# Weyl's Theorem

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We want to prove the following theorem concerning a real semisimple Lie group $G$ and its Lie algebra $\mathfrak{g}$.

**Theorem 1** (Weyl)**.** *The Killing form of the semisimple Lie algebra* $\mathfrak{g}$ *is negative definite if and only if the corresponding Lie group* $G$ *is compact as a manifold.*

Let us fix conventions. The *Killing form* on a Lie algebra is defined as

$$\langle X, Y \rangle = \mathrm{Tr}(\mathrm{ad}(X) \circ \mathrm{ad}(Y)).$$

*Proof.* "$\Leftarrow$". The proof of the converse part is fairly easy if one assumes the possibility of integration on the Lie group $G$. [1] Out of any scalar product on $\mathfrak{g}$ one may then construct a $G$-invariant scalar product $(\cdot,\cdot)$ by averaging over the group (it must be the Killing form up to a positive factor). With respect to this scalar product the linear map $\mathrm{ad}(X)\colon \mathfrak{g} \to \mathfrak{g}$ is antisymmetric; we have $(\mathrm{Ad}(g)X, \mathrm{Ad}(g)Y) = (X, Y)$ and so after differentiation $(\mathrm{ad}(Z)X, Y) + (X, \mathrm{ad}(Z)Y) = 0$. The eigenvalues of an antisymmetric operator $A$ are purely imaginary. Let $Au = \lambda u$. Then $\lambda(u,u) = (u, Au) = (A^{\mathrm{T}}u, u) = -(Au, u) = -\lambda^*(u,u)$. The zero eigenvalues are excluded since for semisimple $\mathfrak{g}$ ad is a bijection. Thus a composition of two such operators has negative eigenvalues and the scalar product is always negative for nonzero elements of $\mathfrak{g}$.

"$\Rightarrow$". For the other direction we shall use some Riemannian geometry. We shall construct a $G$-bi-invariant Riemann metric $\rho$ on $G$ by translating the negative of the Killing form on $\mathfrak{g} = T_eG$ to $T_gG$ as follows

$$\rho(\xi,\eta)(g) := -\langle T_g L_{g^{-1}}\xi, T_g L_{g^{-1}}\eta\rangle.$$

We can easily compute the curvatures of $\rho$. First we shall use the Koszul formula for a Riemann metric $\rho$ and the Levi-Civita connection $\nabla$.

$$2\rho(\nabla_X Y, Z) = X\rho(Y,Z) + Y\rho(X,Z) - Z\rho(X,Y) + \\ + \rho([X,Y],Z) - \rho([X,Z],Y) - \rho([Y,Z],X).$$

For left invariant vector fields $X, Y, Z$ on $G$ and a bi-invariant metric $\rho$ the formula simplifies: the first three summands obviously vanish and the last two cancel. We are left with

$$2\rho(\nabla_X Y, Z) = \rho(Z, [X,Y]),$$

[1] For a compact Lie group one proceeds as follows: Out of the left-invariant Maurer-Cartan form $\omega$ one may construct a top-dimensional form $\nu$ by using the wedge product. The form $\nu$ is a left invariant volume element on $G$ (which is actually also right invariant for compact $G$).

i.e.

$$\nabla_X Y = \frac{1}{2}[X,Y].$$

The geodesics are obviously the integral curves of left-invariant vector fields, i.e. one parameter subroups, in particular, $G$ is complete. Let us compute the Riemann curvature

$$R(X,Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z =$$
$$= \frac{1}{4}([X,[Y,Z]] - [Y,[X,Z]] - 2[[X,Y],Z]) = -\frac{1}{4}[[X,Y],Z],$$

where the last step comes from using the Jacobi identity. We can also rewrite this as

$$R(X,Y)Z = \frac{1}{4}\operatorname{ad}(Z)\circ\operatorname{ad}(X)Y.$$

Now let us compute the Ricci curvature

$$\operatorname{Ricci}(X,Y) := \operatorname{Tr}(Z \mapsto R(X,Z)Y) = \frac{1}{4}\rho(X,Y).$$

We see that it is a multiple of the Killing form. We also see that $\operatorname{Ricci}(X,X) = 1/4\rho(X,X) \geq (n-1)/r^2\rho(X,X) > 0$, so the prerequisites of the Bonnet-Myers theorem are satisfied. We see that $G$ is bounded by $r$ and therefore compact. [2] ■

**Theorem 2** (Bonnet, Myers)**.** *Let $(M,g)$ be a complete Riemann manifold,* $\dim M = n$*. Suppose that the Ricci curvature of $M$ satisfies*

$$\operatorname{Ricci}(X,X)(p) \geq \frac{n-1}{r^2} g(X,X) > 0$$

*for all $p \in M$ and all $X \in T_pM$. Then $M$ is compact and the geodesic distances of the points of $M$ are bounded by $\pi r$ from above.*

*Proof loosely following do Carmo.* Let $p,q \in M$ be arbitrary. Since $M$ is complete there exists (Hopf-Rinow theorem) a minimizing geodesic segment $\gamma\colon [0,1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$. It suffices to show, that

$$\ell = \ell(\gamma) := \int_0^1 g(\gamma',\gamma')^{1/2}\,\mathrm{d}t \leq \pi r.$$

Then, because $M$ is bounded and complete it is also compact. We will proceed by contradiction. Assume that $\ell(\gamma) > \pi r$. Set $e_1 = \gamma'/\ell$ and extend it to an orthonormal basis $(e_1,\dots,e_n)$ of $T_{\gamma(t)}M$. Define the vector fields $v_j$ along $\gamma$ by

$$v_j = \sin(\pi t)e_j, \quad j \in \{2,\dots,n\}.$$

[2] We remind the reader of the form of $\rho(X,X)$ for compact matrix groups: for $\mathfrak{su}(n)$ it is $-2n\operatorname{Tr}(X^2)$, for $\mathfrak{so}(n)$ it is $(2-n)\operatorname{Tr}(X^2)$ and for $\mathfrak{sp}(2n)$ it is $-2(n+1)\operatorname{Tr}(X^2)$.

Note that $v_j(0) = v_j(1) = 0$ so $v_j$'s induce proper variations of $\gamma$ namely $\delta(s,t)$. Concretely, we have

$$\delta_j(0,t) = \gamma(t), \quad \frac{\partial \delta_j}{\partial s} = v_j.$$

Let us denote their energies by

$$E_j(s) = \frac{1}{2}\int_0^1 g(\delta', \delta')\,\mathrm{d}t.$$

For the first and second variation we have

$$E_j'(0) = -\int_0^1 g(v_j, \frac{D}{\mathrm{d}t}\frac{\mathrm{d}\gamma}{\mathrm{d}t})\,\mathrm{d}t = 0$$
$$E_j''(0) = -\int_0^1 g(v_j, \frac{D^2 v_j}{\mathrm{d}t^2} + R(\frac{\mathrm{d}\gamma}{\mathrm{d}t}, v_j)\frac{\mathrm{d}\gamma}{\mathrm{d}t})\,\mathrm{d}t.$$

Let us compute the second variation explicitly

$$E_j''(0) = -\int_0^1 g(\sin(\pi t)e_j, (\sin(\pi t)e_j)'' + \ell^2 R(e_1, \sin(\pi t)e_j)e_1)\,\mathrm{d}t =$$
$$= \int_0^1 \sin^2(\pi t)(\pi^2 - \ell^2 K(e_1, e_j))\,\mathrm{d}t,$$

where $K(e_1, e_j)$ is the sectional curvature in the plane spanned by $e_1$ and $e_j$. Summing the previous expression through $j = 2 \ldots n$ we get

$$\sum_{j=2}^n E_j''(s) = \int_0^1 ((n-1)\pi^2 - \ell^2 \operatorname{Ricci}(e_1, e_1)(\gamma(t)))\sin^2(\pi t)\,\mathrm{d}t$$

and since $\operatorname{Ricci}(e_1, e_1) \geq (n-1)/r^2$ we get

$$\sum_{j=2}^n E_j''(s) \leq \int_0^1 (n-1)(\pi^2 - \frac{\ell^2}{r^2})\sin^2(\pi t)\,\mathrm{d}t < 0.$$

This produces a contradiction since $\gamma$ is a minimising geodesic. ■