

*“You observe that in the life of the intellect there is also a law of inertia. Everything continues to move along its old rectilinear path, and every change, every transition to new and modern ways, meets strong resistance.”*

*Felix Klein (1849-1925)*

## Chapter 5

### Curve Fitting

Collected data points  $(x_i, y_i)$  for  $i = 0, 1, 2, \dots, n$  can be plotted on some type of coordinate paper and then some type of equation can be constructed that will represent the data points in an analytical fashion to produce a graphical  $(x, y)$ -representation of the data. For example, in the previous chapter we showed how one can construct polynomial functions which pass through the data points. In this chapter we will not restrict ourselves to polynomials which pass through the data points. We consider instead how to construct equations which represent the data in some “best” way. The numerical procedure for constructing a curve representing the data points is called curve fitting and the equation or equations for representing the curve is called an empirical curve fit. In curve fitting the more experience you have in recognizing certain “basic curve shapes” the easier it becomes to construct an empirical curve representing the data. The figure 5-1 illustrates certain basic curve shapes that the reader should learn to recognize.

#### Special Graph Paper

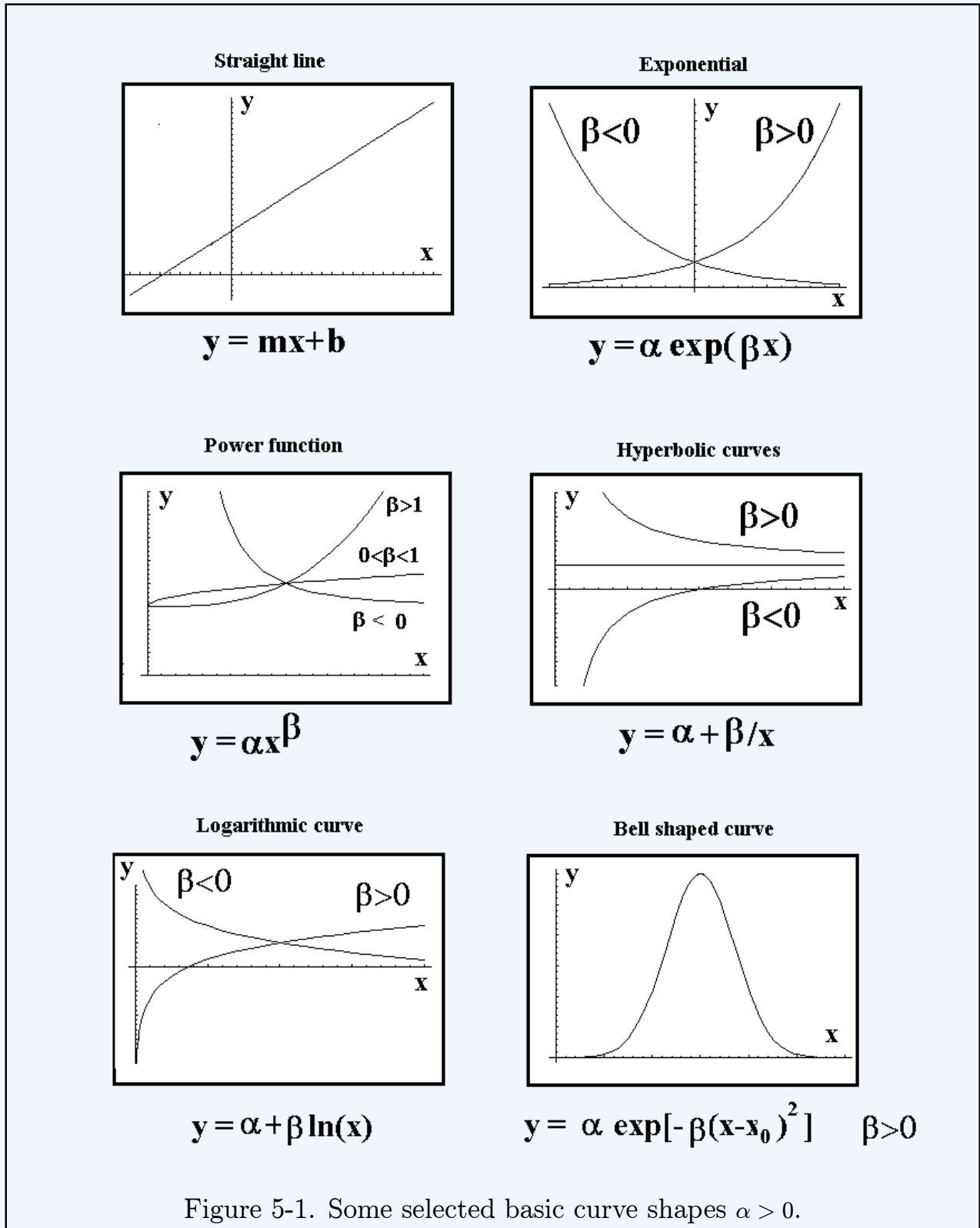
The straight line is the easiest curve to recognize and consequently many special types of graph paper are constructed in order to transform special types of curves into a straight line. For example, take the natural logarithm of the exponential curves  $y = \alpha e^{mx}$  to obtain

$$\ln y = \ln \alpha + mx. \quad (5.1)$$

Now let  $Y = \ln y$  and  $b = \ln \alpha$  to obtain

$$Y = mx + b. \quad (5.2)$$

This demonstrates that data of the form of an exponential curve will plot as a straight line on semi-log paper where one axes is logarithmic. Various examples of semi-log paper are illustrated in the figures 5-2, 5-3 and 5-4.



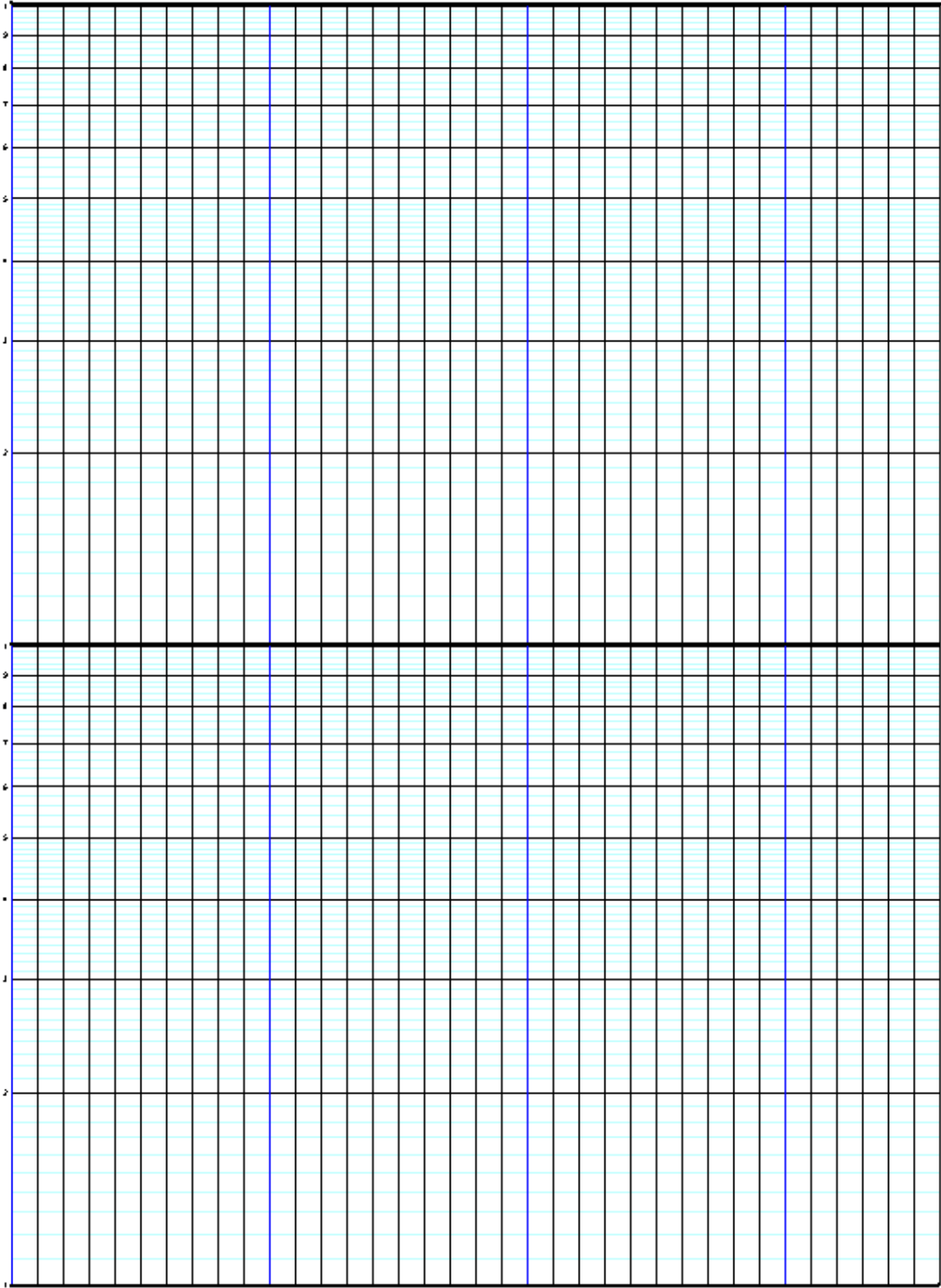


Figure 5-2. 2-Cycle, 34 division Semi-log paper.

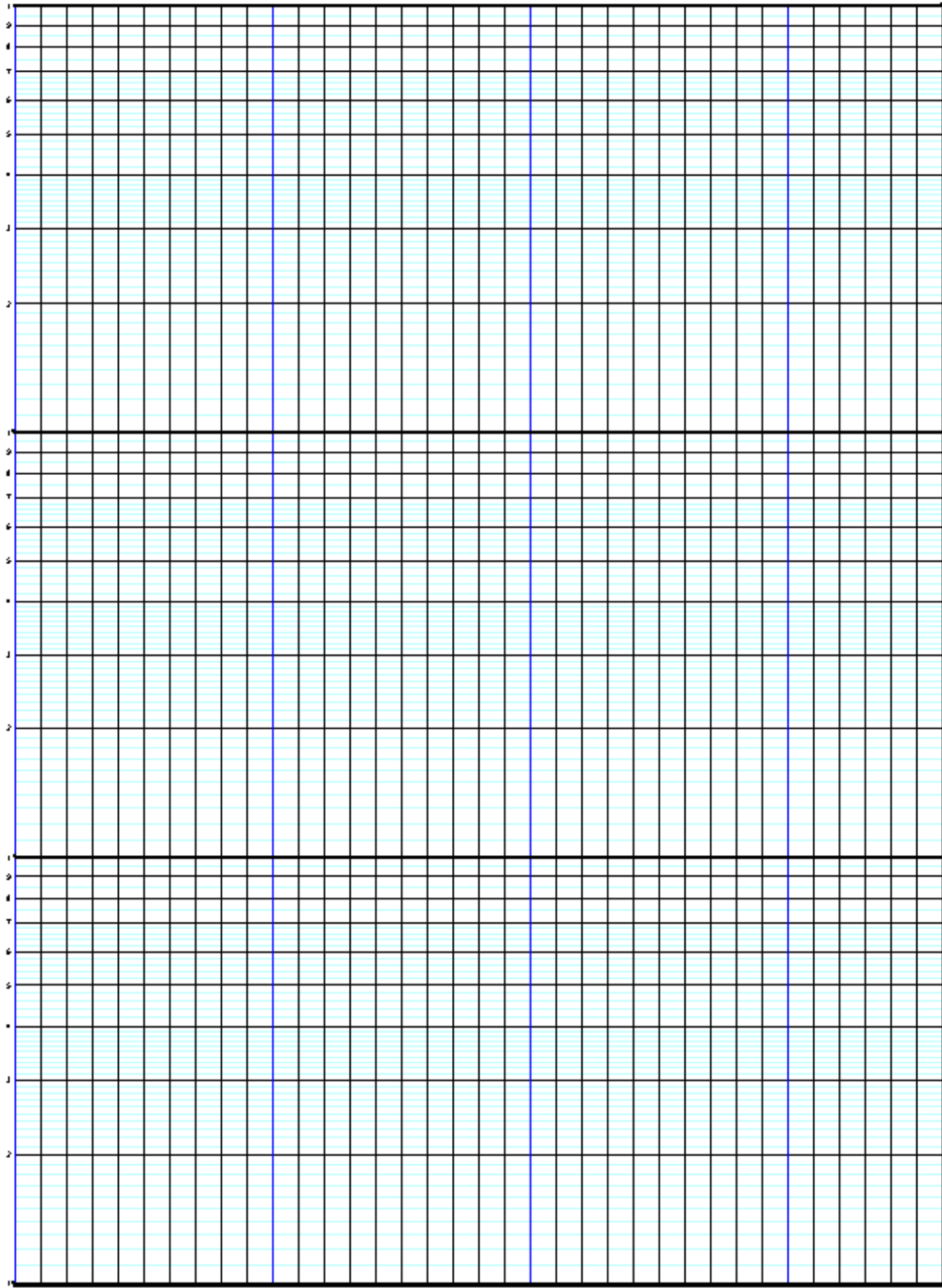


Figure 5-3. 3-Cycle, 34 division Semi-log paper.

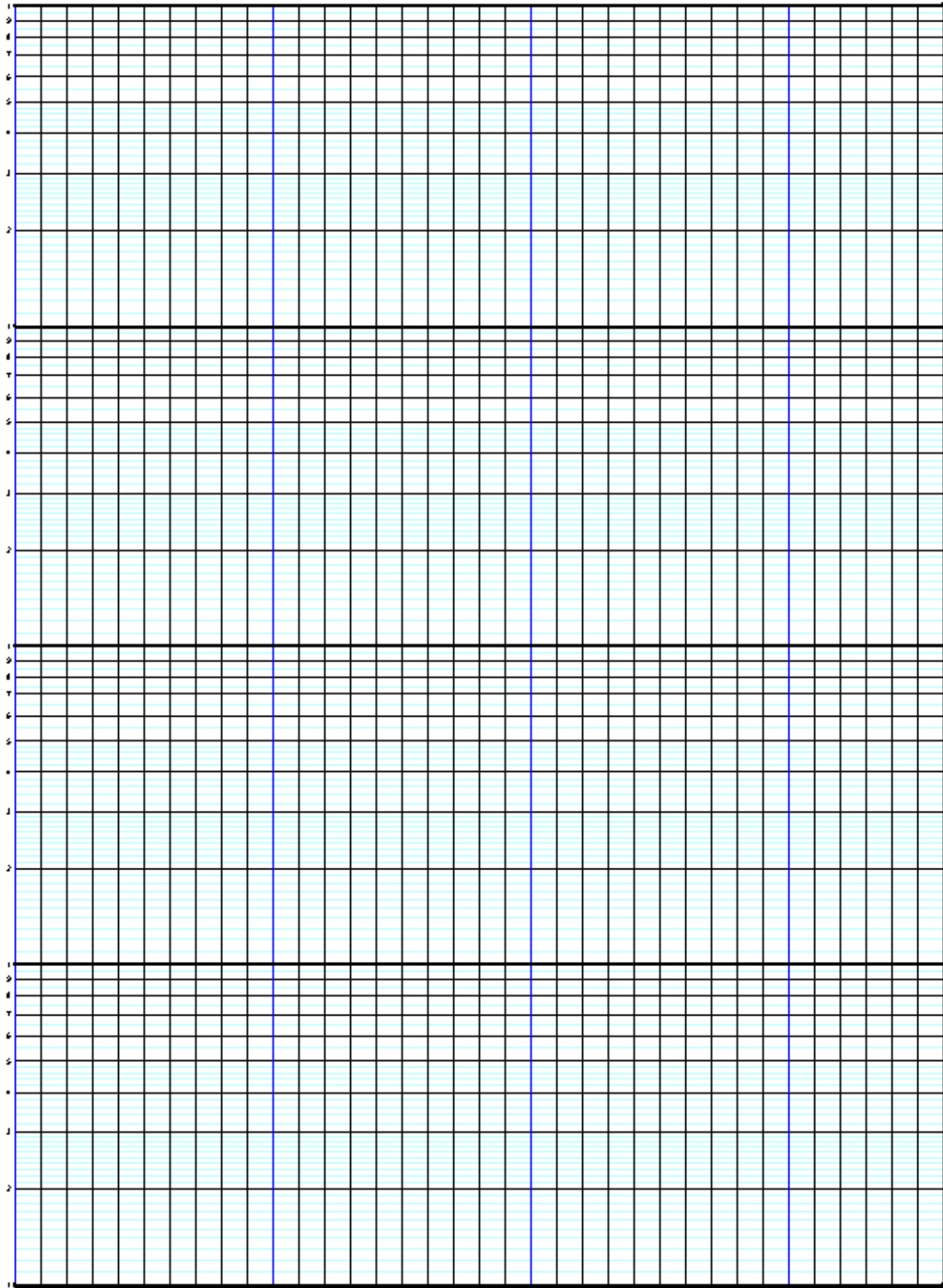


Figure 5-4. 4-Cycle, 34 division Semi-log paper.

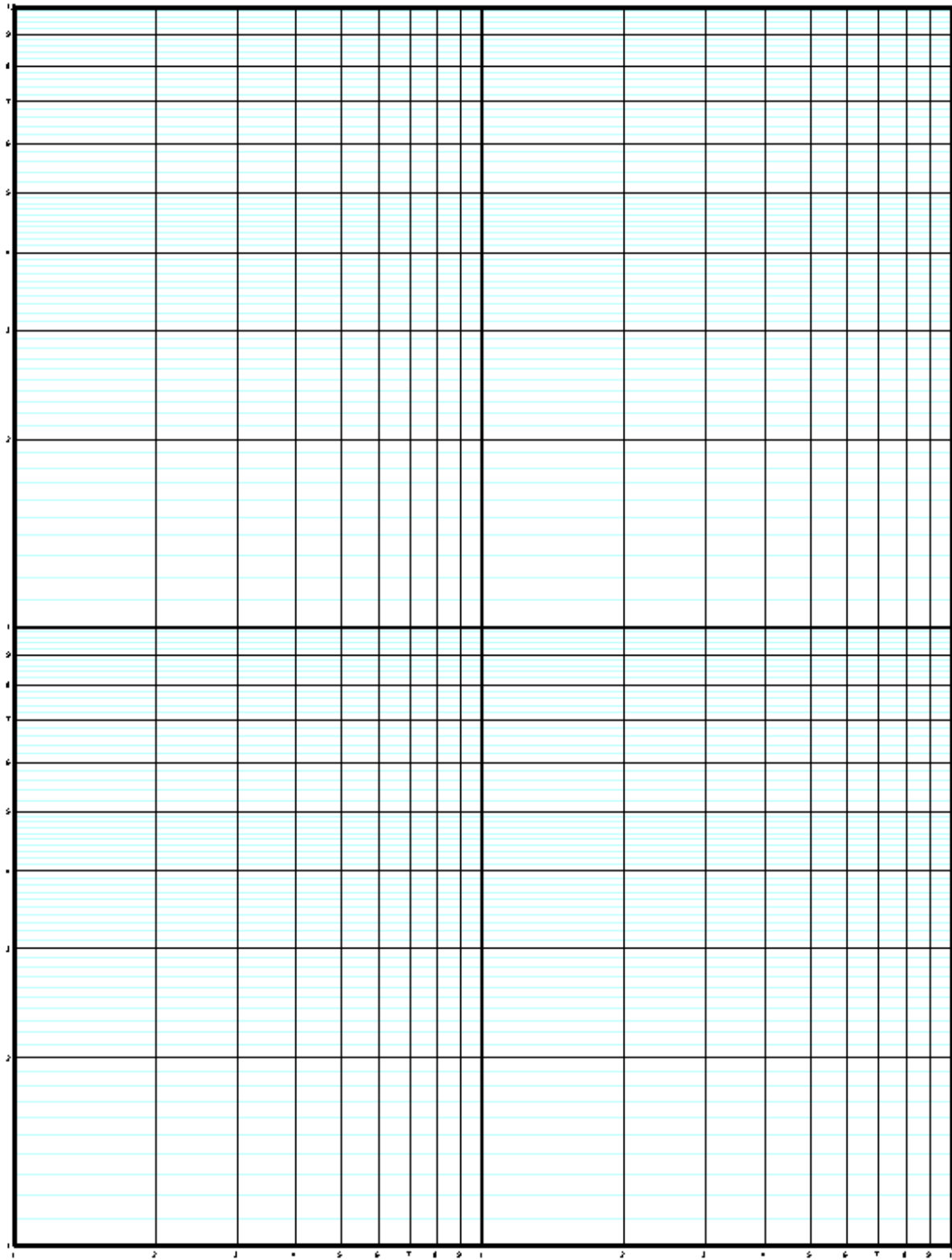


Figure 5-5. 2 × 2 Cycle log-log paper.

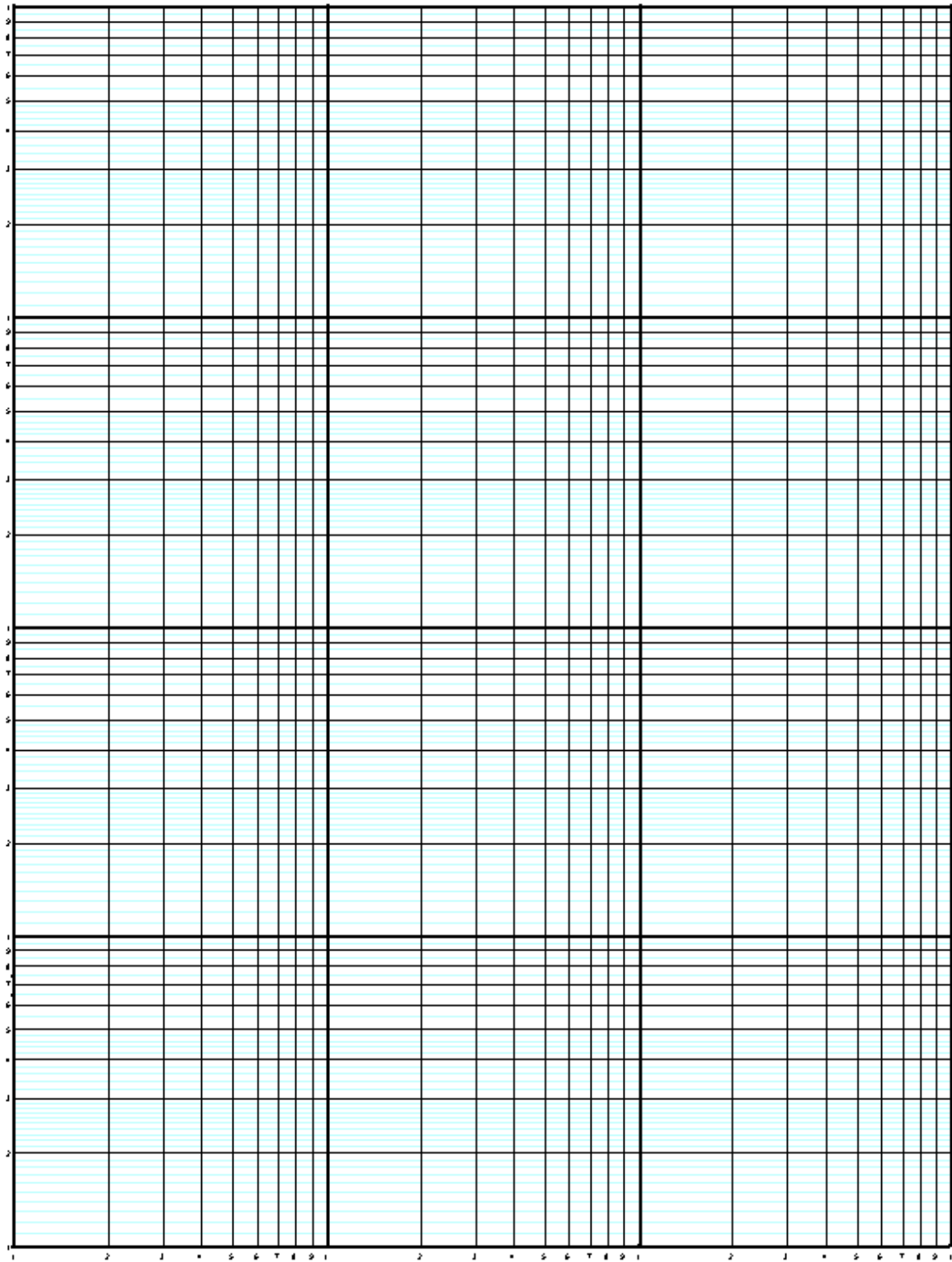


Figure 5-6. 4 × 3 Cycle log-log paper.

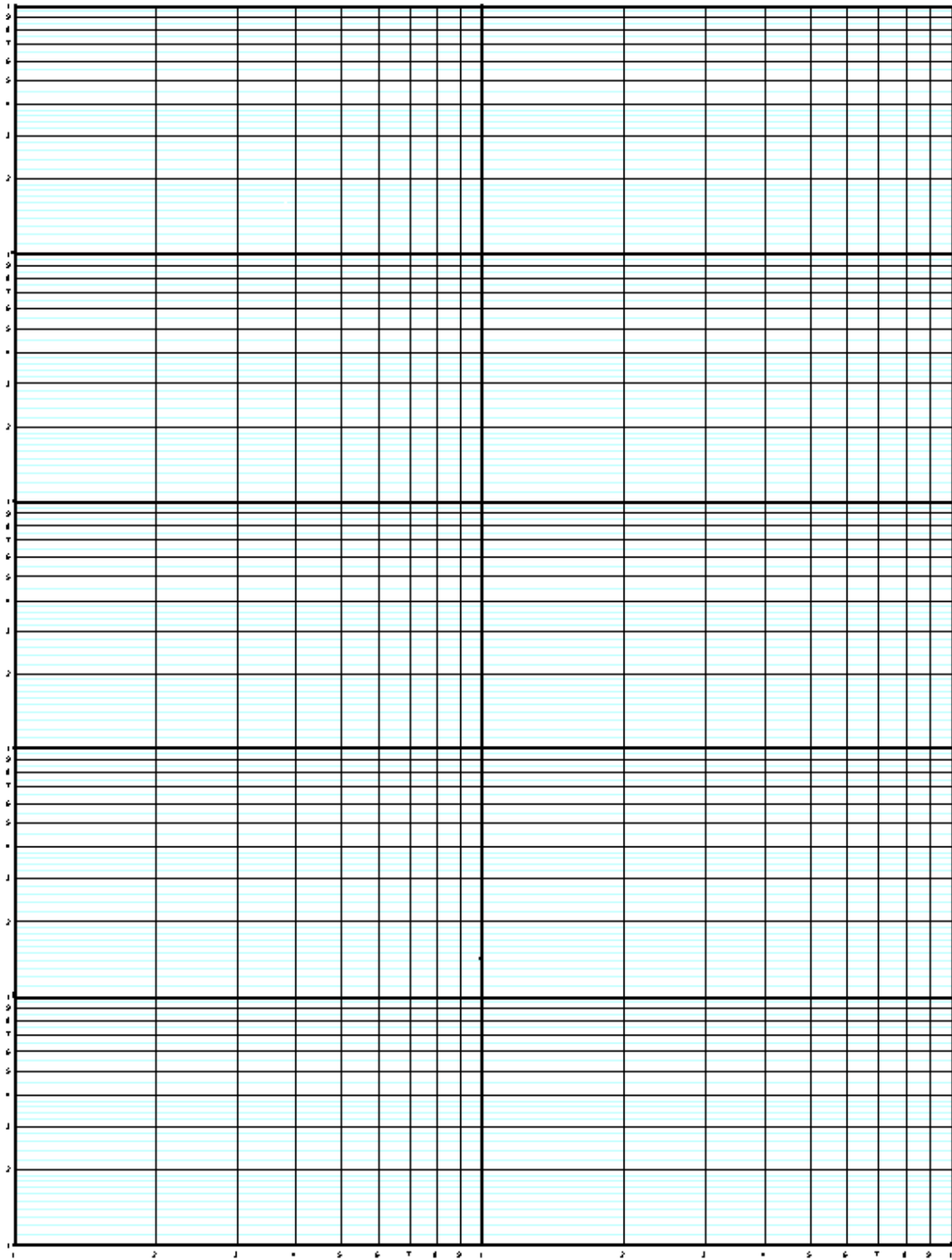


Figure 5-7. 5 × 2 Cycle log-log paper.



Semi-log paper is classified by the number of logarithmic cycles on the ordinate axis and the number of divisions on the abscissa axis. The figure 5-2 illustrates 2-cycles by 34 divisions, while the figures 5-3 and 5-4 illustrate 3-cycles by 34 divisions and 4-cycles by 34 divisions respectively. Semi-log paper comes in a variety of cycles and divisions.

**Example 5-1. (Semi-log paper.)**

Verify that the data

| $x$ | $y$          |
|-----|--------------|
| 1   | 0.406006     |
| 2   | 0.0549469    |
| 4   | 0.00100039   |
| 6   | 0.0000184326 |

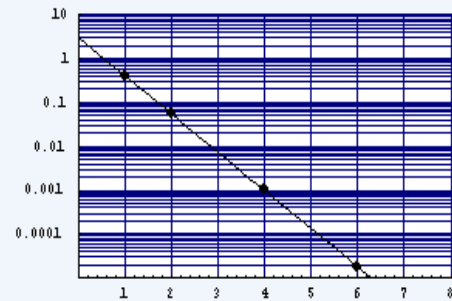
can be fitted to an exponential curve  $y = \alpha e^{mx}$ . Find values for the constants  $\alpha$  and  $m$  associated with the above data points

**Solution:**

We plot the data on semi-log paper and obtain the straight line illustrated. The slope of this line is given by

$$m = \frac{\text{Change in } \ln y}{\text{Change in } x}$$

$$m = \frac{\ln(0.0000184326) - \ln(0.406006)}{5} = -2.0.$$



The value of  $\alpha$  is given by the  $y$ -intercept. This is where the line intersects the  $y$ -axis when  $x = 0$ . We find the values

$$\alpha = 3.0 \quad \text{and} \quad m = -2.0.$$

This gives the exponential curve  $y = 3e^{-2x}$ . ■

Data from the power function  $y = \alpha x^m$  can be represented in the form of a straight line by taking the natural logarithm of both sides of the equation to obtain

$$\ln y = \ln \alpha + m \ln x. \quad (5.3)$$

Now let  $Y = \ln y$ ,  $X = \ln x$  and  $b = \ln \alpha$  so that equation (5.3) takes on the form of the straight line

$$Y = mX + b. \quad (5.4)$$

This shows that the power function becomes a straight line when plotted on log-log paper where both the abscissa and ordinate axes are logarithmic. Some selected examples of log-log paper are given in the figures 5-5, 5-6 and 5-7. Note that log-log paper is classified according to the number of cycles on each axis. The log-log classification is usually denoted by the notation “ $n \times m$  cycles”. Here  $n$  is the number of cycles on the ordinate axis and  $m$  is the number of cycles on the abscissa axis. The log-log paper comes in a variety of integer and fractional cycles.

**Example 5-2. (Log-log paper.)**

Verify that the data

| $x$ | $y$     |
|-----|---------|
| 1   | 4.00000 |
| 3   | 0.76980 |
| 5   | 0.35777 |
| 7   | 0.21598 |

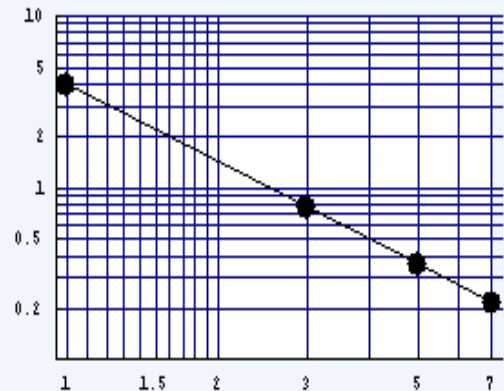
can be fitted to the power curve  $y = \alpha x^m$ . Find values for the constants  $\alpha$  and  $m$  associated with the above data points.

**Solution:**

We plot the data on Log-log paper and obtain the straight line illustrated. The slope of this line is given by

$$m = \frac{\text{Change in } \ln y}{\text{Change in } \ln x}$$

$$m = \frac{\ln(0.21598) - \ln(4.0)}{\ln(7) - \ln(1)} = -1.5.$$



The value of  $\alpha$  results when  $x = 1$  and so can be read off the log-log graph. We find the values  $\alpha = 4.0$  and  $m = -1.5$ . This gives the power function  $y = 4x^{-1.5}$ . ■