 = dx^1 \wedge \dots \wedge dx^n$ is the volume element on X and coefficients A_σ and $B_{\sigma\nu}$ are functions on $J^{s-1} Y$. A section γ is the trajectory of the physical system if it is a solution of the system of equation $E \circ J^s \gamma = 0$. A detailed explanation for mechanics see [20]), for field theory see e.g. [10].

The problem arises whether the equations of motion are variational, i.e. derivable from some variational principle, i.e. whether they are equations of extremals (stationary points) of certain functional (in detail see e.g. in [3, 4, 20] and many others)

$$S[\gamma] = \int_{\Omega} J^r \gamma^* \lambda, \quad \lambda = L(x^i, y^\sigma, y_j^\sigma, \dots, y_{j_r}^\sigma) \omega_0,$$

$\lambda = L \omega_0$ is the *Lagrangian*, $\Omega \subset X$ is a compact set, the pair (π, λ) is the *Lagrange structure*. The condition for extremals reads

$$\left. \frac{dS[\gamma_u]}{du} \right|_{u=0} = \left[\frac{d}{du} \int_{\alpha_{u0}(\Omega)} J^r \gamma_u^* \lambda \right]_{u=0} = \int_{\Omega} J^r \gamma^* \partial_{J^r \xi} \lambda = 0, \quad (10)$$

where $[\gamma_u]$, $u \in (\varepsilon, \varepsilon)$, is the family of sections $\gamma_u = \alpha_u \circ \gamma \circ \alpha_{u0}^{-1}$ defined on $\alpha_{u0}(\Omega)$ and ξ (the *variation*) is the π -projectable vector field on Y with one-parameter group (α_u, α_{u0}) . In such a case the corresponding dynamical form is the Euler-Lagrange form of the Lagrangian:

$$E_\lambda = p_1 d\theta_\lambda,$$

where θ_λ is the *Lepage equivalent of the Lagrangian* (the $\pi_{r+1,0}$ -horizontal form for which $h\theta_\lambda = \lambda$ up to a projection). For E_λ it holds (by direct calculation)

$$E_\lambda = \left[\sum_{\ell=0}^r (-1)^\ell d_{j_1} \dots d_{j_\ell} \left(\frac{\partial L}{\partial y_{j_1 \dots j_\ell}^\sigma} \right) \right] \omega^\sigma \wedge \omega_0, \quad (11)$$

where d_j denotes the total derivative operator with respect to x_j . For the r -th order Lagrangian its Euler-Lagrange form is of the order $2r$ in general. As examples typical for physics we present the *Cartan form* for $r = 1$, $n = 1$, $L = L(t, q^\sigma, \dot{q}^\sigma)$, and *Poincaré-Cartan form* for $r = 1$, $n > 1$, $L = L(x^i, y^\sigma, y_j^\sigma)$:

$$E_\lambda = \left(\frac{\partial L}{\partial q^\sigma} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^\sigma} \right) \omega^\sigma \wedge dt, \quad E_\lambda = \left(\frac{\partial L}{\partial y^\sigma} - \frac{d}{dx^i} \frac{\partial L}{\partial y_i^\sigma} \right) \omega^\sigma \wedge \omega_0.$$

The inverse problem means to find the solution of the system of equations $E_\lambda \circ J^{2r}\gamma^* = 0$, see (11), with respect to L , i.e. to find out whether there exist functions L such that equations (11) are fulfilled for given components of the dynamical form. If so, the dynamical form E is called *(locally) variational*. In such a case one can obtain by direct calculation based on the well-known Poincaré lemma the *Vainberg-Tonti Lagrangian*:

$$\Lambda = \left(y^\sigma \int_0^1 E_\sigma \circ \chi \, du \right) \omega_0, \quad (12)$$

$\chi : [0, 1] \times W_r \ni (u, x^i, y^\sigma, y_j^\sigma, \dots, y_{j_1 \dots j_r}^\sigma) \rightarrow (x^i, uy^\sigma, uy_j^\sigma, \dots, uy_{j_1 \dots j_r}^\sigma) \in J^r Y$, for $J_s = (j_1 \dots j_s)$, $1 \leq i, j, j_1, \dots, j_s \leq n$, $1 \leq s \leq r$, $1 \leq \sigma \leq m$, $W_r \subset J^r Y$ being a specific open set, e.g. an open ball.

The last problem is then to find the Lagrangian of the minimal possible order, called the *minimal Lagrangian*. It can be obtained by *order reducibility procedure*, see [20, 27].

4. Variational sequence and the inverse problem.

The importance of finite order variational sequence in physics was mentioned in Sec. 2. Now we briefly describe the essence of the variational sequence and its representation by forms, primarily with the use of its scheme. (For a deeper understanding of details, especially of local and global aspects of the issue, it is necessary to turn to the original sources already cited.) Simply speaking, the variational sequence representation in calculus of variations is an exact sequence of mappings (closely related to exterior derivative operator) of spaces of differential forms, which also includes the well-known *Euler-Lagrange mapping* assigning to a Lagrangian its Euler-Lagrange form. The kernel of this mapping is the subspace of trivial Lagrangians. The next mapping in the sequence, *Helmholtz-Sonin mapping*, assigns to a dynamical form its *Helmholtz-Sonin form*, while its kernel is formed by variational dynamical forms. The problem of finite order variational sequence representation by forms in field theories was generally and completely solved in [17]. There the reader will find a detailed solution with all proofs, including the proof of the uniqueness of the representation sequence of required properties.

The variational sequence on $J^r Y$ is defined as a quotient sequence of the well-known De Rham sequence on $J^r Y$ by its subsequence of forms of certain type of contactness, i.e. forms "variationally irrelevant".

In the scheme on Fig. 1, the symbol Ω_k^r , $0 \leq k \leq \dim Y$, $r \leq 1$, is the more common notation of $\Lambda_k(TJ^r Y)$ in variational sequence theory. The *De Rham sequence* of mappings $d_k : \Omega_k^r \rightarrow \Omega_{k+1}^r$ is exact ($\text{im } d_{k-1} \subseteq \ker d_k$). $\Omega_{k,c}^r$ is the space of k -contact k -forms, and $\Theta_k^r = \Omega_{k,c}^r + d\Omega_{k-1,c}^r$. The sequence of mappings $d_k : \Theta_k^r \rightarrow \Theta_{k+1}^r$ is the exact subsequence of De Rham sequence. Symbol Ω_k^r / Θ_k^r is the quotient of Ω_k^r with respect to Θ_k^r . The *short sequence*

$\{0\} \rightarrow^0 \Theta_k^r \rightarrow^1 \Omega_k^r \rightarrow^2 \Omega_k^r / \Theta_k^r \rightarrow^3 \{0\}$, where $0 \rightarrow^0 0_{\Theta_k^r}$, $\Theta_k^r \ni \eta \rightarrow^1 \eta \in \Omega_k^r$, $\Omega_k^r \ni \eta \rightarrow^2 [\eta] \in \Omega_k^r / \Theta_k^r$, $\Omega_k^r / \Theta_k^r \ni [\eta] \rightarrow^3 0$, is exact as well.

By the *variational sequence* itself we mean the sequence of mappings ($W \subset J^r Y$ is an open set)

$$E_k^r : \Omega_k^r W / \Theta_k^r W \ni [\eta] \longrightarrow E_k^r([\eta]) = [d\eta] \in \Omega_{k+1}^r W / \Theta_{k+1}^r W. \quad (13)$$

The variational sequence is also exact. The mappings E_n^r and E_{n+1}^r (*Euler-Lagrange mapping* and *Helmholtz-Sonin mapping*) are directly related to the calculus of variations. The relation of variational sequences of different order is given by immersions $\Omega_k^r / \Theta_k^r \rightarrow \Omega_k^{r+1} / \Theta_k^{r+1}$. These mappings are injective. Finally, the following sequence is exact:

$$\{0\} \longrightarrow \Omega_k^r / \Theta_k^r \longrightarrow \Omega_k^{r+1} / \Theta_k^{r+1} \longrightarrow (\Omega_k^{r+1} / \Theta_k^{r+1}) / (\Omega_k^r / \Theta_k^r)$$

Representation by forms: Representation of the variational sequence is the sequence of mappings for appropriately chosen representatives of classes $[\eta] \in \Omega_k^r / \Theta_k^r$ related to calculus of variations. The corresponding *representation mapping* $\mathcal{R}_k : \Omega_k^r / \Theta_k^r \ni [\eta] \rightarrow \mathcal{R}_k([\eta]) \in \Omega_k^s$ ($s \geq r$ in general) is based on following natural requirements: **R-linearity**, *exactness* (the corresponding sequence of spaces of representatives should be exact), *projection property* ($\mathcal{R}_k([\mathcal{R}_k([\eta])]) = \mathcal{R}_k([\eta])$). This means that we obtain the image of a class $[\eta]$ by applying a certain operator $\mathcal{I}$ to an arbitrarily chosen element of this class, and the result is always the same and uniquely determined ($\mathcal{I}^2 = \mathcal{I}$). The mappings $\mathcal{R}_k$ are as follows:

$$\begin{aligned} \mathcal{R}_k([\eta]) &= \mathcal{I}(\eta) = h\eta \text{ for } 0 \leq k \leq n, \quad s = r + 1, \\ \mathcal{R}_k([\eta]) &= \mathcal{I}(\eta) \text{ for } n + 1 \leq k = n + q \leq P, \quad s = 2r + 1, \\ \mathcal{R}_k([\eta]) &= \mathcal{I}(\eta) = \eta \text{ for } P + 1 \leq k \leq N, \quad s = r. \end{aligned} \quad (14)$$

Thus, for $0 \leq k \leq n$, $\mathcal{I}$ is horizontalization. For $P + 1 \leq k \leq$, $\mathcal{I}$ is identity. For $n + 1 \leq k = n + q \leq P$, $\mathcal{I}$ means the Euler operator

$$\mathcal{I} : \Omega_{n+q}^r W \ni \eta \longrightarrow \frac{1}{q} \omega^\sigma \wedge \sum_{|J|=0}^r (-1)^{|J|} d_J \left(\partial_{\partial/\partial y_J^\sigma} p_q \eta \right) \in \Omega_{n+q}^{2r+1} W. \quad (15)$$

It holds $(\pi^{2r+1})^* \eta - \mathcal{I}(\eta) \in \Theta_{n+q}^{2r+1} W$, $\mathcal{I}^2 = \mathcal{I}$ up to a projection, $\ker \mathcal{I} = \Theta_{n+q}^r W$. For mappings in the representation sequence ($W \subset J^r Y$)

$$\mathcal{E}_k : \mathcal{R}_k(\Omega_k^r W / \Theta_k^r W) \ni \mathcal{R}_k[\eta] \longrightarrow \mathcal{E}_k(\mathcal{R}_k([\eta])) \in \mathcal{R}_{k+1}(\Omega_{k+1}^r W / \Theta_{k+1}^r W)$$

the commutative diagram holds: $\mathcal{E}_k \circ \mathcal{R}_k([\eta]) = \mathcal{R}_{k+1} \circ E_k^r([\eta]) = \mathcal{R}_{k+1}([d\eta])$ (Fig. 2).

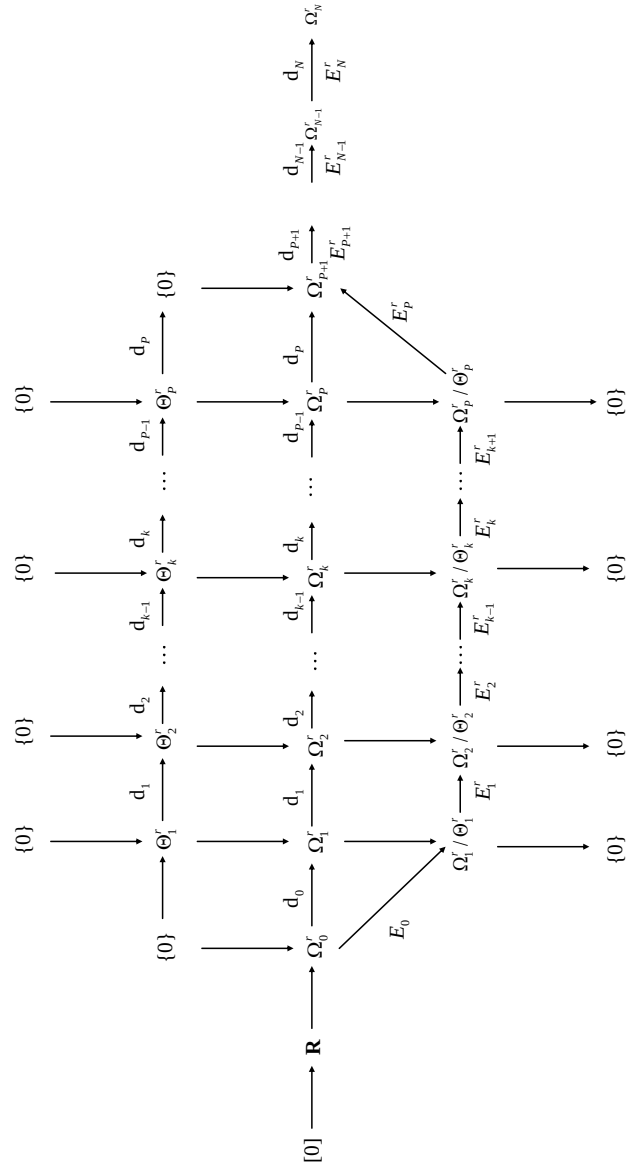


FIG. 1. Variational sequence scheme.

$$\begin{array}{ccccccc}
\longrightarrow & \Omega_k^r W / \Theta_k^r W & \xrightarrow{E_k^r} & \Omega_{k+1}^r W / \Theta_{k+1}^r W & \longrightarrow \\
& \downarrow \mathcal{R}_k & & \downarrow \mathcal{R}_{k+1} & \\
\longrightarrow & \mathcal{R}_k(\Omega_k^r W / \Theta_k^r W) & \xrightarrow{\varepsilon_k} & \mathcal{R}_{k+1}(\Omega_{k+1}^r W / \Theta_{k+1}^r W) & \longrightarrow
\end{array}$$

FIG. 2. Commutation diagram for E_k^r and $\mathcal{E}_k^r$.

The representation can be summarized in the following commuting diagram (Fig. 3). Vertical arrows assign to a form $\eta \in \Omega_k^r$ the unique form representing its class $[\eta]$.

$$\begin{array}{ccccccc}
\longrightarrow & \Omega_k^r W & \xrightarrow{d_k} & \Omega_{k+1}^r W & \longrightarrow \\
& \downarrow \mathcal{I} & & \downarrow \mathcal{I} & \\
\longrightarrow & \mathcal{R}_k(\Omega_k^r W / \Theta_k^r W) & \xrightarrow{\varepsilon_k} & \mathcal{R}_{k+1}(\Omega_{k+1}^r W / \Theta_{k+1}^r W) & \longrightarrow
\end{array}$$

FIG. 3. Representation sequence – k -th term.

By the direct calculation in charts using the rule (14) we can immediately see the meaning of the representation sequence for practical applications resulting from its exactness. The terms for n , $(n+1)$ and $(n+2)$ are important for calculus of variations in physics:

- The representative $\mathcal{R}_n([\lambda])$ of the class of a Lagrangian λ is the Lagrangian itself.
- The representative $\mathcal{R}_{n+1}([d\lambda])$ of the class $[d\lambda]$ is the Euler-Lagrange form of λ .
- λ is the trivial Lagrangian ($\lambda \in \ker \mathcal{R}_n$, if and only if $\mathcal{R}_{n+1}([d\lambda]) = 0$).
- The representative of the class of Euler-Lagrange dynamical form $\mathcal{R}_{n+1}([E_\lambda])$ is the form E_λ itself.
- The representative $\mathcal{R}_{n+2}([dE])$ of the class of exterior derivative of a dynamical form E is the Helmholtz-Sonin form H_E .

- The dynamical form E is variational, i.e. $E \in \ker \mathcal{R}_{n+1}$ if and only if $H_E = 0$.

The last item represents the solution of the inverse problem: Let ϱ be an arbitrary $(n+1)$ -form. The representative of the class $[\varrho]$, $\mathcal{R}_{n+1}([\varrho])$, is unique due to above formulated requirements for the representation and it is of the type of a dynamical form E (say of the s -th order in general, $E = E_\sigma \omega^\sigma \wedge \omega_0$.) Its Helmholtz-Sonin form, i.e. the representative of the class $[dE]$, $\mathcal{R}_{n+2}([dE]) = \mathcal{I}(d\mathcal{I}(E)) = \mathcal{I}(d\mathcal{I}(\varrho))$, is

$$\begin{aligned} \mathcal{I}(d\mathcal{I}(E)) = & \frac{1}{2} \sum_{\ell=0}^{2r+1} \left[\left(\frac{\partial E_\nu}{\partial y_{j_1 \dots j_\ell}^\sigma} - (-1)^\ell \frac{\partial E_\sigma}{\partial y_{j_1 \dots j_\ell}^\nu} - \right. \right. \\ & \left. \left. - \sum_{p=\ell+1}^r (-1)^p \binom{p}{\ell} d_{j_{\ell+1}} \dots d_{j_p} \frac{\partial E_\sigma}{\partial y_{j_1 \dots j_\ell j_{\ell+1} \dots j_p}^\nu} \right) \omega_{j_1 \dots j_\ell}^\sigma \right] \wedge \omega^\nu \wedge \omega_0. \end{aligned} \quad (16)$$

For history of the inverse problem up to year 2010 see a very well written paper [11].

5. Variational physics.

Using the properties of exactness of the variational sequence and its representation by forms presented in the previous section we have obtained the criteria (Helmholtz conditions of variationality) for deciding whether (or verifying that) a given system of equations of motion is variational or not. Now we will present an analysis of the equations of motion of key physical theories from the point of view of solving the inversion problem. We show their variationality, construct the Vainberg-Tonti Lagrangian and the corresponding minimal order Lagrangian.

The equations of motion of physical systems, whether in mechanics or in field theories, are standardly second-order affine in accelerations, i.e. in coordinates $\ddot{q}^\sigma$ (mechanics) or y_{ij}^σ , $1 \leq \sigma \leq m$, $1 \leq i, j \leq n$. The corresponding dynamical form is therefore defined on J^2Y . The Helmholtz conditions (16) obtain the following form:

$$\begin{aligned} \frac{\partial E_\nu}{\partial y_{ij}^\sigma} - \frac{\partial E_\sigma}{\partial y_{ij}^\nu} &= 0, \\ \frac{\partial E_\nu}{\partial y_i^\sigma} + \frac{\partial E_\sigma}{\partial y_i^\nu} - 2d_j \frac{\partial E_\sigma}{\partial y_{ij}^\nu} &= 0, \\ \frac{\partial E_\nu}{\partial y^\sigma} - \frac{\partial E_\sigma}{\partial y^\nu} + d_j \frac{\partial E_\sigma}{\partial y_j^\nu} - d_i d_j \frac{\partial E_\sigma}{\partial y_{ij}^\nu} &= 0. \end{aligned} \quad (17)$$

For mechanics

$$\frac{\partial E_\nu}{\partial \ddot{q}^\sigma} - \frac{\partial E_\sigma}{\partial \ddot{q}^\nu} = 0,$$

$$\begin{aligned} \frac{\partial E_\nu}{\partial \dot{q}^\sigma} + \frac{\partial E_\sigma}{\partial \dot{q}^\nu} - 2 \frac{d}{dt} \frac{\partial E_\sigma}{\partial \dot{q}^\nu} &= 0, \\ \frac{\partial E_\nu}{\partial q^\sigma} - \frac{\partial E_\sigma}{\partial q^\nu} + \frac{d}{dt} \frac{\partial E_\sigma}{\partial \dot{q}^\nu} - \frac{d^2}{dt^2} \frac{\partial E_\sigma}{\partial \ddot{q}^\nu} &= 0. \end{aligned} \quad (18)$$

There is an alternative and elegant way leading to Helmholtz-Sonin conditions of variationality in mechanics, see [20]. However, we consider the variational sequence as quite general and the most effective mean for solving the inverse problem.

Algebraic note: Index lifting or lowering by metric tensor: Only briefly: In general for a vector $F \in V$, V being a finite-dimensional vector space endowed with covariant metric tensor g , the corresponding covector is $F^\flat = i_F g$, where i_F is the contraction of g by F . And vice versa, for a covector $\omega \in V^*$ the corresponding vector is $\omega^\sharp = i_\omega \tilde{g}$, where $\tilde{g}$ is the induced contravariant metric tensor. The situation is extremely simple in Euclidean spaces in orthonormal bases, where the vector and covector components are equal, i.e. $F_j^\flat = F^j$, $\omega^\sharp_j = \omega_j$.

5.1 Classical mechanics. The typical form of equations of motion of the mechanical system with m degrees of freedom is $M g_{\sigma\nu} \ddot{q}^\nu = F_\sigma$, where $G = (g_{\sigma\nu})$, $1 \leq \sigma, \nu \leq m$, is the matrix of the covariant metric tensor and quantities F_σ are the components of the 1-form $F^\flat = F_\sigma dq^\sigma$ representing the forces acting on the mechanical system in consideration. (The vector field representing the resulting force is F .) Then the corresponding dynamical form reads

$$E = (F_\sigma - M g_{\sigma\nu} \ddot{q}^\nu) \omega^\sigma \wedge dt = (F_\sigma - M g_{\sigma\nu} \ddot{q}^\nu) dq^\sigma \wedge dt.$$

Because of the fact that the metric tensor is in principle symmetric and it depends only on coordinates, the first Helmholtz conditions in the set (18) are automatically fulfilled. The remaining conditions have the form

$$\frac{\partial F_\sigma}{\partial \dot{q}^\nu} + \frac{\partial F_\nu}{\partial \dot{q}^\sigma} = -2M \frac{\partial g_{\sigma\nu}}{\partial q^\mu} \dot{q}^\mu, \quad \frac{\partial F_\sigma}{\partial q^\nu} - \frac{\partial F_\nu}{\partial q^\sigma} = \frac{1}{2} \frac{d}{dt} \left(\frac{\partial F_\sigma}{\partial \dot{q}^\nu} - \frac{\partial F_\nu}{\partial \dot{q}^\sigma} \right). \quad (19)$$

It follows from the first set of equations (19) that F_σ is the 2-nd degree polynomial in variables $\dot{q}^\nu$ i.e.

$$F_\sigma = \Gamma_{\sigma\nu\mu} \dot{q}^\nu \dot{q}^\mu + \phi_{\sigma\nu} \dot{q}^\nu + \psi_\sigma, \quad (20)$$

where functions $\phi_{\sigma\nu}$ and ψ_σ can depend on time and coordinates only. Putting this expression into (19) we obtain

$$\Gamma_{\sigma\nu\mu} = -\frac{M}{2} \left(\frac{\partial g_{\sigma\mu}}{\partial q^\nu} + \frac{\partial g_{\nu\sigma}}{\partial q^\mu} - \frac{\partial g_{\mu\nu}}{\partial q^\sigma} \right), \quad \phi_{\sigma\nu} + \phi_{\nu\sigma} = 0, \quad (21)$$

$$\frac{\partial \phi_{\sigma\mu}}{\partial q^\nu} + \frac{\partial \phi_{\mu\nu}}{\partial q^\sigma} + \frac{\partial \phi_{\nu\sigma}}{\partial q^\mu} = 0, \quad \frac{\partial \psi_\sigma}{\partial q^\nu} - \frac{\partial \psi_\nu}{\partial q^\sigma} - \frac{\partial \phi_{\sigma\nu}}{\partial t} = 0. \quad (22)$$

The elements of the set $(\Gamma) = (\Gamma_{\sigma\nu\mu})$ with symmetries resulting from symmetries of G are the well-known Christoffel symbols. The quantities $(\phi) = (\phi_{\sigma\nu})$

have the character of the antisymmetric covariant tensor field (m independent components). For $m = 3$, i.e. $(Y, \pi, X) = (\mathbf{R} \times \mathbf{R}^3, p, \mathbf{R})$ (p being the cartesian projection on the first factor) the obtained conditions have the well-known sense: In the Euclidean space and Cartesian coordinates G is the unit matrix and then $(\Gamma) = (0)$ and $F_i = \phi_{ij}\dot{x}^j + \psi_i$, $1 \leq i, j \leq 3$. Thanks to second equation in (21) and transformation properties in such a special case the independent components of ϕ can be then interpreted as components of a vector field $\vec{\Phi} = (\Phi^1, \Phi^2, \Phi^3)$, $\phi_{23} = -\phi_{32} = \Phi^1$ and cycl. Then the first equation of (22) gives $\text{div } \vec{\Phi} = 0$, and thus $\vec{\Phi}$ has the form $\vec{\Phi} = \text{rot } \vec{A}$. Moreover, the second condition in (22) gives $\frac{\partial \vec{\Phi}}{\partial t} + \text{rot } \vec{\psi} = \vec{0}$, i.e. $\text{rot} \left(\frac{\partial \vec{A}}{\partial t} + \vec{\psi} \right) = \vec{0}$. Then there exists a function φ of time and coordinates such that $\frac{\partial \vec{A}}{\partial t} + \vec{\psi} = -\text{grad } \varphi$. The vector field $\vec{F}$ obeying the variationality conditions (20–22) can be then (denoting $\vec{v} = \dot{\vec{r}} = (\dot{q}^1, \dot{q}^2, \dot{q}^3)$),

$$\vec{F} = \vec{v} \times \text{rot } \vec{A} - \frac{\partial \vec{A}}{\partial t} - \text{grad } \varphi. \quad (23)$$

This is, of course, the well-known expression for a variational force. The Vainberg–Tonti Lagrangian

$$\Lambda = \left[q^\sigma \int_0^1 (u^2 (\Gamma_{\sigma\nu\mu} \circ \chi) \dot{q}^\nu \dot{q}^\mu) + u(\phi_{\sigma\nu} \circ \chi) \dot{q}^\nu + \psi_\sigma \circ \chi \right] du \, dt$$

depends of the concrete form of the metric tensor g and the free functions obeying conditions (22). (A general procedure to reduce the Vainberg–Tonti Lagrangian in mechanics to obtain the minimal one can be found in [20].)

Variationality of fictitious forces. The fictitious forces manifesting in non-inertial reference frames occur in equations of motion as additional forces guaranteeing the applicability of the 2nd Newton’s law. Consider again the fibred space $(\mathbf{R} \times \mathbf{R}^3, p, \mathbf{R})$ with the Euclidean metric tensor, i.e. in Cartesian coordinates (orthonormal base) we have $g_{\sigma\nu} = \delta_{\sigma\nu}$. Suppose that a non-inertial reference frame exhibits with respect to inertial ones the translational motion (acceleration $\vec{a}(t)$) and rotational motion (angular velocity $\vec{\omega}(t)$). The 2nd Newton law for a particle (mass M) then includes the fictitious force $\vec{F}^*$,

$$\vec{F}^* = -M (\vec{a} + 2\vec{\omega} \times \vec{v} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \vec{\varepsilon} \times \vec{r}),$$

where $\vec{r} = (q^1, q^2, q^3)$ and $\vec{v} = (\dot{q}^1, \dot{q}^2, \dot{q}^3)$ is the position vector and the velocity of the particle, respectively, and $\vec{\varepsilon} = \dot{\vec{\omega}}$. ($\vec{F}^*$ is the sum of the translational, Coriolis, centrifugal and Euler forces, $\vec{a}$ and $\vec{\omega}$ are only functions of time.) Suppose zero real forces. The equations of motion for which the variationality needs to be verified are

$$M \frac{d^2 \vec{r}}{dt^2} = \vec{F}^* \implies E = (F_\sigma^* - M \delta_{\sigma\nu} \ddot{q}^\nu) dq^\sigma \wedge dt.$$

Denoting

$$\begin{aligned}\vec{A} &= M\vec{\omega} \times \vec{r}, \text{ then } \frac{\partial \vec{A}}{\partial t} = M\vec{\varepsilon} \times \vec{r}, \\ \varphi &= M\vec{a}\vec{r} - \frac{M}{2}(q^1 \ q^2 \ q^3) \begin{pmatrix} \omega_2^2 + \omega_3^2 & -\omega_1\omega_2 & -\omega_1\omega_3 \\ -\omega_1\omega_2 & \omega_1^2 + \omega_3^2 & -\omega_2\omega_3 \\ -\omega_1\omega_3 & -\omega_2\omega_3 & \omega_1^2 + \omega_2^2 \end{pmatrix} \begin{pmatrix} q^1 \\ q^2 \\ q^3 \end{pmatrix} = \\ &= Ma_\sigma q^\sigma - \frac{1}{2}M\Omega_{\sigma\nu}q^\sigma q^\nu,\end{aligned}$$

we can easily verify that for $\vec{F}^*$ it holds the relation (23).

For the Vainberg–Tonti Lagrangian $\Lambda = \mathcal{L} dt$ and the minimal Lagrangian $\lambda = L dt$ we have

$$\mathcal{L} = \left(-\frac{1}{2}M\vec{r}\ddot{\vec{r}} + \vec{A}\vec{v} - \varphi \right), \quad L = \frac{1}{2}M\vec{v}^2 + \vec{A}\vec{v} - \varphi, \quad L = \mathcal{L} + \frac{d}{dt} \left(\frac{1}{2}M\vec{r}\dot{\vec{r}} \right).$$

Using (11) for $n = 1$ it is easy to verify that both Lagrangians Λ ($r=2$) and λ ($r=1$) lead to the same dynamical form E .

Note: It is usual to denote $\text{rot } \vec{A} = Q\vec{B}$, $-\left(\frac{\partial \vec{A}}{\partial t} + \varphi\right) = Q\vec{E}$, Q being a constant. Then we can write for variational forces

$$\vec{F} = Q\vec{E} + Q\vec{v} \times \vec{B}, \quad \text{div } \vec{B} = 0, \quad \text{rot } \vec{E} = -\frac{\partial \vec{B}}{\partial t}. \quad (24)$$

Without attributing any meaning to the quantities $\vec{E}$ and $\vec{B}$ at this point, we have obtained the known result (obtained by other means): We see that the variational forces in classical mechanics are of the Lorentzian type.

5.2 Classical electrodynamics. Basic equations of motion in electrodynamics are of course the well-known Maxwell equations. Their vacuum form is, with usual meaning of quantities and additional relations

$$\begin{aligned}\vec{B} &= \text{rot } \vec{A}, \quad \vec{E} = -\text{grad } \varphi - \frac{\partial \vec{A}}{\partial t}, \\ \text{div } \vec{E} &= \frac{\rho}{\varepsilon_0}, \quad \text{rot } \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \quad \text{div } \vec{B} = 0, \quad \text{rot } \vec{B} = \varepsilon_0\mu_0 \frac{\partial \vec{E}}{\partial t} + \vec{j}.\end{aligned} \quad (25)$$

For proving their variationality and finding both, Vainberg-Tonti and minimal Lagrangian, we will start from their four-dimensional formulation in the contravariant form (see the classical textbook [30]). The character of second and third of them is given by (24) with the relevant interpretation of quantities $\vec{E}$ and $\vec{B}$.¹

¹Note that the magnetic induction is not the vector in the true sense of the word, but, thanks to its transformational properties, is the antisymmetric 2nd order mixed tensor.

The underlying fibred manifold is $(Y, \pi, X) = (\mathbf{R}^4 \times \mathbf{R}^4, p, \mathbf{R}^4)$ endowed with Minkowski metric tensor $g = G_{ij}dx^i \otimes dx^j$ (diagonal in orthonormal bases connected e.g. with standard coordinates $(x^0, x^1, x^2, x^3) \in X$, where $x^0 = ct$ is the time coordinate and (x^i) , $i = 1, 2, 3$, are Cartesian coordinates, $\text{diag } G = (1, -1, -1, -1)$). Coordinates on fibres, A_0, A_1, A_2, A_3 , $A_0 = c^{-1}\varphi$, are components of 1-form $A = A_i dx^i$. (The corresponding vector $\vec{A} = A^\sharp$ (four-potential) has components (A^0, A^1, A^2, A^3) , $A^i = \tilde{g}^{ij}A_j$, where $\tilde{g}$ is the induced contravariant metric tensor, $\tilde{G} = G^{-1}$. So, $A^0 = A_0$, $A^i = -A_i$ for $1 \leq i \leq 3$.) Coordinates on (J^2Y, π_2, X) are $(x^i, A_i, A_{i,j}, A_{i,jk})$, $0 \leq i, j, k \leq 3$.

The Maxwell equations with sources (first and last in (25)) expressed in the four-dimensional contravariant version have the form (see [30])

$$E^i = -\frac{1}{c} \left(j^i + \frac{1}{\mu_0} \frac{\partial \mathcal{F}^{ij}}{\partial x^j} \right) = 0, \quad (26)$$

where $\mathcal{F} = (\mathcal{F}^{ij})$, $0 \leq i, j \leq 3$, is the contravariant electromagnetic field tensor defined as

$$\mathcal{F}^\flat = \mathcal{F}_{k\ell} dx^k \wedge dx^\ell, \quad \mathcal{F}_{k\ell} = \frac{\partial A_\ell}{\partial x^k} - \frac{\partial A_k}{\partial x^\ell}, \quad \mathcal{F}^{ij} = g^{ik}g^{j\ell}\mathcal{F}_{k\ell}. \quad (27)$$

For E^i we now verify the conditions of variationality (17). It holds

$$\begin{aligned} E^i &= -\frac{1}{c} \left[j^i + \frac{1}{\mu_0} g^{ik}g^{j\ell}(A_{\ell,kj} - A_{k,\ell j}) \right]. \\ \frac{\partial E^i}{\partial A_{p,rs}} &= -\frac{1}{\mu_0 c} g^{ik}g^{j\ell}(\delta_{\ell p}\delta_{kr}\delta_{js} - \delta_{kp}\delta_{\ell r}\delta_{js}) = -\frac{1}{\mu_0 c} (g^{ir}g^{sp} - g^{ip}g^{sr}) \\ \frac{\partial E^p}{\partial A_{i,rs}} &= -\frac{1}{\mu_0 c} g^{pk}g^{j\ell}(\delta_{\ell i}\delta_{kr}\delta_{js} - \delta_{ki}\delta_{\ell r}\delta_{js}) = -\frac{1}{\mu_0 c} (g^{pr}g^{si} - g^{pi}g^{sr}) \end{aligned}$$

Taking into account the symmetry $A_{j,rs} = A_{j,sr}$ for all j, r, s we can see that the last set of conditions (17) is fulfilled. The remaining conditions are then fulfilled trivially.

The Vainberg-Tonti Lagrangian is

$$\begin{aligned} \Lambda &= \left(A_i \int_0^1 (E^i \circ \chi) du \right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3, \\ \chi(x^i, A_j, A_{j,k}, A_{j,k\ell}) &= (x^i, uA_j, uA_{j,k}, uA_{j,k\ell}). \\ \Lambda &= -\frac{1}{c} A_i \left(j^i + \frac{1}{2\mu_0} g^{ik}g^{j\ell}(A_{\ell,kj} - A_{k,\ell j}) \right) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3. \end{aligned}$$

Subtracting the trivial Lagrangian

$$-\frac{1}{2\mu_0 c} g^{ik}g^{j\ell} d_j [A_i(A_{\ell,k} - A_{k,\ell})] \omega_0 =$$

$$= -\frac{1}{2\mu_0 c} g^{ik} g^{j\ell} [A_{i,j}(A_{\ell,k} - A_{k,\ell}) + A_i(A_{\ell,kj} - A_{k,\ell j})] \omega_0$$

and taking into account that

$$A_{i,j} \mathcal{F}^{ij} = -A_{j,i} \mathcal{F}^{ij} \Rightarrow A_{i,j} \mathcal{F}^{ij} = -\frac{1}{2} \mathcal{F}_{ij} F^{ij}$$

resulting from the antisymmetry of $\mathcal{F}$ we obtain the minimal order Lagrangian identical with [30]

$$\lambda = -\frac{1}{c} \left(j^i A_i + \frac{1}{4\mu_0} \mathcal{F}_{ik} \mathcal{F}^{ik} \right) \omega_0. \quad (28)$$

Using calculations similar to the previous ones, it can be easily verified that both, Λ and λ , lead to the original dynamical (Euler-Lagrange) form E and thus to the well-known equations of motion [30].

5.3 Wave equation: The presentation of wave theory is standardly based directly on the equation of motion – the wave equation (for electromagnetic field resulting from three-dimensional Maxwell equations). Suppose the scalar wave $w = w(t, x^1, x^2, x^3)$, function of time and Cartesian coordinates,

$$\Delta w - \frac{1}{v^2} \frac{\partial^2 w}{\partial t^2} = 0, \quad v \text{ is the phase speed.}$$

The adequate geometric basis for this theory is the fibred manifold $(Y, \pi, X) = (\mathbf{R}^4 \times \mathbf{R}, \pi, \mathbf{R}^4)$, $n = 4, m = 1$, π is the Cartesian projection $Y \rightarrow X$, the charts are (V, ψ) , $\psi = (x^0, x^1, x^2, x^3, w)$, $x^0 = t$, $(\pi(V), \varphi)$, $\varphi = (t, x^1, x^2, x^3)$, associated fibred charts on $J^1 Y$ and $J^2 Y$ are $\psi_1 = (x^i, w, w_i)$ and $\psi_2 = (x^i, w, w_i, w_{ij})$, $i, j = 0, 1, 2, 3$. The corresponding dynamical form is

$$E = E_1 \omega^1 \wedge \omega_0 = \left(-\frac{1}{v^2} w_{00} + w_{11} + w_{22} + w_{33} \right) \omega^1 \wedge dt \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

The conditions (17) are trivially fulfilled. The Vainberg-Tonti Lagrangian is

$$\Lambda = \left[w \int_0^1 \left(-\frac{1}{v^2} w_{00} + w_{11} + w_{22} + w_{33} \right) u \, du \right] \omega_0 = \frac{1}{2} w \left(-\frac{1}{v^2} w_{00} + w_{11} + w_{22} + w_{33} \right) \omega_0.$$

This is the 2nd order Lagrangian. The minimal (1st order) Lagrangian we obtain by subtracting the trivial Lagrangian

$$\frac{1}{2} \left[-\frac{1}{v^2} d_0(ww_0) + d_1(ww_1) + d_2(ww_2) + d_3(ww_3) \right] \omega_0,$$

$$\lambda = \frac{1}{2} \left(-\frac{1}{v^2} w_0^2 + w_1^2 + w_2^2 + w_3^2 \right) \omega_0.$$

Verifying the variationality in this case was trivial. Note that both Lagrangians, λ and Λ lead to the wave equation:

$$\frac{\partial L}{\partial w} - d_i \frac{\partial L}{\partial w_i} = \frac{\partial \mathcal{L}}{\partial w} - d_i \frac{\partial \mathcal{L}}{\partial w_i} + d_i d_j \frac{\partial \mathcal{L}}{\partial w_{ij}} = \left(-\frac{1}{v^2} w_{00} + w_{11} + w_{22} + w_{33} \right) = E_1.$$

5.4 Quantum mechanics: The equation of motion of a quantum mechanical system is the Schrödinger equation. We will start from its expression in coordinate representation and, for simplicity, consider the one-dimensional motion of a free particle. (In the case of a particle in three-dimensional space, the underlying fibred manifold differs only in the dimension of the base X , and the equation of motion is symmetric with respect to the interchange of Cartesian coordinates.) The essence of the problem will be captured even in this simplified model.

The state of the particle is given by the wave function (complex function of time and coordinate, $x^1 = t, x^2 = x$) $\phi(t, x) = v(t, x) + iw(t, x)$, $v(t, x) = \text{Re } \phi(t, x)$, $w(t, x) = \text{Im } \phi(t, x)$. The underlying fibred manifold is $(\mathbf{R}^2 \times \mathbf{R}^2, \pi, \mathbf{R}^2)$, local charts on (Y, π, X) , $X, (J^1 Y, \pi_1, X), (J^2 Y, \pi_2, X)$: $\psi = (t, x, v, w)$, $\varphi = (t, x)$, $\psi_1 = (t, x, v, w, v_t, w_t, v_x, w_x)$, $\psi_2 = (t, x, v, w, v_t, w_t, v_x, w_x, v_{tt}, v_{tx}, v_{xx}, w_{tt}, w_{tx}, w_{xx})$. Schrödinger equation reads

$$i\hbar \frac{\partial \psi}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} = 0 \implies -\hbar \frac{\partial w}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 v}{\partial x^2} = 0, \quad \hbar \frac{\partial v}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2 w}{\partial x^2} = 0,$$

$$E = E_1 \omega^1 \wedge dt \wedge dx + E_2 \omega^2 \wedge dt \wedge dx,$$

$$E = \left(-\hbar w_t + \frac{\hbar^2}{2m} v_{xx} \right) \omega^1 \wedge dt \wedge dx + \left(\hbar v_t + \frac{\hbar^2}{2m} w_{xx} \right) \omega^2 \wedge dt \wedge dx.$$

The calculation proving that the dynamical form E again fulfills the conditions (17) is very easy. The Vainberg-Tonti Lagrangian reads

$$\Lambda = \left[v \int_0^1 \left(-\hbar w_t + \frac{\hbar^2}{2m} v_{xx} \right) u \, du + w \int_0^1 \left(\hbar v_t + \frac{\hbar^2}{2m} w_{xx} \right) u \, du \right] dt \wedge dx,$$

$$\Lambda = \left(\frac{\hbar^2}{4m} (v v_{xx} + w w_{xx}) - \frac{\hbar}{2} (v w_t - w v_t) \right) dt \wedge dx.$$

Subtracting the trivial Lagrangian $\frac{\hbar^2}{4m} d_x(v v_x + w w_x) dt \wedge dx$ and taking into account that that $v = \frac{1}{2}(\phi + \phi^*)$, $w = \frac{1}{2i}(\phi - \phi^*)$ we obtain the 1st order (minimal) Lagrangian

$$\begin{aligned} \lambda &= \left[-\frac{\hbar^2}{4m} (v_x^2 + w_x^2) - \frac{\hbar}{2} (v w_t - w v_t) \right] dt \wedge dx, \\ \lambda &= \left[-\frac{\hbar^2}{4m} \phi_x \phi_x^* + \frac{i\hbar}{4} (\phi^* \phi_t - \phi \phi_t^*) \right] dt \wedge dx. \end{aligned} \quad (29)$$

The direct calculation leading to the verification of equations of motion from Λ and λ is again simple, e.g. for λ :

$$\frac{\partial L}{\partial v} - d_t \frac{\partial L}{\partial v_t} - d_x \frac{\partial L}{\partial v_x} = -\hbar w_t + \frac{\hbar^2}{2m} v_{xx},$$

$$\frac{\partial L}{\partial w} - d_t \frac{\partial L}{\partial w_t} - d_x \frac{\partial L}{\partial w_x} = \hbar v_t + \frac{\hbar^2}{2m} w_{xx}.$$

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