Weyl’s Theorem

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We want to prove the following theorem concerning a real semisimple Lie group $G$ and its Lie algebra $\mathfrak{g}$.

**Theorem 1 (Weyl).** The Killing form of the semisimple Lie algebra $\mathfrak{g}$ is negative definite if and only if the corresponding Lie group $G$ is compact as a manifold.

Let us fix conventions. The Killing form on a Lie algebra is defined as

$$\langle X, Y \rangle = \operatorname{Tr}(\text{ad}(X) \circ \text{ad}(Y)).$$

**Proof.** "$\Leftarrow"$. The proof of the converse part is fairly easy if one assumes the possibility of integration on the Lie group $G$. Out of any scalar product on $\mathfrak{g}$ one may then construct a $G$-invariant scalar product $(\cdot, \cdot)$ by averaging over the group (it must be the Killing form up to a positive factor). With respect to this scalar product the linear map $\text{ad}(X)$ : $\mathfrak{g} \rightarrow \mathfrak{g}$ is antisymmetric; we have $(\text{Ad}(g)X, \text{Ad}(g)Y) = (X, Y)$ and so after differentiation $(\text{ad}(Z)X, Y) + (X, \text{ad}(Z)Y) = 0$. The eigenvalues of an antisymmetric operator $A$ are purely imaginary. Let $Au = \lambda u$. Then $\lambda(u, u) = (u, Au) = (A^T u, u) = -(Au, u) = -\lambda^*(u, u)$. The zero eigenvalues are excluded since for semisimple $\mathfrak{g}$ ad is a bijection. Thus a composition of two such operators has negative eigenvalues and the scalar product is always negative for nonzero elements of $\mathfrak{g}$.

"$\Rightarrow"$. For the other direction we shall use some Riemannian geometry. We shall construct a $G$-bi-invariant Riemann metric $\rho$ on $G$ by translating the negative of the Killing form on $\mathfrak{g} = T_eG$ to $T_gG$ as follows

$$\rho(\xi, \eta)(g) := -\langle T_gL_{g^{-1}}\xi, T_gL_{g^{-1}}\eta \rangle.$$

We can easily compute the curvatures of $\rho$. First we shall use the Koszul formula for a Riemann metric $\rho$ and the Levi-Civita connection $\nabla$.

$$2\rho(\nabla_X Y, Z) = X\rho(Y, Z) + Y\rho(X, Z) - Z\rho(X, Y) +$$

$$+ \rho([X, Y], Z) - \rho([X, Z], Y) - \rho([Y, Z], X).$$

For left invariant vector fields $X, Y, Z$ on $G$ and a bi-invariant metric $\rho$ the formula simplifies: the first three summands obviously vanish and the last two cancel. We are left with

$$2\rho(\nabla_X Y, Z) = \rho(Z, [X, Y]),$$

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1 For a compact Lie group one proceeds as follows: Out of the left-invariant Maurer-Cartan form $\omega$ one may construct a top-dimensional form $\nu$ by using the wedge product. The form $\nu$ is a left invariant volume element on $G$ (which is actually also right invariant for compact $G$).
i.e.
\[ \nabla_X Y = \frac{1}{2} [X, Y]. \]

The geodesics are obviously the integral curves of left-invariant vector fields, i.e. one parameter subgroups, in particular, \( G \) is complete. Let us compute the Riemann curvature

\[
R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z =
\frac{1}{4} (\{X, \{Y, Z\}\} - \{Y, \{X, Z\}\} - 2\{X, Y, Z\}) = -\frac{1}{4} \{X, Y, Z\},
\]

where the last step comes from using the Jacobi identity. We can also rewrite this as

\[
R(X, Y)Z = \frac{1}{4} \text{ad}(Z) \circ \text{ad}(X) Y.
\]

Now let us compute the Ricci curvature

\[
\text{Ricci}(X, Y) := \text{Tr}(Z \mapsto R(X, Z)Y) = \frac{1}{4} \rho(X, Y).
\]

We see that it is a multiple of the Killing form. We also see that \( \text{Ricci}(X, X) = 1/4 \rho(X, X) \geq (n-1)/r^2 \rho(X, X) > 0 \), so the prerequisites of the Bonnet-Myers theorem are satisfied. We see that \( G \) is bounded by \( r \) and therefore compact.\[\square\]

**Theorem 2** (Bonnet, Myers). Let \((M, g)\) be a complete Riemann manifold, \(\dim M = n\). Suppose that the Ricci curvature of \(M\) satisfies

\[
\text{Ricci}(X, X)(p) \geq \frac{n-1}{r^2} g(X, X) > 0
\]

for all \(p \in M\) and all \(X \in T_p M\). Then \(M\) is compact and the geodesic distances of the points of \(M\) are bounded by \(\pi r\) from above.

**Proof loosely following do Carmo.** Let \(p, q \in M\) be arbitrary. Since \(M\) is complete there exists (Hopf-Rinow theorem) a minimizing geodesic segment \(\gamma: [0,1] \to M\) such that \(\gamma(0) = p\) and \(\gamma(1) = q\). It suffices to show, that

\[
\ell = \ell(\gamma) := \int_0^1 g(\gamma', \gamma')^{1/2} dt \leq \pi r.
\]

Then, because \(M\) is bounded and complete it is also compact. We will proceed by contradiction. Assume that \(\ell(\gamma) > \pi r\). Set \(e_1 = \gamma'/\ell\) and extend it to an orthonormal basis \((e_1, \ldots, e_n)\) of \(T_{\gamma(t)} M\). Define the vector fields \(v_j\) along \(\gamma\) by

\[
v_j = \sin(\pi t) e_j, \quad j \in \{2, \ldots, n\}.
\]

\[\text{We remind the reader of the form of } \rho(X, X) \text{ for compact matrix groups: for } su(n) \text{ it is } -2n \text{ Tr}(X^2), \]

\[\text{for } so(n) \text{ it is } (2-n) \text{ Tr}(X^2) \text{ and for } sp(2n) \text{ it is } -2(n+1) \text{ Tr}(X^2)\].

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Note that $v_j(0) = v_j(1) = 0$ so $v_j$’s induce proper variations of $\gamma$ namely $\delta(s,t)$. Concretely, we have

$$\delta_j(0,t) = \gamma(t), \quad \frac{\partial \delta_j}{\partial s} = v_j.$$ 

Let us denote their energies by

$$E_j(s) = \frac{1}{2} \int_0^1 g(\delta', \delta') \, dt.$$ 

For the first and second variation we have

$$E_j'(0) = -\int_0^1 g(v_j, \frac{D}{dt} \frac{d \gamma}{dt}) \, dt = 0$$

$$E_j''(0) = -\int_0^1 g(v_j, \frac{D^2 v_j}{dt^2} + R(\frac{d \gamma}{dt}, v_j) \frac{d \gamma}{dt}) \, dt.$$ 

Let us compute the second variation explicitly

$$E_j''(0) = -\int_0^1 g(\sin(\pi t) e_j, (\sin(\pi t) e_j)'') + \ell^2 R(e_1, \sin(\pi t) e_1) e_1) \, dt =$$

$$= \int_0^1 \sin^2(\pi t)(\pi^2 - \ell^2 K(e_1, e_j)) \, dt,$$

where $K(e_1, e_j)$ is the sectional curvature in the plane spanned by $e_1$ and $e_j$. Summing the previous expression through $j = 2 \ldots n$ we get

$$\sum_{j=2}^n E_j''(s) = \int_0^1 \left((n-1)\pi^2 - \ell^2 \text{Ricci}(e_1, e_1)(\gamma(t))\right) \sin^2(\pi t) \, dt$$

and since $\text{Ricci}(e_1, e_1) \geq (n-1)/\ell^2$ we get

$$\sum_{j=2}^n E_j''(s) \leq \int_0^1 (n-1)(\pi^2 - \frac{\ell^2}{\ell^2}) \sin^2(\pi t) \, dt < 0.$$ 

This produces a contradiction since $\gamma$ is a minimising geodesic. \[\blacksquare\]