Representation Theory[1]

- **Representation of a group**: A set of square, non-singular matrices $\{D(g)\}$ associated with the elements of a group $g \in G$ such that if $g_1 g_2 = g_3$ then $D(g_1)D(g_2) = D(g_3)$. That is, D is a *homomorphism*. The (m, n) entry of the matrix $D(g)$ is denoted $D_{mn}(g)$.

- **Identity representation matrix**: If e is the identity element of the group, then $D(e) = \mathbf{1}$ (the identiy matrix).

- **Identity (trivial) representation**: $D(g) = 1$ for all $g \in G$.

- **Faithful representation**: All $D(g)$ are distinct (D is an *isomorphism*).

- **Dimension of a representation**: The order d of the matrices ($d \times d$ matrices have dimension d).

- **Characters of a representation**: Set of traces $\chi(g) = \operatorname{tr} D(g)$. Note that $\chi(e) = d$, where d is the dimension of the representation and e is the identity element.

- **Equivalent representations**: Two representations D and D' are *equivalent* if they are related by a **similarity transformation** (invertible matrix) S: i.e., $D'(g) = SD(g)S^{-1}$ for all $g \in G$. Note that D' is a representation of the group for every such S, and that the traces $\chi'(g) = \operatorname{tr} D'(g) = \chi(g)$ are unaffected (trace = sum of eigenvalues, which are invariant under S).

- **Inequivalent representations**: Representations D and D' for which it is impossible to find a similarity transform S relating them.

- **Unitary representation**: A representation such that $D(g)$ is unitary for all g; i.e., $D(g)^\dagger = D(g)^{-1}$, where $\dagger$ denotes the conjugate-transpose (adjoint).
  - For a *finite* group, *every* representation is equivalent to a unitary representation by some similarity transformation, so *we can restrict ourselves to unitary representations without loss of generality.*

- **Reducible representation**: A representation that is *equivalent* to a representation having a block-diagonal form:

$$\left\{ \begin{pmatrix} D^{(1)}(g) & \mathbf{0} \\ \mathbf{0} & D^{(2)}(g) \end{pmatrix} \right\},$$

  for all $g \in G$, where both $D^{(1)}$ and $D^{(2)}$ are representations.

- **Irreducible representation**: A representation that is not reducible; i.e. it is impossible to find a similarity transformation that reduces *all* of its matrices *simultaneously* to block form.
  - A reducible representation can be reduced (decomposed) into a number of irreducible representations.
  - We only care about inequivalent, irreducible representations. The set of irreducible representations is well-known for any group we will encounter.

- **Class of elements**: A non-empty subset of elements $C \subseteq G$ forms a *class* (or *conjugacy class*) if it consists of elements that are all conjugate to one another, and which are not conjugate to anything *not* in C. Two elements g_1 and g_2 of G are **conjugate** if there exists a $g \in G$ such that $g_1 = g^{-1} g_2 g$.
  - D has the *same trace* for all elements of C, since $D(g_1)$ and $D(g_2)$ are related by a similarity transformation $D(g_1) = D(g)^{-1}D(g_2)D(g)$.
  - If an element g_0 in the group commutes with all of the elements of G then it forms a class by itself, since $g^{-1}g_0 g = g^{-1}g g_0 = g_0$. Thus, the identity e is always in its own class.

[1]For proofs and more information, see e.g.: T. Inui, Y. Tanabe, and Y. Onodera, *Group Theory and Its Applications in Physics* (Springer: New York, 1996).

- **The Great Orthogonality Theorem**: Denote the inequivalent irreducible representations of G by $D^{(\alpha)}$, where $\alpha = 1, \cdots, n_r$. Then:

$$\sum_{g \in G} D^{(\alpha)}_{mn}(g)^* D^{(\alpha')}_{m'n'}(g) = \frac{|G|}{d_\alpha} \delta_{\alpha\alpha'} \delta_{mm'} \delta_{nn'},$$

where $|G|$ is the number of elements in G, d_α is the dimension of the representation $D^{(\alpha)}$, and δ_{ij} is the Kronecker delta ($= 1$ if $i = j$, $= 0$ otherwise).

- **Character table**: The table of characters (traces) associated with each class (columns of the table) and each irreducible representation (rows of the table). The entries of the table obey the following rules, and in fact can often be constructed directly from these rules *without knowing the representations*:

  – Number of irreducible representations n_r = number of classes n_c.

  – $\sum_\alpha d_\alpha^2 = |G|$. (This severely restricts the dimensions of the representations.)

  – From the trace ($\sum_{m=m'} \sum_{n=n'}$) of the orthogonality theorem we find that the *rows* of the character table are orthogonal to one another, when scaled by the number of elements in each class:

$$\begin{aligned} |G|\delta_{\alpha\alpha'} &= \sum_{g \in G} \chi^{(\alpha)}(g)^* \chi^{(\alpha')}(g) \\ &= \sum_{C_i} \chi^{(\alpha)}(C_i)^* \chi^{(\alpha')}(C_i) |C_i|, \end{aligned}$$

  where the C_i are the classes (with $|C_i|$ elements), using the fact that every element of a class has the same trace.

  – It also turns out that the *columns* of the character table are orthogonal:

$$\sum_\alpha \chi^{(\alpha)}(C_i)^* \chi^{(\alpha)}(C_j) = \delta_{ij} \frac{|G|}{|C_i|}.$$

  – Finally, we can define the product of two classes $C_i C_j = (\sum_{g_i \in C_i} g_i) \cdot (\sum_{g_j \in C_j} g_j)$. It turns out that this product always consists of classes C_k whose elements all appear $m_{ij}^{(k)}$ times for some integer $m_{ij}^{(k)}$: $C_i C_j = \sum_k m_{ij}^{(k)} C_k$. Then, it turns out that the following relation holds:

$$\begin{aligned} &|C_i|\chi^{(\alpha)}(C_i) \cdot |C_j|\chi^{(\alpha)}(C_j) \\ &\quad = d_\alpha \sum_k m_{ij}^{(k)} |C_k| \chi^{(\alpha)}(C_k). \end{aligned}$$

  This relation is sometimes needed, in addition to the previous rules, to determine the character table uniquely.

- **Partner function**: A set $\{\phi_i^{(\alpha)}(\mathbf{x})\}$of functions that transform according to $D^{(\alpha)}$, with $i = 1, \cdots, d_\alpha$. That is, if $\hat{O}_g$ is the operator that transforms ϕ according to $g \in G$, then

$$\hat{O}_g \phi_j^{(\alpha)} = \sum_i \phi_i^{(\alpha)} D_{ij}^{(\alpha)}(g)$$

for all $g \in G$.

  – Different partner functions are *orthogonal*: If $\phi_i^{(\alpha)}$ and $\psi_{i'}^{(\alpha')}$ are partner functions in a Hilbert space, then $\langle \phi_i^{(\alpha)} | \psi_{i'}^{(\alpha')} \rangle = 0$ if $i \neq i'$ or $\alpha \neq \alpha'$.

- **Projection operator**: Any function $\psi(\mathbf{x})$ can be decomposed $\psi = \sum_\alpha \sum_i c_i^{(\alpha)} \phi_i^{(\alpha)}$ as a sum of components $\phi_i^{(\alpha)}$ that are partner functions of $D^{(\alpha)}$, with some expansion coefficients $c_i^{(\alpha)}$. These components can be found via $c_i^{(\alpha)} \phi_i^{(\alpha)} = \hat{P}_i^{(\alpha)} \psi$, where $\hat{P}_i^{(\alpha)}$ is the *projection operator*

$$\hat{P}_i^{(\alpha)} = \frac{d_\alpha}{|G|} \sum_{g \in G} D_{ii}^{(\alpha)}(g)^* \hat{O}_g.$$

The operator $\hat{P}^{(\alpha)} = \sum_i \hat{P}_i^{(\alpha)} = \frac{d_\alpha}{|G|} \sum_g \chi^{(\alpha)}(g)^* \hat{O}_g$ projects ψ onto its components that transform according to the representation $D^{(\alpha)}$.