“*Words differently arranged have a different meaning and meanings differently arranged have a different effect.*”

Blaise Pascal (1623-1662)

Chapter 6

Difference Equations and Z-transforms

There are many concepts in science and engineering that can be approached from either a discrete or a continuous viewpoint. For example, consider how you might record the temperature outside at some specific place as a function of time. One technique would be to purchase a chart recorder capable of measuring and plotting the temperature as a function of time. This would give a continuous record of the temperature over some interval of time. Another way to record the temperature would be to measure the temperature, at the specified place, at discrete time intervals. The contrast between these two methods is that one method measures temperature continuously while the other method measures the temperature in a discrete fashion.

In any laboratory experiment, one must make a decision as to how data from the experiment is to be collected. Whether discrete measurements or continuous measurements are recorded depends upon many factors as well as the type of experiment being considered. The techniques used to analyze the data collected depends upon whether the data is continuous or discrete.

The investment of money at compound interest is an example of a physical problem which requires analysis of discrete values. Say, \$1,000.00 is to be invested at R percent interest compounded quarterly. How do we determine the discrete values representing the amount of money available at the end of each compound period? To solve this problem, we let P_0 denote the amount of money initially invested, R the percent interest yearly with $\frac{1}{4}\frac{R}{100} = i$ the quarterly interest and P_n the principal due at the end of the nth compound period. We can then set up the equations for the determination of P_n. We have

$$
\begin{aligned}
P_0 &= \text{Initial amount invested} \\
P_1 &= P_0 + P_0 i = P_0(1+i) \\
P_2 &= P_1 + P_1 i = P_1(1+i) = P_0(1+i)^2 \\
P_3 &= P_2 + P_2 i = P_2(1+i) = P_0(1+i)^3 \\
&\vdots \\
P_n &= P_{n-1} + P_{n-1} i = P_{n-1}(1+i) = P_0(1+i)^n
\end{aligned}
$$

For $i = \frac{1}{4}\frac{R}{100}$ and $P_0 = 1{,}000.00$, figure 6-1 illustrates a graph of P_n vs time, for a 30 year period, where one year represents four payment periods. In this figure values of R for 4%, 5.5%, 7%, 8.5% and 10% were used in the above calculations.

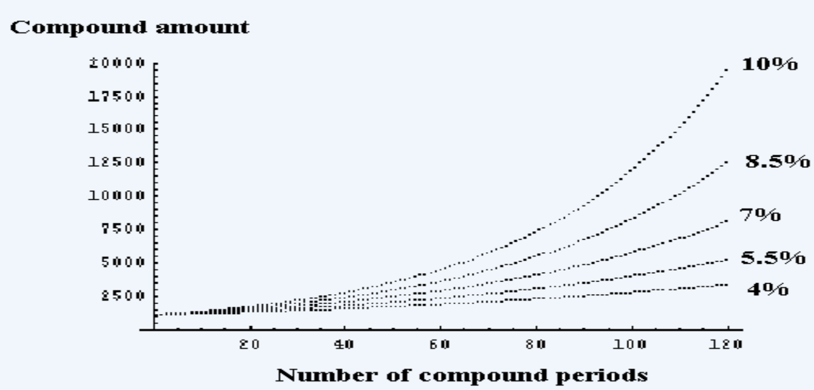

Figure 6-1. Return from $1,000 investment compounded quarterly over 30 year period.

In this chapter we investigate some techniques that can be used in the analysis of discrete phenomena like the compound interest problem just considered.

The study of calculus has demonstrated that derivatives are the mathematical quantities that represent continuous change. We find that if we replace derivatives (continuous change) by differences (discrete change), then linear ordinary differential equations become linear difference equations. We shall investigate these difference equations and find ways to construct solutions to such equations.

In the following discussions, note that the various techniques developed for analyzing discrete systems are very similar to many of the methods used for studying continuous systems.

Differences and Difference Equations

Consider the function $y = f(x)$ illustrated in the figure 6-1 which is evaluated at the equally spaced $x-$values of $x_0, x_1, x_2, \ldots, x_n, x_{n+1}, \ldots$ where $x_{n+1} = x_n + h$ for $n = 0, 1, 2, \ldots,$ where h is the distance between two consecutive points.

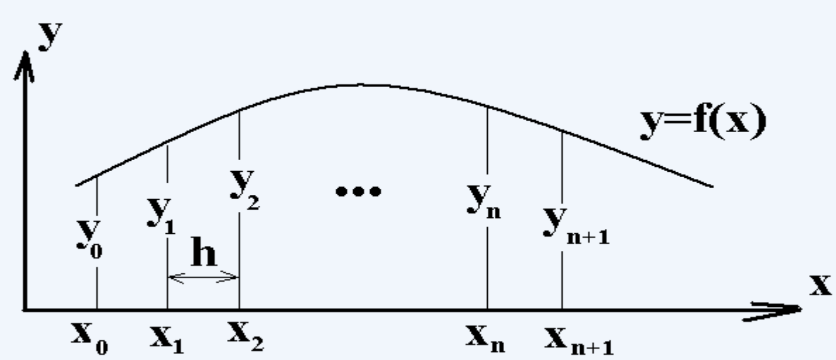

Figure 6.2. Discrete values of $y = f(x)$.

Let $y_n = f(x_n)$ and consider the approximation of the derivative $\frac{dy}{dx}$ at the discrete value x_n. By using the definition of a derivative we may write the approximation as

$$\frac{dy}{dx}\Big|_{x=x_n} \approx \frac{y_{n+1} - y_n}{h}.$$

This is called a forward difference approximation. By letting $h = 1$ in the above equation, we can define the first forward difference of y_n as

$$\Delta y_n = y_{n+1} - y_n. \tag{6.1}$$

There is no loss in generality in letting $h = 1$, since we can always rescale the x-axis by defining the new variable X defined by the transformation equation $x = x_0 + Xh$, then when $x = x_0, x_0 + h, x_0 + 2h, \ldots, x_0 + nh, \ldots$ the scaled variable X takes on the values $X = 0,\ 1,\ 2, \ldots, n, \ldots.$

Define the second forward difference as a difference of the first forward difference. A second difference is denoted by the notation $\Delta^2 y_n$ and

$$\begin{aligned} &\Delta^2 y_n = \Delta(\Delta y_n) = \Delta y_{n+1} - \Delta y_n = (y_{n+2} - y_{n+1}) - (y_{n+1} - y_n) \\ \text{or} \quad &\Delta^2 y_n = y_{n+2} - 2y_{n+1} + y_n. \end{aligned} \tag{6.2}$$

Higher ordered difference are defined in a similar manner. A nth order forward difference is defined as differences of $(n-1)$st forward differences for $n = 2, 3, \ldots.$

Analogous to the differential operator $D = \frac{d}{dx}$, there is a stepping operator E defined as follows:

$$
\begin{aligned}
Ey_n &= y_{n+1} \\
E^2 y_n &= y_{n+2} \\
&\cdots \\
E^m y_n &= y_{n+m}.
\end{aligned}
\tag{6.3}
$$

From the definition given by equation (6.1) we can write the first ordered difference

$$\Delta y_n = y_{n+1} - y_n = E y_n - y_n = (E-1) y_n$$

which illustrates that the difference operator Δ can be expressed in terms of the stepping operator E and

$$\Delta = E - 1. \tag{6.4}$$

This operator identity, enables us to express the second-order difference of y_n as

$$
\begin{aligned}
\Delta^2 y_n &= (E-1)^2 y_n \\
&= (E^2 - 2E + 1) y_n \\
&= E^2 y_n - 2E y_n + y_n \\
&= y_{n+2} - 2y_{n+1} + y_n.
\end{aligned}
$$

Higher order differences such as $\Delta^3 y_n = (E-1)^3 y_n,\ \Delta^4 y_n = (E-1)^4 y_n, \dots$ and higher ordered differences are quickly calculated by applying the binomial expansion to the operators operating on y_n.

Difference equations are equations which involve differences. For example, the equation

$$L_2(y_n) = \Delta^2 y_n = 0$$

is an example of a second-order difference equation, and

$$L_1(y_n) = \Delta y_n - 3y_n = 0$$

is an example of a first-order difference equation. The symbols $L_1(),\ L_2()$ are operator symbols. Using the operator E, the above equations can be written as

$$
\begin{aligned}
L_2(y_n) &= \Delta^2 y_n = (E-1)^2 y_n = y_{n+2} - 2y_{n+1} + y_n = 0 \quad \text{and} \\
L_1(y_n) &= \Delta y_n - 3y_n = (E-1) y_n - 3y_n = y_{n+1} - 4y_n = 0,
\end{aligned}
$$

respectively.

There are many instances where variable quantities are assigned values at uniformly spaced time intervals. We shall be interested in studying these discrete variable quantities by using differences and difference equations. An equation which relates values of a function y and one or more of its differences is called a difference equation. In dealing with difference equations one assume that the function y evaluated at x_n and its differences $\Delta y_n,\ \Delta^2 y_n, \ldots$, are all defined for every number x in some set of values $\{x_0,\ x_0+h,\ x_0+2h, \ldots, x_0+nh, \ldots\}$. A difference equation is called linear and of order m if it can be written in the form

$$L(y_n) = a_0(n)y_{n+m} + a_1(n)y_{n+m-1} + \cdots + a_{m-1}(n)y_{n+1} + a_m(n)y_n = g(n), \tag{6.5}$$

where the coefficients $a_i(n),\ i = 0,\ 1,\ 2, \ldots, m$, and the right-hand side $g(n)$ are known functions of n. If $g(n) \neq 0$, the difference equation is said to be nonhomogeneous and if $g(n) = 0$, the difference equation is called homogeneous.

The difference equation (6.5) can be written in the operator form

$$L(y_n) = [a_0(n)E^m + a_1(n)E^{m-1} + \cdots + a_{m-1}(n)E + a_m(n)]y_n = g(n),$$

where E is the stepping operator.

A mth-order linear initial value problem associated with a mth-order linear difference equation consists of a linear difference equation of the form given in the equation (6.5) together with a set of m initial values of the type

$$y_0 = \alpha_0, \quad y_1 = \alpha_1, \quad y_2 = \alpha_2, \quad \ldots, \quad y_{m-1} = \alpha_{m-1},$$

where $\alpha_0,\ \alpha_1, \ldots, \alpha_{m-1}$ are specified constants.

Difference equations may be solved using techniques which are very similar to the solution methods associated with ordinary differential equations. All the concepts and theorems derived for linear differential equations have analogs in the study of linear difference equations. Instead of presenting each derivation, we list the following summary of these important results. In this summary D.E. can represent either "differential equation" or "difference equation".

1. An nth-order, linear, homogeneous D.E. possesses n independent solutions.
2. The general solution of an nth-order, linear D.E. has n arbitrary constants.
3. The general solution of an nth-order, linear, nonhomogeneous D.E. can be formed by adding the general complementary solution of the homogeneous equation to any particular solution of the nonhomogeneous equation.

4. If two independent solutions of a linear, homogeneous D.E. are known, then any linear combination of these solutions is also a solution.
5. If n-independent solutions to a linear, nth-order, homogeneous D.E. are known, then any linear combination of these solutions produces the general solution.
6. An nth-order linear initial value problem associated with a D.E. possesses a unique solution.

The above analogies between difference equations and differential equations can be anticipated if one writes the forward difference approximation of a derivative in operator form as $Dy = \lim_{h\to 0} \frac{\Delta y}{h}$, where $D = \frac{d}{dx}$ and Δ is the difference operator.

Example 6-1.

Show $\Delta a^k = (a-1)a^k$, for a constant and k an integer.

Solution: Let $y_k = a^k$, then by definition

$$\Delta y_k = y_{k+1} - y_k = a^{k+1} - a^k = (a-1)a^k.$$

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Example 6-2.

The function

$$k^{[N]} = k(k-1)(k-2)\cdots[k-(N-2)][k-(N-1)], \qquad k^{[0]} \equiv 1$$

is called a factorial polynomial, see equation (4.72). Here $k^{[N]}$ is a product of N terms.

Show $\Delta k^{[N]} = N\, k^{[N-1]}$ for N a positive integer and fixed.

Solution: Observe that the factorial polynomials are

$$k^{[0]} = 1, \quad k^{[1]} = k, \quad k^{[2]} = k(k-1), \quad k^{[3]} = k(k-1)(k-2), \quad \cdots$$

Use $y_k = k^{[N]}$ and calculate the forward difference

$$\begin{aligned}\Delta y_k = &\; y_{k+1} - y_k = (k+1)^{[N]} - k^{[N]} \\ = &\; (k+1)\underbrace{(k)(k-1)\cdots[k+1-(N-1)]}_{k^{[N-1]}} - \underbrace{k(k-1)(k-2)\cdots[k-(N-2)]}_{k^{[N-1]}}[k-(N-1)]\end{aligned}$$

which simplifies to

$$\Delta y_k = \{(k+1) - [k-(N-1)]\}\, k^{[N-1]} = N k^{[N-1]}.$$

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Example 6-3.

Verify the forward difference relation

$$\Delta(U_kV_k) = U_k\Delta V_k + V_{k+1}\Delta U_k$$

Solution: Let $y_k = U_kV_k$, then we can write

$$\begin{aligned}\Delta y_k &= y_{k+1} - y_k \\ &= U_{k+1}V_{k+1} - U_kV_k + [U_kV_{k+1} - U_kV_{k+1}] \\ &= U_k[V_{k+1} - V_k] + V_{k+1}[U_{k+1} - U_k] \\ &= U_k\Delta V_k + V_{k+1}\Delta U_k.\end{aligned}$$

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Special Differences

The table 6.1 contains a list of some well known forward differences which are useful in many applications. The verification of these differences is left as an exercise.

Table 6.1 Some common forward differences		
1.	$\Delta a^k = (a-1)a^k$	
2.	$\Delta k^{[N]} = N\,k^{[N-1]}$ N fixed	$k^{[N]}$ is factorial function See equation (4.72)
3.	$\Delta\sin(\alpha+\beta k) = 2\sin(\beta/2)\cos(\alpha+\beta/2+\beta k)$	α, β constants
4.	$\Delta\cos(\alpha+\beta k) = -2\sin(\beta/2)\sin(\alpha+\beta/2+\beta k)$	α, β constants
5.	$\Delta\binom{k}{N} = \binom{k}{N-1}$ N fixed	$\binom{k}{N}$ are binomial coefficients
6.	$\Delta(k!) = k(k!)$	
7.	$\Delta(U_kV_k) = U_k\Delta V_k + V_{k+1}\Delta U_k$	
8.	$\Delta\left(\frac{1}{k_{[N]}}\right) = \frac{-N}{k_{[N+1]}}$, N fixed	$k_{[N]}$ is factorial function See equation (4.72)
9.	$\Delta k^2 = 2k+1$	
10.	$\Delta\log k = \log(1+1/k)$	

Finite Integrals

Associated with finite differences are finite integrals. If $\Delta y_k = f_k$, then the function y_k, whose difference is f_k, is called the finite integral of f_k. The inverse of the difference operation Δ is denoted Δ^{-1} and one can write $y_k = \Delta^{-1} f_k$, if $\Delta y_k = f_k$. For example, consider the difference of the factorial function $k^{[n]}$, defined by equation (4.72). If $\Delta k^{[n]} = nk^{[n-1]}$, then $\Delta^{-1} nk^{[n-1]} = k^{[n]}$. Associated with the difference table 6.1 is the finite integral table 6.2. The derivation of the entries is left as an exercise.

Table 6.2 Some selected finite integrals

1.	$\Delta^{-1} a^k = \dfrac{a^k}{a-1} \quad a \neq 1$	
2.	$\Delta^{-1} k^{[n])} = \dfrac{k^{[n+1]}}{n+1}$	$k^{[n]}$ is factorial function See equation (4.72)
3.	$\Delta^{-1} \sin(\alpha + \beta k) = \dfrac{-1}{2\sin(\beta/2)} \cos(\alpha - \beta/2 + \beta k)$	α, β constants
4.	$\Delta^{-1} \cos(\alpha + \beta k) = \dfrac{1}{2\sin(\beta/2)} \sin(\alpha - \beta/2 + \beta k)$	α, β constants
5.	$\Delta^{-1} \binom{k}{n} = \binom{k}{n+1} \quad \text{n}fixed$	$\binom{k}{n}$ are binomial coefficients
6.	$\Delta^{-1} (a + bk)^{[n]} = \dfrac{(a+bk)^{[n+1]}}{b(n+1)}$	a, b constants.

Summation of Series

Let $y_{k+1} - y_k = f_k$, then one can substitute $k = 0, 1, 2, \ldots$ to obtain

$$
\begin{aligned}
y_1 - y_0 &= f_0 \\
y_2 - y_1 &= f_1 \\
y_3 - y_2 &= f_2 \\
&\vdots \\
y_n - y_{n-1} &= f_{n-1} \\
y_{n+1} - y_n &= f_n
\end{aligned}
\tag{6.6}
$$

Adding these equations one obtains

$$
\sum_{i=0}^{n} f_i = y_{n+1} - y_0 = \Delta^{-1} f_i\Big]_{i=0}^{n+1} = y_i\Big]_{i=0}^{n+1} \qquad \text{where} \quad \Delta y_k = f_k.
$$

One can verify that by adding the equations (6.6) from some point $i = m$ to n, one obtains the more general result

$$\sum_{i=m}^{n} f_i = y_{n+1} - y_m = \Delta^{-1} f_i\Big]_{i=m}^{n+1} = y_i\Big]_{i=m}^{n+1}. \tag{6.7}$$

Example 6-4.

Evaluate the sum

$$S = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)$$

Solution: Let $f_k = k(k+1) = k^2 + k$ and show one can write f_k as the factorial function $f_k = (k+1)^{[2]}$. Therefore,

$$S = \sum_{i=1}^{n} f_i = \sum_{i=1}^{n} (i+1)^{[2]} = \Delta^{-1} f_i\Big]_{i=1}^{n+1} = \frac{(i+1)^{[3]}}{3}\Bigg]_{i=1}^{n+1} = \frac{(n+2)^{[3]}}{3} - \frac{2^{[3]}}{3}$$

which simplifies to $S = \frac{1}{3} n(n+1)(n+2)$.

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Difference Equations with Constant Coefficients

Difference equations arise in a variety of situations. The following are some examples of where difference equations arise in applications. In assuming a power series solution to differential equations, the coefficients must satisfy certain recurrence formula which are nothing more than difference equations. In the study of stability of numerical methods there occurs difference equations which must be analyzed. In the computer simulation of various types of real-world processes, difference equations frequently occur. Difference equations also are studied in the areas of probability, statistics, economics, physics, and biology. We begin our investigation of difference equations by studying those with constant coefficients as these are the easiest to solve.

Example 6-5.

Given the difference equation

$$y_{n+1} - y_n - 2y_{n-1} = 0$$

with the initial conditions $y_0 = 1,\ y_1 = 0$. Find values for y_2 through y_{10}.

Solution: In the given difference equation, replace n by $n+1$ in all terms, to obtain

$$y_{n+2} = y_{n+1} + 2y_n,$$

then one can verify

$$\begin{aligned}
n &= 0, && y_2 = y_1 + 2y_0 = 2 \\
n &= 1, && y_3 = y_2 + 2y_1 = 2 \\
n &= 2, && y_4 = y_3 + 2y_2 = 6 \\
n &= 3, && y_5 = y_4 + 2y_3 = 10 \\
n &= 4, && y_6 = y_5 + 2y_4 = 22 \\
n &= 5, && y_7 = y_6 + 2y_5 = 42 \\
n &= 6, && y_8 = y_7 + 2y_6 = 86 \\
n &= 7, && y_9 = 7_8 + 2y_7 = 170 \\
n &= 8, && y_{10} = y_9 + 2y_8 = 342.
\end{aligned}$$

■

The study of difference equations with constant coefficients closely parallels the development of ordinary differential equations. Our goal is to determine functions $y_n = y(n)$, defined over a set of values of n, which reduce the given difference equation to an identity. Such functions are called solutions of the difference equation. For example, the function $y_n = 3^n$ is a solution of the difference equation $y_{n+1} - 3y_n = 0$ because $3^{n+1} - 3 \cdot 3^n = 0$ for all $n = 0,\ 1,\ 2, \ldots$. Recall that for linear differential equations with constant coefficients we assumed a solution $y(x) = \exp(\omega x)$. We did this to obtain the characteristic equation and characteristic roots associated with the differential equation. When $x = n$, we obtain $y(n) = y_n = \exp(\omega n) = \lambda^n$, where $\lambda = \exp(\omega)$ is a constant. This suggests in our study of difference equations with constant coefficients that we should assume a solution of the form $y_n = \lambda^n$, where λ is a constant. Analogous to ordinary linear differential equations with constant coefficients, we find that a linear, nth-order, homogeneous difference equation with constant coefficients has associated with it a characteristic equation with characteristic roots $\lambda_1, \lambda_2, \ldots, \lambda_n$. The characteristic equation is found by assuming a solution $y_n = \lambda^n$, where λ is a constant. The various cases that can arise are illustrated by the following examples.