GEOMETRICAL THEORY OF NON-HOLONOMIC SYSTEMS APPLIED: TWO ROLLING CYLINDERS

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Geometrical theory of non-holonomic systems, see [1]

I. Unconstrained mechanical systems

A *fibred manifold*: $\pi : Y \to X$, as underlay structure in mechanics: $\dim Y = m+1$, $\dim X = 1$.
Dynamical form: 1-contact $\pi_{2,0}$-horizontal 2-form on J^2Y, expressed in every fibred chart by $E = E_\sigma(t, q^\rho, \dot q^\rho, \ddot q^\rho)\, dq^\sigma \wedge dt$, $1 \le \sigma, \rho \le m$.
Path of a dynamical form: a section γ of π such that $E \circ J^2\gamma = 0$. Having on mind physical applications, only dynamical forms affine in the second derivatives (accelerations) will be considered, i.e.
$E_\sigma(t, q^\rho, \dot q^\rho, \ddot q^\rho) = A_\sigma(t, q^\rho, \dot q^\rho) + B_{\sigma\nu}(t, q^\rho, \dot q^\rho)\ddot q^\nu$. Equations of paths are of the form $A_\sigma + B_{\sigma\nu}\ddot q^\nu = 0$ along $J^2\gamma$ and are called *(unconstrained) equations of motion*.
Lepagean 2-form associated to E: a 2-form α on J^1Y such that $p_1\alpha = E$. It holds $\alpha = A_\sigma \omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot q^\nu + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu$, where $F_{\sigma\nu} = -F_{\nu\sigma}$ are some free functions on J^1Y.
Lepagean class of E or *first-order mechanical system associated to* E: the equivalence class $[\alpha]$: $\alpha^1 \sim \alpha^2$ iff $\alpha^1 - \alpha^2$ is a 2-contact 2-form on J^1Y.
A section γ of π is a *path of* $E = p_1\alpha$ iff $J^1\gamma^* i_\xi \alpha = 0$ for every π_1-vertical vector field ξ on J^1Y.
A path of a dynamical form E will be also called a *path of the mechanical system* $[\alpha]$ *associated to* E.

II. Constrained mechanical systems

Constraint manifold Q: a submanifold of J^1Y fibred over Y, $\dim Q = 2m+1-k$, $1 \le k \le m-1$, locally defined by a system of first-order ordinary differential equations, called *non-holonomic constraints*: $f^i(t, q^\sigma, \dot q^\sigma) = 0$, $\mathrm{rank}(\partial f^i / \partial \dot q^\sigma) = k$, $1 \le i \le k$. These equations can be (locally) solved and rewritten in a *normal form:*
$\dot q^{m-k+i} - g^i(t, q^\sigma, \dot q^l) = 0$. *Canonical embedding*:
$\iota : Q \to J^1Y$, $\iota(t, q^\sigma, \dot q^l) \to (t, q^\sigma, \dot q^l, g^i(t, q^\nu, \dot q^s))$.
We use new basis of 1-forms on Q: $(dt, \omega^l, \bar\varphi^i, d\dot q^l)$, where
$\bar\varphi^i = -\partial g^i / \partial \dot q^l \omega^l + dq^{m-k+i} - g^i dt$. Distribution $\mathcal{C}$ on Q who's anihilator is $\mathcal{C}^0 = \mathrm{span}\{\bar\varphi^i\}$ is called *canonical distribution*.

1. Reduced equations of motion:
$\iota^*\alpha^1 \sim \iota^*\alpha^2$ iff $\iota^*\alpha^1 - \iota^*\alpha^2 = \bar F_{ls}\omega^l \wedge \omega^s + \bar\varphi^i \wedge \chi_i$ where $\bar F_{ls} = -\bar F_{sl}$ are some functions on Q and χ_i is a 1-form on Q.
Constrained system on Q *related to the mechanical system* $[\alpha]$: the equivalence class $[\alpha_Q] = [\iota^*\alpha]$,
$\alpha_Q = \bar A_l \omega^l \wedge dt + \bar B_{ls}\omega^l \wedge d\dot q^s + \bar F_{ls}\omega^l \wedge \omega^s + \iota^*\varphi^i \wedge \chi_i$, where $\bar A_l = \bar A_l(t, q^\sigma, \dot q^l)$, $\bar B_{ls} = \bar B_{ls}(t, q^\sigma, \dot q^l)$ are functions on Q (see [1]).
A section γ is called a *path of constrained system* $[\alpha_Q]$ if $f^i \circ J^1\gamma = 0$ together with one of the equivalent conditions: $J^1\gamma^* i_{\xi_\mathcal{C}} \alpha_Q = 0$ for every vector field $\xi_\mathcal{C}$ belonging to $\mathcal{C}$ or $\bar A_l + \bar B_{ls}\ddot q^s = 0$ along $J^2\gamma$. These equations are called *reduced equations of motion*.

2. Dynamical equations of motion:
Chetaev constraint force on J^1Y: $\Phi = \mu_i \partial f^i / \partial \dot q^\sigma dq^\sigma \wedge dt$, functions $\mu_i(t, q^\sigma, \dot q^\sigma)$ are called *Lagrange multipliers*.
Deformation of mechanical system $[\alpha]$*:*
$\alpha_\Phi = (A_\sigma - \mu_i \partial f^i / \partial \dot q^\sigma)\omega^\sigma \wedge dt + B_{\sigma\nu}\omega^\sigma \wedge d\dot q^\nu + F_{\sigma\nu}\omega^\sigma \wedge \omega^\nu$.
Constrained system is represented by class of equivalence $[\alpha_\Phi]$: $\alpha^1_\Phi \sim \alpha^2_\Phi$ iff $\alpha^1_\Phi - \alpha^2_\Phi = \eta_{\sigma\nu}\omega^\sigma \wedge \omega^\nu$.
A section γ is called a *path of constrained system* $[\alpha_\Phi]$ if $f^i \circ J^1\gamma = 0$ together with one of the equivalent conditions: $J^1\gamma^* i_\xi \alpha_\Phi = 0$ for every vector field ξ on J^1Y or $\bar A_\sigma + \bar B_{\sigma\nu}\ddot q^\nu - \mu^i \partial f^i / \partial \dot q^\sigma = 0$ along $J^2\gamma$, which are called *deformed equations of motion*. Deformed equations of motion together with non-holonomic conditions are called *dynamical equations of motion*.

Example: Two rolling cylinders

The cylinder V (radius R, mass M), symmetric about its central axis, is hollow and we suppose there is no difference between the inside and outside radius of the cylinder. It is constrained to roll without sliding on an inclined plane forming an angle φ with the horizontal one. The angle $\varphi = \varphi(t)$ a is given function of time. Cylinder W (radius r, $R > r$, mass m), symmetric about its central axis and placed inside cylinder V, rolls on interior face of hollow cylinder V without sliding. The moment of inertia about the center axis of cylinder V resp. W is $J_M = MR^2$ resp. $J_m = 1/2mr^2$. The system is in homogeneous gravitational field.
There are six degrees of freedom and we get $Y = R \times R^6$, $\dim Y = 7$, (t) is the asociated coordinate on base $X = R$, $(t, \psi, \beta, x, z, X, Z)$ fibered coordinates on Y and $(t, \psi, \beta, x, z, X, Z, \dot\psi, \dot\beta, \dot x, \dot z, \dot X, \dot Z)$ asociated fibered coordinates on $J^1Y = R \times R^6 \times R^6$. Coordinates (X, Z) resp. (x, z) define the position of center of mass of V resp W. β resp. ψ is the angle of total rotation of V resp. W regarding to z axis (see Fig.1).

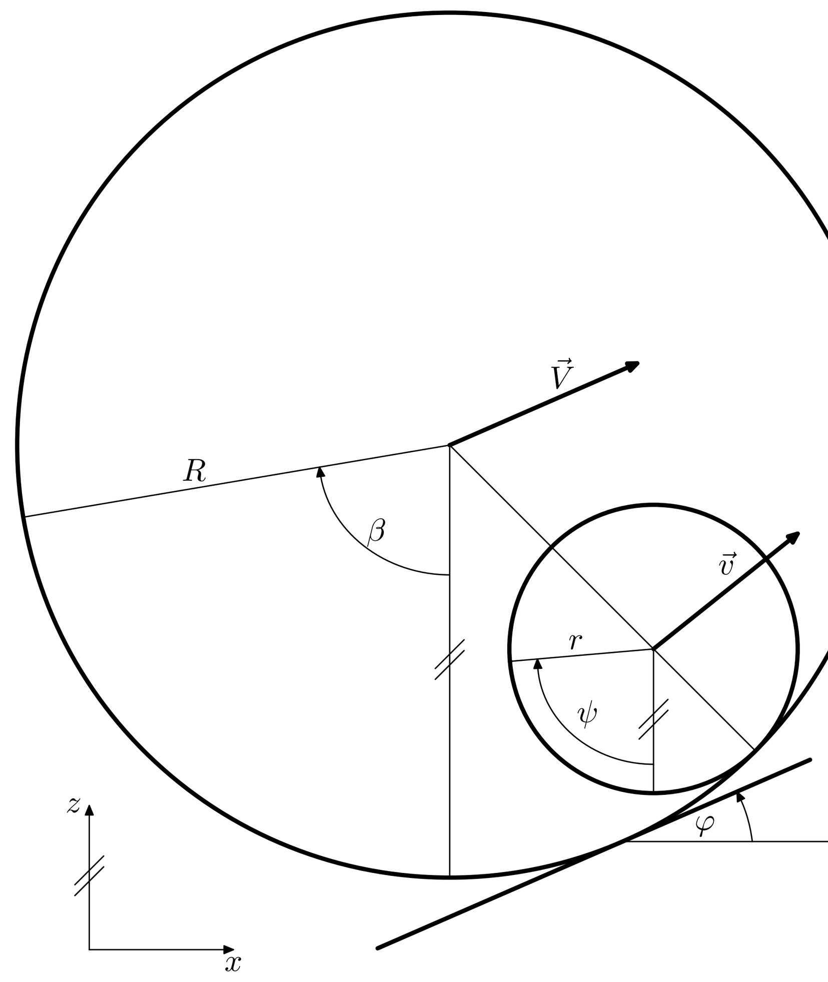

Fig.1 *The scheme of the two rolling cylinders.*

I. Unconstrained mechanical system:

Lagrangian: $L = \frac{1}{2}m(\dot x^2 + \dot z^2) + \frac{1}{2}M(\dot X^2 + \dot Z^2) + \frac{1}{2}J_m\dot\psi^2 + J_M\dot\beta^2 - mgz - MgZ$.
Equations of motion:
$-J_m\ddot\psi = 0, -J_M\ddot\beta = 0, -m\ddot x = 0, -m\ddot z - mg = 0, -M\ddot X = 0, -M\ddot Z - Mg = 0.$
Unconstrained machanical system $[\alpha]$ is represented by Lepagean 2-form:
$\alpha = -mg\omega^4 \wedge dt - Mg\omega^6 \wedge dt - J_m\omega^1 \wedge d\dot\psi - J_M\omega^2 \wedge d\dot\beta -$
$-m(\omega^3 \wedge d\dot x + \omega^4 \wedge d\dot z) - M(\omega^5 \wedge d\dot X + \omega^6 \wedge d\dot Z)$, where $\omega^1 = d\psi - \dot\psi dt$, $\omega^2 = d\beta - \dot\beta dt$, $\omega^3 = dx - \dot x dt$, $\omega^4 = dz - \dot z dt$, $\omega^5 = dX - \dot X dt$, $\omega^6 = dZ - \dot Z dt$. Dynamical form:
$E = -J_m\ddot\psi d\psi \wedge dt - J_M\ddot\beta d\beta \wedge dt - m\ddot x dx \wedge dt - (m\ddot z + mg)dz \wedge dt -$
$-M\ddot X dX \wedge dt - (M\ddot Z + Mg)dZ \wedge dt.$

II. Nonholonomic constraint: Rolling without sliding.

It holds:

$\dot x \equiv g^1 = \dot x_1 + R\dot\beta\cos\varphi$, $\dot z \equiv g^2 = \dot y_1 + R\dot\beta\sin\varphi$,
$\dot X \equiv g^3 = R\dot\beta\cos\varphi$, $\dot Z \equiv g^4 = R\dot\beta\sin\varphi$,
where $\vec v_1 = (\dot x_1, \dot y_1)$ is relative velocity of centers of mass of cylinders.

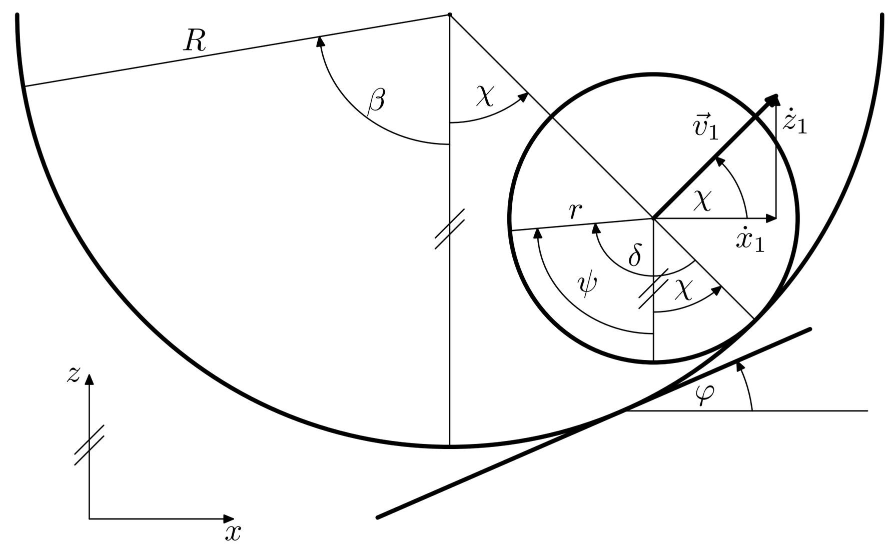

Fig.2 *A specification of* $\vec v_1$*.*

See Fig.2: if cylinders roll without sliding it holds $\delta = \psi + \chi$, $r\delta = R(\beta + \chi)$. Now we get $\chi = \frac{r}{R-r}\psi - \frac{R}{R-r}\beta$ and

$$\begin{aligned} \dot x_1 &= (R-r)\dot\chi\cos\chi = (r\dot\psi - R\dot\beta)\bar C, \\ \dot z_1 &= (R-r)\dot\chi\sin\chi = (r\dot\psi - R\dot\beta)\bar S, \end{aligned}$$

where $\bar C = \cos\left(\frac{r}{R-r}\psi - \frac{R}{R-r}\beta\right)$, $\bar S = \sin\left(\frac{r}{R-r}\psi - \frac{R}{R-r}\beta\right)$.
We get four non-holonomic constraints:

$$\begin{aligned} f^1 &\equiv \dot x - (r\dot\psi - R\dot\beta)\bar C - R\dot\beta\cos\varphi = 0, \quad f^2 \equiv \dot z - (r\dot\psi - R\dot\beta)\bar S - R\dot\beta\sin\varphi = 0, \\ f^3 &\equiv \dot X - R\dot\beta\cos\varphi = 0, \qquad\qquad\qquad\quad f^4 \equiv \dot Z - R\dot\beta\sin\varphi = 0. \end{aligned}$$

By these four non-holonomic conditions the submanifold Q is defined, $\dim Q = 9$.

III. Constrained system

$\alpha_Q = \bar A_1\omega^1 \wedge dt + \bar A_2\omega^2 \wedge dt + \bar B_{11}\omega^1 \wedge d\dot\psi + \bar B_{22}\omega^2 \wedge d\dot\beta + \bar B_{12}\omega^1 \wedge d\dot\beta + \bar B_{21}\omega^2 \wedge d\dot\psi$,
where $\bar A_1 = -mgr\bar S - mrR\dot\varphi\dot\beta S$,
$\bar A_2 = gR\{m\bar S - (M+m)\sin\varphi\} + mS[D^2\frac{R}{R-r} + R^2\dot\varphi\dot\beta]$, $\bar B_{11} = -mr^2 - J_m$,
$\bar B_{22} = -MR^2 - J_M - 2mR^2(1-C)$, $\bar B_{12} = \bar B_{21} = mrR(1-C)$, and $D = r\dot\psi - R\dot\beta$,
$C = \cos\left(\frac{r}{R-r}\psi - \frac{R}{R-r}\beta - \varphi\right)$, $S = \sin\left(\frac{r}{R-r}\psi - \frac{R}{R-r}\beta - \varphi\right)$.

IV. Reduced equations of motion:

$$\begin{aligned} &(\bar A_1 + \bar B_{11}\ddot\psi + \bar B_{12}\ddot\beta) \circ J^2\bar\gamma = 0, (\bar A_2 + \bar B_{22}\ddot\beta + \bar B_{21}\ddot\psi) \circ J^2\bar\gamma = 0, \\ &\dot x = D\bar C + R\dot\beta\cos\varphi,\ \dot z = D\bar S + R\dot\beta\sin\varphi,\ \dot X = R\dot\beta\cos\varphi,\ \dot Z = R\dot\beta\sin\varphi. \end{aligned}$$

There is no analytical solution of reduced equations of motion available.

V. Deformed equations of motion:

Chetaev constraint force: $\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4$

$$\begin{aligned} \Phi_1 &= \mu_1(-r\bar C, R\bar C - R\cos\varphi, 1, 0, 0, 0) \\ \Phi_2 &= \mu_2(-r\bar S, R\bar S - R\sin\varphi), 0, 1, 0, 0) \\ \Phi_3 &= \mu_3(0, -R\cos\varphi, 0, 0, 1, 0), \Phi_4 = \mu_4(0, -R\sin\varphi, 0, 0, 0, 1) \end{aligned}$$

Deformed equations of motion are:

$$\begin{aligned} -J_m\ddot\psi - \mu_1 r\bar C - \mu_2 r\bar S &= 0, \\ -J_M\ddot\beta + \mu_1\left[R\bar C - R\cos\varphi\right] + & \\ +\mu_2\left[R\bar S - R\sin\varphi\right] - \mu_3 R\cos\varphi - \mu_4 R\sin\varphi &= 0, \\ -m\ddot x + \mu_1 = 0, -m\ddot z - mg + \mu_2 &= 0, \\ -M\ddot X + \mu_3 = 0, -M\ddot Z - Mg + \mu_4 &= 0. \end{aligned}$$

From the physical point of view there are forces of interaction and frictional forces at the contact point of the two cylinders. Let us write down Newton's equations of motion. Let (X, Y, Z) resp. (x, y, z) be coordinates in R^3 of center of mass of cylinder V resp. W. Let us denote $\vec N = (N_x, N_y, N_z)$ resp. $-\vec N = (-N_x, -N_y, -N_z)$ resultant of frictional and interaction forces that acts at contact point of the two cylinders onto cylinder W resp V. At the contact point of cylinder V and inclined plane there acts onto V resultant force $\vec P = (P_x, P_y, P_z)$. We require $N_y = 0$, $P_y = 0$. Let us denote $\vec M_W$ momentum of force $\vec N$ with respect to centre of mass of cylinder W and $\vec M_{V1}$ resp. $\vec M_{V2}$ momentum of force $-\vec N$ resp. $\vec P$ with respect to center of mass of cylinder V. Let $\vec r$ resp. $\vec R_1$ resp. $\vec R_2$ be radius vector of point, where $\vec N$ resp. $-\vec N$ resp. $\vec P$ is applied. It holds $\vec r = r(\bar S, 0, -\bar C), \vec R_1 = R(\bar S, 0, -\bar C)$, $\vec R_2 = R(\sin\varphi, 0, -\cos\varphi)$. Taking into account that $Y = y = konst$, we get Newton's equations of motion for the system of the two cylinders that can roll with sliding:

$$\begin{aligned} J_m\ddot\psi - r(\bar C N_x + \bar S N_z) &= 0, \\ J_M\ddot\beta + R(\bar C N_x + \bar S N_z - P_x\cos\varphi - P_z\sin\varphi) &= 0, \\ m\ddot x + N_x = 0, m\ddot z + mg + N_z &= 0, \\ M\ddot X - N_x + P_x = 0, M\ddot Z + Mg - N_z + P_z &= 0. \end{aligned}$$

To roll without sliding we have to add four non-holonomic condition we have mentioned above. Comparing this Newton's equations of motion with the system of deformed equations of motion we realize that
$\mu_1 = -N_x, \mu_2 = -N_z, \mu_3 = N_x - P_x, \mu_4 = N_z - P_z$. Hence Chetaev constraint force has the meaning of the physical forces and moments of this forces.

VI. Numerical solution of reduced equations of motion

The characteristics of mechanical system: $M = 1.0kg, m = 0.6kg, R = 1.0m, r = 0.6m$.
The initial conditions:
$z(0) = r, Z(0) = R, x(0) = X(0) = 0m, \psi(0) = \beta(0) = 0, \dot\psi(0) = 0s^{-1}, \dot\beta(0) = 1s^{-1}$. We have found solution for two types of function φ: $\varphi = 0$ and $\varphi = -\pi/10t$. Numerical solution was obtained using the program Maple. We can see the results of simulation for $\varphi = 0$ resp. $\varphi = -\pi/10t$ on Figs.3-6 resp. Figs.7-10.

Reference: [1] O. Krupková, Mechanical Systems with Nonholonomic Constraints, J. Math. Phys. 38 , 5098-5138, 1997

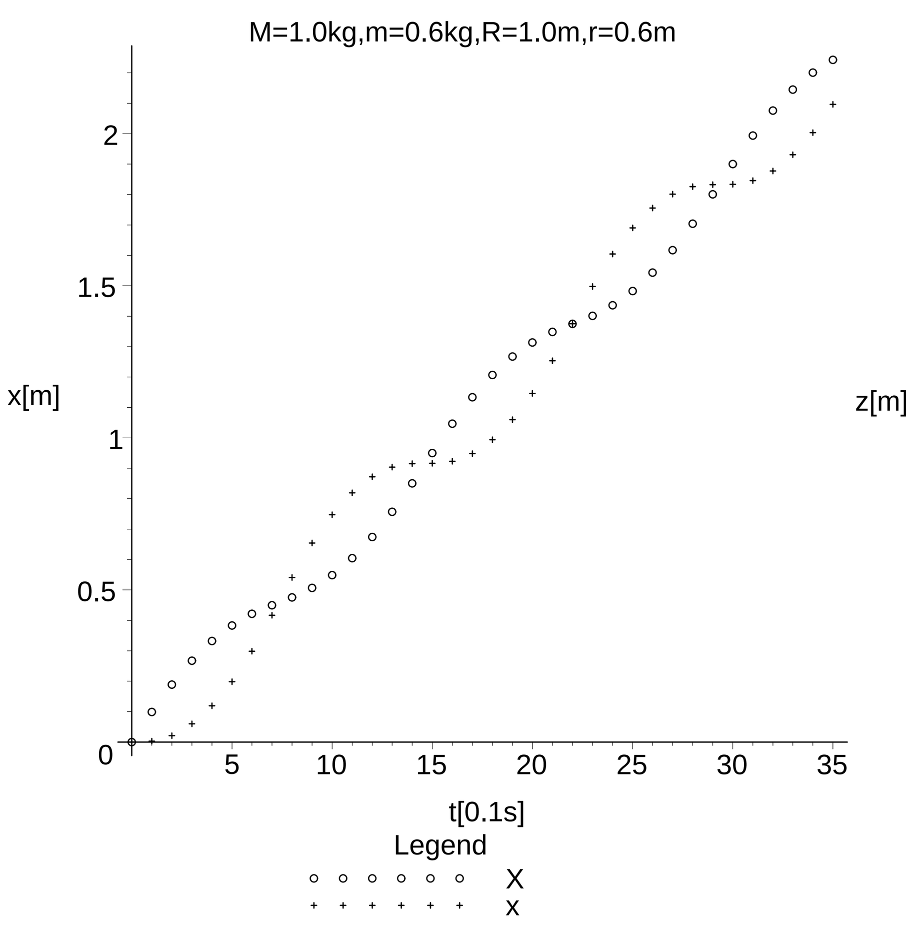

Fig.3 *Numerical solution* $X(t)$, $x(t)$ *for* $\varphi(t) = 0$

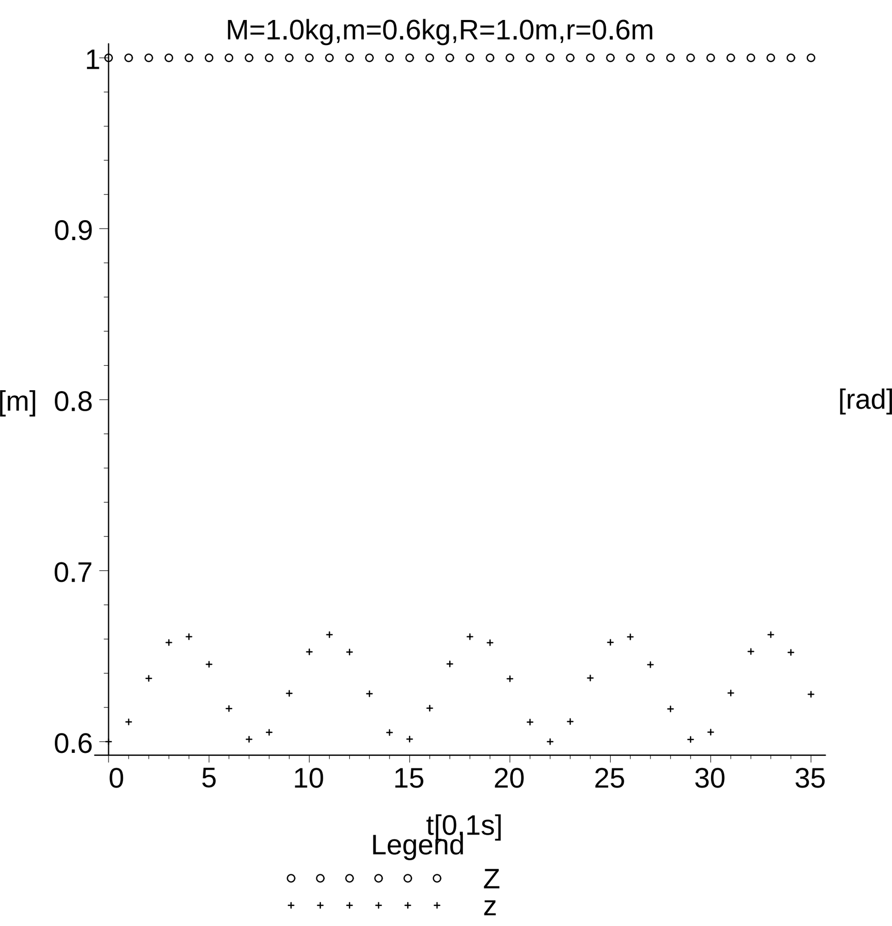

Fig.4 *Numerical solution* $Z(t)$, $z(t)$ *for* $\varphi(t) = 0$

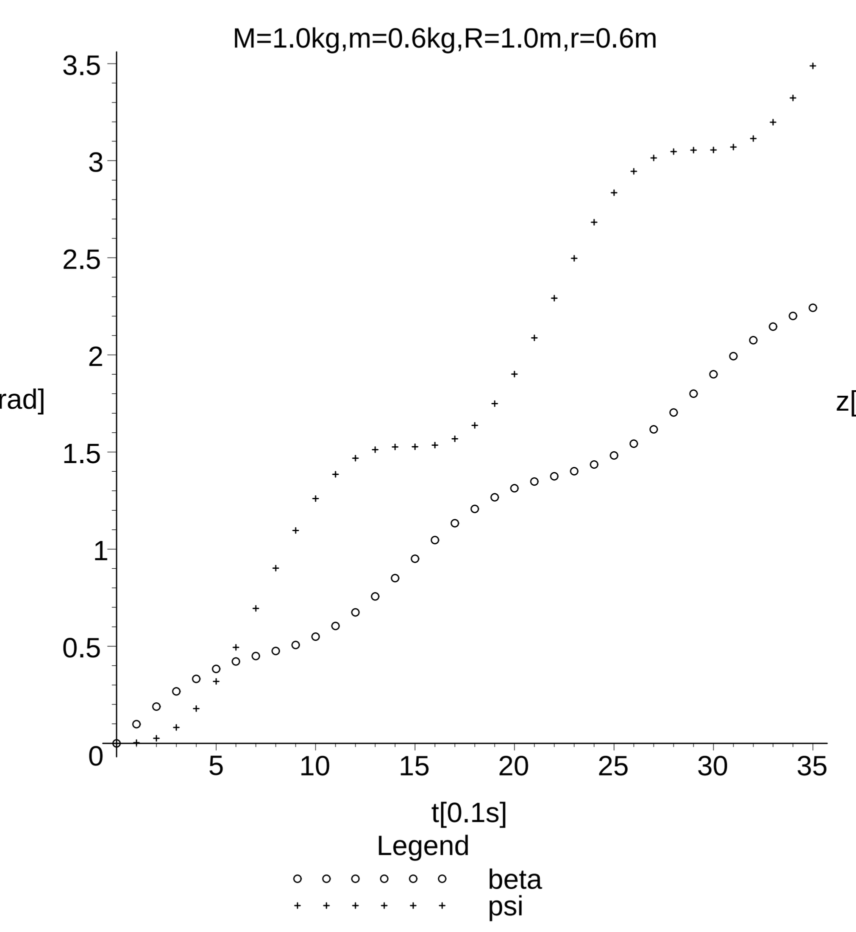

Fig.5 *Numerical solution* $\beta(t)$, $\psi(t)$ *for* $\varphi(t) = 0$

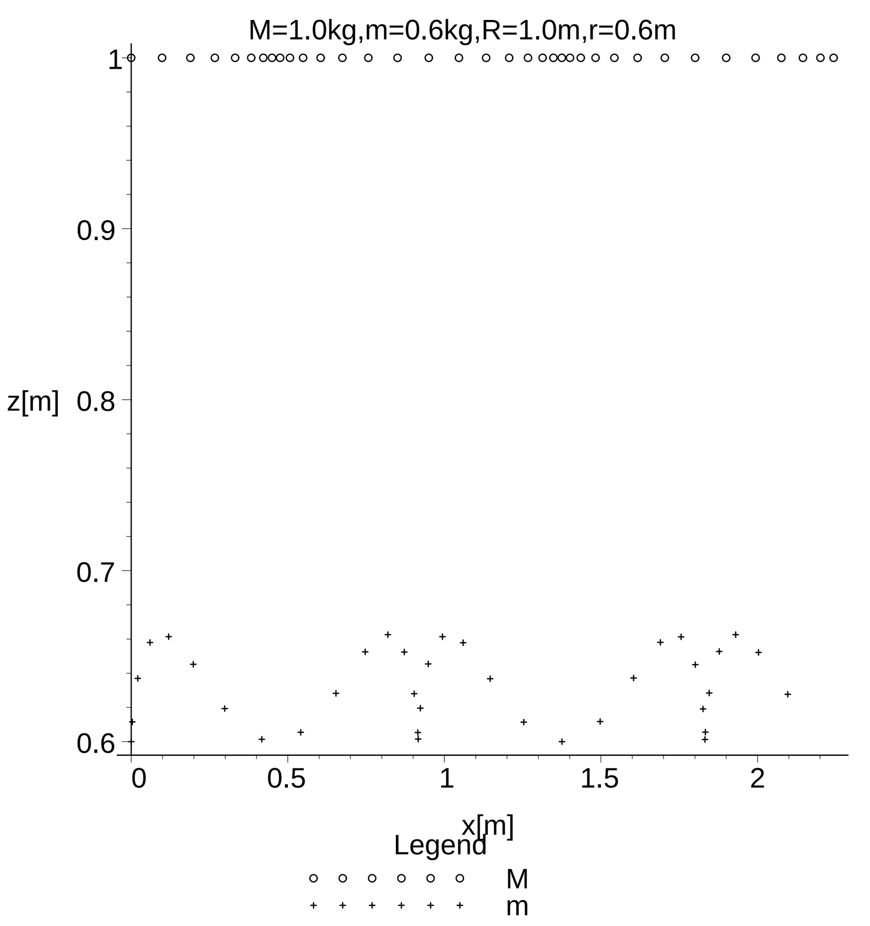

Fig.6 *Num. solution trajectory of centers of mass,* $\varphi(t) = 0$

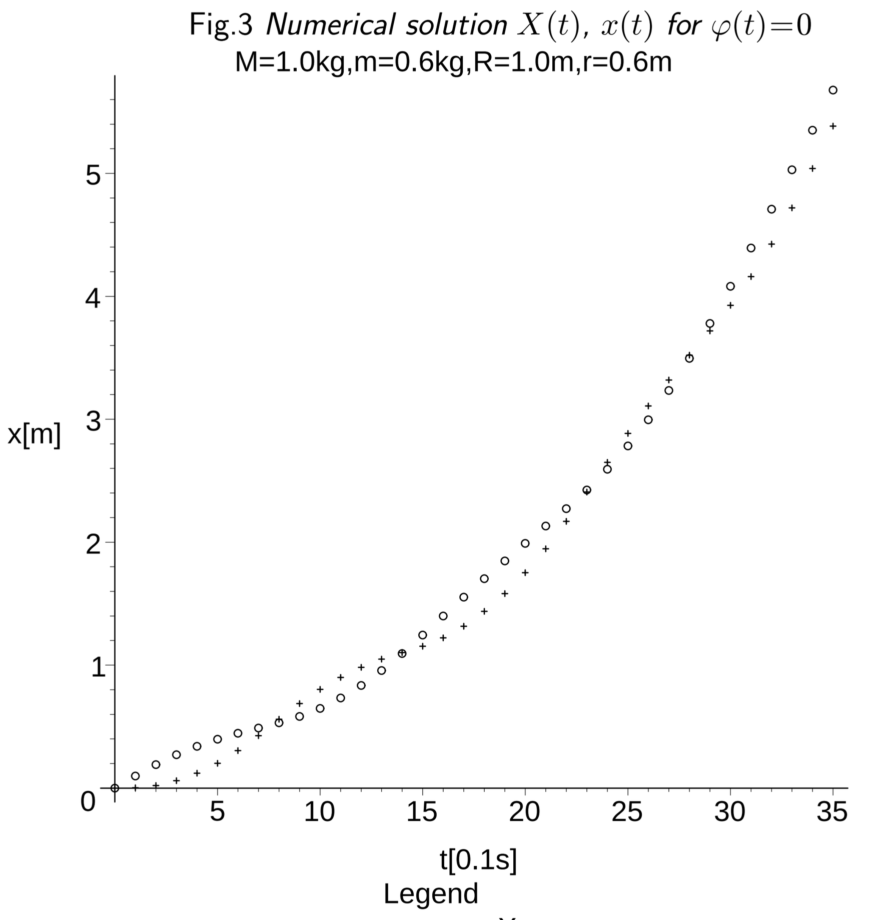

Fig.7 *Numerical solution* $X(t)$, $x(t)$ *for* $\varphi(t) = -t/10$

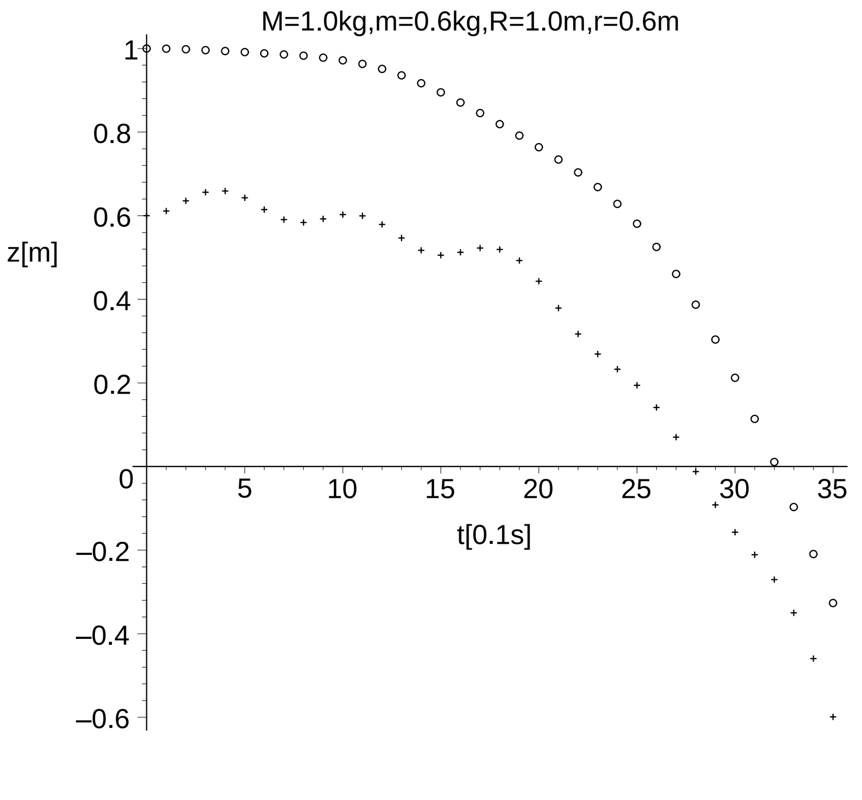

Fig.8 *Numerical solution* $Z(t)$, $z(t)$ *for* $\varphi(t) = -t/10$

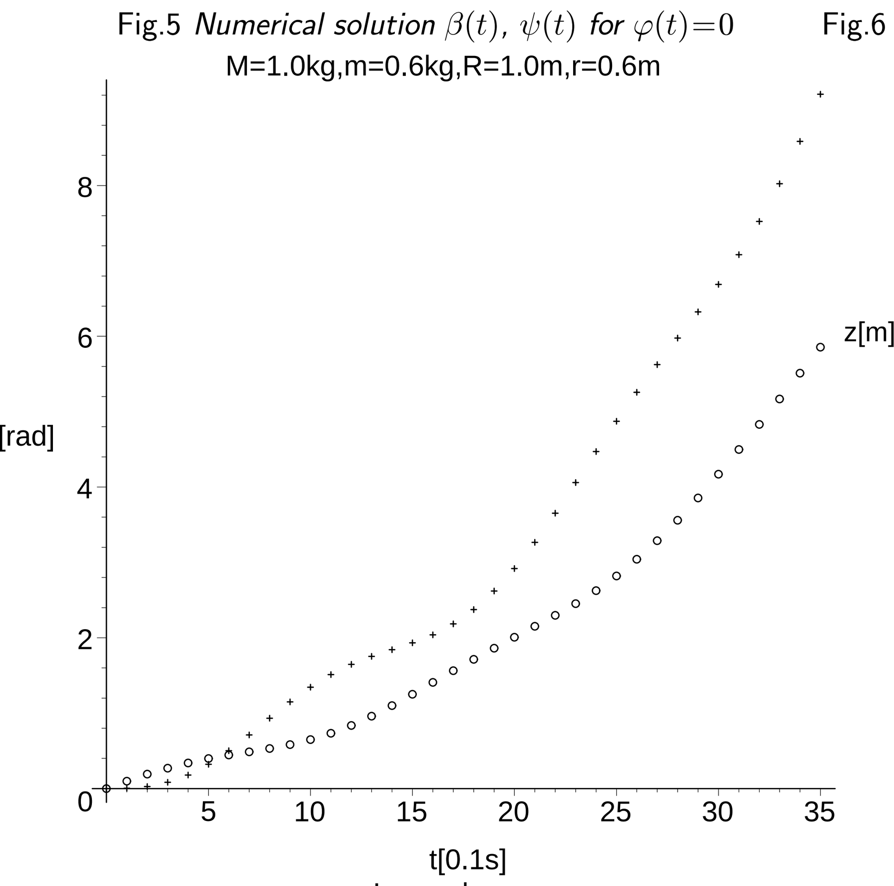

Fig.9 *Numerical solution* β, ψ *for* $\varphi(t) = -t/10$

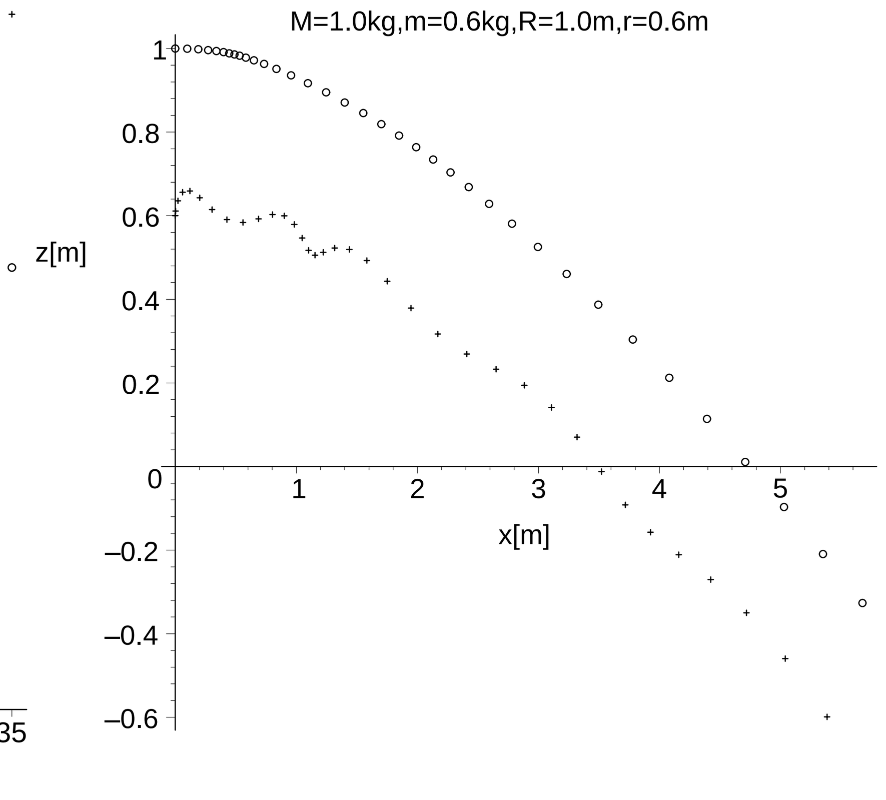

Fig.10 *Num. solution, trajectory of centers of mass,* $\varphi(t) = -t/10$