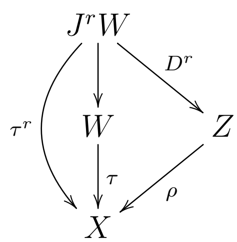
The Geometry of the Hamilton-Jacobi Equation

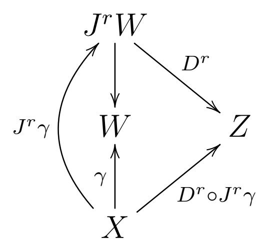
Michael Krbek

1. Differential operators. Let $\tau\colon W\to X$ and $\rho\colon Z\to X$ be two vector bundles over the same base X and let $\Gamma(W)$, resp. $\Gamma(Z)$ denote the set of smooth local sections of W, resp. Z. A mapping $D\colon \Gamma(W)\to\Gamma(Z)$ is said to be a ***differential operator***, if there exist an integer $r\geq 0$ and a morphism of vector bundles $D^r\colon J^rW\to Z$ over the identity id_X such that for every section $\gamma\in\Gamma(W)$

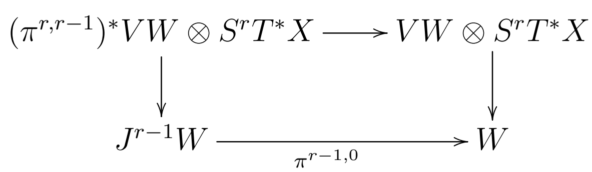
$$D(\gamma)=D^r\circ J^r\gamma.$$

The minimal such integer r is called the ***order*** of the differential operator D.

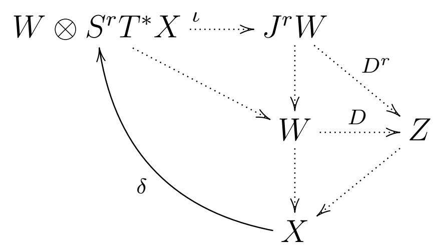




2. Symbol of a differential operator. If $\tau\colon W\to X$ is vector bundle, then so is the prolongation $J^rW\to X$, in any case the bundle $J^rW\to J^{r-1}W$ is affine, the underlying vector bundle being $(\pi^{r,r-1})^*VW\otimes S^rT^*X$, VW being the vertical bundle (in the case of vector bundles, it can be identified with W itself).



This allows to define the ***symbol*** of the differential operator D as a map $\Sigma\colon \Gamma(W \otimes S^r T^* X) \to \Gamma(Z)$ using the diagram



Denote $\pi\colon T^*X \to X$ the canonical projection. Let us consider the situation in local coordinate charts (U, x^i) on X and adapted charts $(\tau^{-1}(U), w^a)$ resp. $(\rho^{-1}(U), z^b)$ resp. $(\pi^{-1}(U), p_j)$.

$$z^b(D(\gamma)) = \sum_{|I|\le r} D_a^{b,I} \frac{\partial^{|I|} w^a(\gamma)}{\partial x^I}$$

and

$$z^b(\Sigma(D)(\delta)) = \sum_{|I|=r} D_a^{b,I} w^a(\delta) p_I(\delta),$$

where $I = (i_1 \dots i_k)$, $1 \le i_1 \le \cdots \le i_k \le n = \dim X$ is a symmetric multiindex, its length is $|I| = k$,

$$\frac{\partial^{|I|}}{\partial x^I} = \frac{\partial^k}{\partial x^{i_1} \dots \partial x^{i_k}}$$

and

$$p_I = p_1^{j_1} \dots p_n^{j_n},$$

where j_ℓ is the number of times the index ℓ occurs in the multiindex I.

3. Example. Let $W = Z = \Lambda^r T^* X$, $(\cdot, \cdot)$ be a (semi)-Riemannian metric of signature $(n-p, p)$ on X. Consider the second order Laplace-Beltrami operator

$$\Box\colon \Gamma(\Lambda^r T^* X) \to \Gamma(\Lambda^r T^* X),$$

where explicitly $\Box = \star\,\mathrm{d}\star\mathrm{d} - \mathrm{d}\star\mathrm{d}\star$.

The symbol of $\Box$ then is

$$\Sigma(\Box)(p) = (p,p)(-1)^{r(n-r)+p}\,\mathrm{id}_{\Gamma(\Lambda^r T^* X)}$$

4. Asymptotic expansions. Let B be a topological vector space (we always have in mind $B = \Gamma(W)$; a sequence of sections converges if it has common support and it together with all derivatives of arbitrary finite order converge uniformly). Consider smooth maps $u, v\colon \mathbb{R} \to B$. Consider the (equivalence) relation $u \approx v$

$$\lim_{t\to\infty} t^N(u(t) - v(t)), \quad \forall N > 0.$$

We define the ***asymptotic expansion*** of u if there exists a series of vectors u_k such that

$$t^N \left(u(t) - \sum_{k \leq N} u_k t^{-k} \right) \to 0.$$

Clearly, if the series exists, it depends only on the equivalence class $[u]$. Let $B_1, \dots B_\ell$, C be topological vector spaces and $B\colon B_1 \times \cdots B_\ell \to C$ a continuous linear map. Then we have

$$B([u_1], \dots, [u_\ell]) = [B(u_1, \dots, u_\ell).$$

An ***asymptotic differential operator*** $[L]\colon \Gamma(W) \to \Gamma(Z)$ is an asymptotic expansion of operators $L_k\colon E \to F$ such that

$$L \approx \sum_k L_k t^{-k},$$

a ***simple asymptotic section*** of W has the form

$$\gamma \approx \exp(\mathrm{i}\, tS) \sum_k \gamma_k / (\mathrm{i}\, t)^k.$$

We wish to solve $L\gamma = 0$ asymptotically for a simple asymptotic section γ. We have

$$[L][\gamma] = \exp(\mathrm{i}\, tS) \sum_k \delta_k / (\mathrm{i}\, t)^k, \quad \delta_k = 0\ \forall k.$$

Define $\Sigma([L]) = \sum_k \Sigma(L_k)$. We therefore demand

$$\Sigma([L])(\mathrm{d}\, S)\gamma_0 = 0 \quad \textit{(the characteristic equation)}$$

and for γ_k, $k > 0$, one may proceed inductively.

5. The method of Hamilton and Jacobi. So in order to get a solution, we must demand

$$\ker \Sigma([L])(\mathrm{d}\, S) \neq 0.$$

The characteristic variety $\mathscr{V} \subset T^*X$ consists of all points p where $\ker \Sigma([L])(p) \neq 0$. Now we have

$$(\mathrm{d}\, S)(x) = 0 \quad \forall x \in X.$$

so the image of $\mathrm{d}\, S$ must lie in $\mathscr{V}$. The map $\mathrm{d}\, S$ can be thought of as $X \to T^*X$; it is a (so called holonomic) section of T^*X. We break the task into two parts:

(a) Find a section $\gamma\colon X \to T^*X$ such that $\gamma(X) \subset \mathscr{V}$.

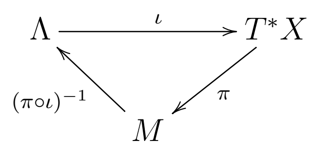
(b) Find a function S such that $\gamma(x) = \mathrm{d}\, S(x)$, $\forall x \in X$.

We have a canonical linear form α on T^*X given by $\alpha_p(v_p) = p(T\pi_p \cdot v_p)$, $p \in T^*X$, $v_p \in T_pT^*X$. So $\gamma \in \Gamma(T^*X)$ iff $\gamma^*\alpha = \gamma$ and $\gamma = \mathrm{d}\, S$ iff $\gamma^*\alpha = \mathrm{d}\, S$. Again we may try to relax this condition in two ways:

(i) Take a closed 2-form ω and demand $\gamma^*\omega = 0$. If $\omega = \mathrm{d}\,\alpha$ we get the previous condition.

(ii) Do not require for γ to be a section of T^*X. Let $\gamma\colon Y \to T^*X$ such that $\gamma^*\omega = 0$, $\dim Y = \dim X$ and γ is an immersion. Such a γ is called a ***Lagrangian submanifold***.

Notice that if $\iota\colon \Lambda \to T^*X$ is a Lagrangian submanifold and $\pi \circ \iota$ a diffeomorphism then $\gamma = \iota \circ (\pi \circ \iota)^{-1}$ is a section of T^*X.



Let $\mathscr{V} \subset T^*X$. We seek a Lagrangian submanifold $\iota\colon \Lambda \to T^*X$ such that

(1) $\iota(\Lambda) \subset \mathscr{V}$,

(2) $\pi \circ \iota$ is a diffeomorphism,

(3) $\iota^*\alpha = \mathrm{d}\, S$.

6. Solving (1)+(2)+(3). (1) This may be done on any ***symplectic manifold*** (M, ω) (we have $M = T^*X$ and $\omega = \mathrm{d}\,\alpha$). Let H be a function on M such that $\mathrm{d}\, H \neq 0$ if $H = 0$. Define the vector field ξ_H by $\omega(\xi_H, \cdot) = -\,\mathrm{d}\, H$.

Theorem: Let $\mathscr{V} \subset M$ be integrable of codimension k. Let $\Lambda_0 \subset \mathscr{V}$ of dimension $n-k$ be isotropic (with respect to ω) and transversal to all ξ_f, $f \in \mathrm{Zero}(\mathscr{V})$. Then there exists (an essentially) unique Lagrangian submanifold Λ, $\Lambda_0 \subset \Lambda \subset \mathscr{V}$.

(2) The solution is clearly possible only locally, two manifolds of the same dimension need not be diffeomorphic but they are always locally diffeomorphic.

(3) The solution is also possible locally, generally there are obstructions in the appropriate de Rham cohomology group.

7. The bicharacteristic symbol.

For each $p \in T^*X$ we have a linear map

$$\Sigma([L])(p)\colon W_{\pi(p)} \to Z_{\pi(p)}.$$

Using the pullback by π we can consider W and Z as vector bundles over $M = T^*X$ and consider the exact sequence

$$0 \longrightarrow \ker \Sigma([L])(p) \longrightarrow W_p \xrightarrow{\Sigma([L])(p)} Z_p \longrightarrow \operatorname{im} \Sigma([L])(p) \longrightarrow 0$$

So $\Sigma\colon W \to Z$ is a vector bundle map (over M!) and we have a map $A\colon W_p \to Z_p$, $\forall p \in M$. We choose local trivializations for$E$ and F over $U \subset M$. For each $w \in W_p$ we choose $\gamma\colon U \to W_p$ such that $\gamma(p) = w$. then $A \circ \gamma\colon U \to Z_p$ and we can compute its differential $\mathrm{d}_p(A \circ \gamma)$, so if $\xi \in T_pU$ then $\mathrm{d}_p(A \circ \gamma)(\xi) \in Z_p$ and using the exact sequence we can project on $\operatorname{im} \Sigma(p)$. For $w \in \ker \Sigma(p)$ we have a map $I\colon \ker \Sigma \otimes TM \to \operatorname{im} \Sigma$.

If $\dim W = \dim Z$, $\dim \ker \Sigma = 1$ and $\Sigma_p \neq 0$ $\forall p \neq 0$ we say that Σ is ***simple***. Using the 1-1 correspondence achieved by ω between vectors and covectors we may define the ***bicharacteristic symbol*** $R\colon \ker \Sigma \otimes T^*M \to \operatorname{im} \Sigma$. The bicharacteristics of R correspond to trajectories in $M = T^*X$. and the space of such bicharacteristics carries a natural contact structure.

8. Generalization to principal bundles with 1-dimensional fiber.

Let $G = \mathbb{R}$ or $G = U(1)$ and $\mathscr{G} \to X$ be principal G-bundle. Consider the manifold $C\mathscr{G}$ of contact elements of $\mathscr{G}$, i.e. the manifold of all hyperplanes in $T\mathscr{G}$. The Hamilton-Jacobi equation is equivalent to a G-invariant submanifold E, $\operatorname{codim} E = 1$ in $C\mathscr{G}$.

The space $\operatorname{Char} E$ of characteristics of E can be also given the structure of a contact manifold (at least locally see Theorem). Let us suppose $\operatorname{Char} E$ is a contact manifold and G acts also $\operatorname{Char} E$, i.e. there exists a discrete normal subgroup $H \subset G$ such that $\operatorname{Char} E$ is a G/H-bundle. The base $\operatorname{Ph} E$ of this G/H-bundle may be thought of as phase space and the curvature of the bundle $\operatorname{Char} E \to \operatorname{Ph} E$ may be thought of as representing the usual symplectic form on phase space.

9. Generalization to arbitrary principal bundles.

Let G be an arbitrary Lie group. The setting is the same as in the case $\dim G = 1$ with one crucial difference — $\operatorname{Ph} E$ can no longer be thought of as carrying a symplectic structure just a weak generalization.

10. References.

1. Guillemin, Sternberg, *Geometric asymptotics*, AMS, 1977, 1990
2. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, 1978, 1989

3. Kolář, Michor, Slovák, *Natural Operations in Differential Geometry*, Springer, 1993

4. Čap, Slovák, *Parabolic Geometries*, unpublished

5. Blaom, arXiv: DG/0404313

6. Blaom, arXiv: DG/0509071