MacDowell-Mansouri Theory of Gravity

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Contents

1. The Lie algebra $\mathfrak{sp}(4,\mathbb{C})$ and its representations

Consider V to be a $2n$-dimensional vector space over the field $\mathbb{C}$ of complex numbers, and

$$J\colon V\times V\to\mathbb{C}$$

to be a nondegenerate, skew-symmetric bilinear form on V. The symplectic Lie group $\mathrm{Sp}(2n,\mathbb{C})$ is then defined to be the group of automorphisms A of V preserving the form J, i.e.

$$J(Av,Aw)=J(v,w),\qquad \text{for all}\quad v,w\in V$$

and its Lie algebra $\mathfrak{g}=\mathfrak{sp}(2n,\mathbb{C})$ then consists of endomorphisms $A\colon V\to V$ such that

$$J(Av,w)+J(v,Aw)=0,\qquad \text{for all}\quad v,w\in V.$$

Since all nondegenerate, skew-symmetric bilinear forms on V are equivalent we may as well choose a suitable J

$$J(e_i,e_{i+n})=1\quad\text{and}\quad J(e_i,e_j)=0\quad\text{for all}\quad i\neq j\pm n,$$

where $e_1,\dots,e_{2n}$ denotes an arbitrary fixed basis of V. In such a basis we may then write

$$J(u,v)=u^TJv,\quad\text{where}\quad J=\begin{pmatrix}0 & I_n\\ -I_n & 0\end{pmatrix}.$$

Here we denote both the bilinear form and its matrix by J and similarly the vectors and the matrices of their components in the basis $e_1,\dots,e_{2n}$ by u,v.

Hence $\mathrm{Sp}(2n,\mathbb{C})$ may be identified with the group of matrices A of order $2n$ satisfying

$$J=A^TJA$$

and the Lie algebra $\mathfrak{g}=\mathfrak{sp}(2n,\mathbb{C})$ with the space of matrices M of order $2n$ satisfiyng

$$M^TJ+JM=0.$$

Writing the matrix M using square matrices of order n, we deduce that

$$M=\begin{pmatrix}A & B\\ C & -A^T\end{pmatrix},$$

where, moreover, $B = B^T$ and $C = C^T$ are symmetric matrices.

The Cartan subalgebra $\mathfrak{h}$ is spanned by the diagonal matrices of such type, i.e. by the n matrices $H_i = E_{i,i} - E_{i+n,i+n}$. The dual $\mathfrak{h}^*$ has a basis $L_i(H_j) = \delta_{i,j}$ correspondingly. Consider the matrices $X_{i,j} = E_{i,j} - E_{n+j,n+i}$ which span the subspace

$$\begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix}$$

for $1 \leq i, j \leq n$ and $i \neq j$, the matrices $Y_{i,j} = E_{i,n+j} + E_{j,n+i}$ which span the subspace

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$$

for $1 \leq i < j \leq n$ and the matrices $Z_{i,j} = E_{n+i,j} + E_{n+j,i}$ which span the subspace

$$\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}$$

also for $1 \leq i < j \leq n$ and further the matrices $U_i = E_{i,n+i}$ and $V_i = E_{n+i,i}$ for $1 \leq i \leq n$. The previously defined matrices are generalized eigenvectors for the action of $\mathfrak{h}$ on $\mathfrak{g}$ with eigenvalues given by Table 1. So the roots of

Eigenvector	Eigenvalue
$X_{i,j}$	$L_i - L_j$
$Y_{i,j}$	$L_i + L_j$
$Z_{i,j}$	$-L_i - L_j$
U_i	$2L_i$
V_i	$-2L_i$

Table 1: Eigenvectors and eigenvalues for $\mathfrak{sp}(4, \mathbb{C})$

$\mathfrak{g}$ are the $2n^2$ vectors $\pm L_i \pm L_j \in \mathfrak{h}^*$. Specifically for $\mathfrak{sp}(4, \mathbb{C})$ we have eight roots depicted in Figure 1.

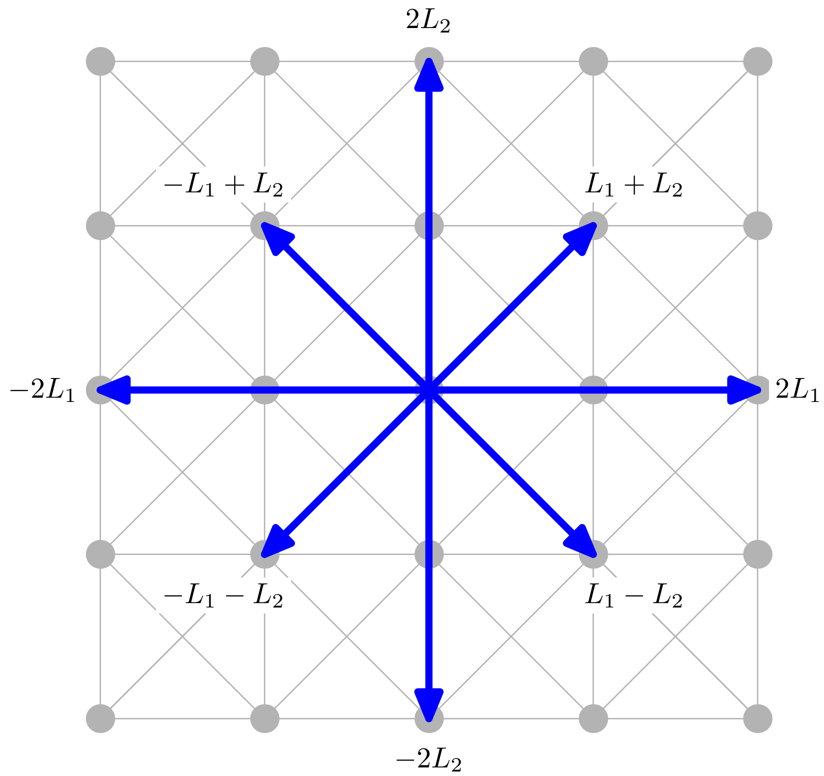


Figure 1: The roots of $\mathfrak{sp}(4,\mathbb{C})$ (the positive primitive roots are in red)

Next, we shall calculate the restriction of the Killing form to $\mathfrak{h}$ which is by definition

$$B(H,G) = \sum_{\alpha} \alpha(H)\alpha(G),$$

where the sum is over all roots $\alpha \in \mathfrak{h}^*$. If we express $H = \sum a_i H_i$ and $G = \sum b_j H_j$ in the basis H_i we get

$$B(H,G) = 4\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(a_i+a_j)(b_i+b_j) + \sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(a_i-a_j)(b_i-b_j) =$$

$$= 4(a,b) + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}(a_i b_i + a_j b_j) = 4(a,b) + 2\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n} a_i b_i + \sum_{j=1}^{n}\sum_{\substack{i=1\\ i\neq j}}^{n} a_j b_j =$$

$$= 4(a,b) + 2(n-1)(a,b) + 2(n-1)(a,b) = 4(n+1)(a,b),$$

where $(a,b) = \sum_{i=1}^{n} a_i b_i$ is the standard inner product in $\mathfrak{h}$.

Next, we must locate the distinguished copies of $\mathfrak{sl}(2,\mathbb{C})$ and the corresponding elements $H_\alpha \in \mathfrak{h}$. The first class of copies is given by

$$[X_{i,j}, X_{j,i}] = H_i - H_j,$$

so the distinguished element in $\mathfrak{h}$ is a multiple of $H_i - H_j$ but as we have $[H_i - H_j, X_{i,j}] = X_{i,j}$ the multiple is 1.

The second class is given by

$$[Y_{i,j}, Z_{i,j}] = H_i + H_j, \quad [H_i + H_j, Y_{i,j}] = 2Y_{i,j}$$

and the third class by

$$[U_i, V_i] = H_i, \quad [H_i, U_i] = 2U_i.$$

The complete results are written in Table 2.

Eigenvalue α	Element H_α
$L_i - L_j$	$H_i - H_j$
$L_i + L_j$	$H_i + H_j$
$-L_i - L_j$	$-H_i - H_j$
$2L_i$	H_i
$-2L_i$	$-H_i$

Table 2: Eigenvalues α and distinguished elements $H_\alpha \in \mathfrak{h}$

This means that the weight lattice of linear forms in $\mathfrak{h}^*$ integral in the arguments H_α is exactly the lattice of integral linear combinations of the L_i.

The group of symmetries of the weights of an arbitrary representation of $\mathfrak{sp}(2n,\mathbb{C})$ is generated by involutions in $\mathfrak{h}^*$ fixing the hyperplane given by $L(H_\alpha) = 0$ for each root α and acting as $-\operatorname{id}$ on the line spanned by α. But

$[\cdot,\cdot]$	H_1	H_2	$X_{1,2}$	$X_{2,1}$	$Y_{1,2}$	$Z_{1,2}$	U_1	U_2	V_1	V_2
H_1	$\cdot$	0	$X_{1,2}$	$-X_{2,1}$	$Y_{1,2}$	$-Z_{1,2}$	$2U_1$	0	$-2V_1$	0
H_2	$\cdot$	$\cdot$	$-X_{1,2}$	$X_{2,1}$	$Y_{1,2}$	$-Z_{1,2}$	0	$2U_2$	0	$-2V_2$
$X_{1,2}$	$\cdot$	$\cdot$	$\cdot$	$H_1 - H_2$	$2U_1$	$-2V_2$	0	$Y_{1,2}$	$-Z_{1,2}$	0
$X_{2,1}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	U_2	V_1	$Y_{1,2}$	0	0	$-Z_{1,2}$
$Y_{1,2}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$H_1 + H_2$	0	0	W	X
$Z_{1,2}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-X_{1,2}$	$-X_{2,1}$	0	0
U_1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	0	H_1	0
U_2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	0	H_2
V_1	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	0
V_2	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table 3: Multiplication table for the distinguished elements in $\mathfrak{sp}(4,\mathbb{C})$

α is perpendicular to the hyperplane $L(H_\alpha) = 0$ (using the induced scalar product $\tilde{B}$ on $\mathfrak{h}^*$), so this is a reflection.

In the case of $\mathfrak{sp}(4,\mathbb{C})$ we get the dihedral group Dih_4 (the group of symmetries of the square) as seen if Figure 2.

We may choose the positive roots as $\Delta^+ = \{L_i + L_j\}_{i\le j} \cup \{L_i - L_j\}_{i<j}$, the primitive roots being $\{L_i - L_{i+1}\}$ for $i = 1,\dots,n-1$ and $2L_n$. The Weyl chamber is

$$W = \left\{ \sum_{i=1}^{n} a_i L_i | a_i \ge a_{i+1} \ge 0, i = \{1,\dots,n-1\} \right\}.$$

The situation is depicted in Figure 3 for $\mathfrak{sp}(4,\mathbb{C})$ again. According to the general theory, there is a unique irreducible representation Γ_α associated with the highest weight α for any α in the intersection of the weight lattice with the (closed) Weyl chamber W. Any such α can be written as a nonnegative integral linear combination of L_1 and $L_1 + L_2$ so that the irreducible representations may as well be characterized by a pair of nonnegative integers (a,b) such that $\alpha = aL_1 + b(L_1 + L_2)$ as $\Gamma_{a,b}$. In Figures 5,4 and 6 the weights of representations are depicted including their multiplicities. Using Young theory for $\mathfrak{sp}(2n,\mathbb{C})$, these representations are equivalent to

$$\Gamma_{1,0} \sim \ydiagram{1} \qquad \Gamma_{0,1} \sim \ydiagram{1,1} \qquad \Gamma_{1,1} \sim \ydiagram{2,1}.$$

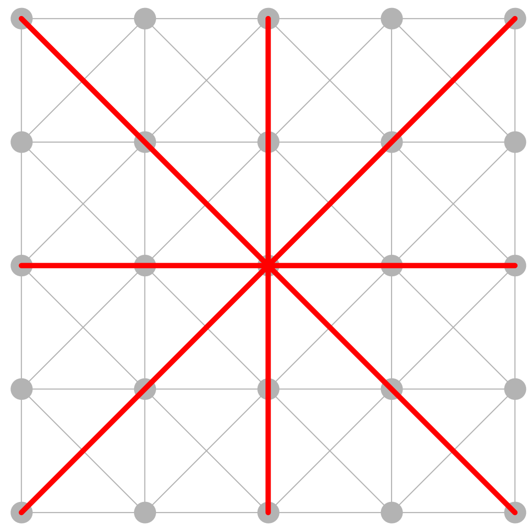


Figure 2: The symmetries of weights (grey dots) generated by reflections about the four lines (in red)

The information about the structure of the Lie algebra may be encoded in its Dynkin diagram whose nodes correspond to positive primitive roots, which are connected by single, double or (in one case) triple lines according to the angle between such roots and in case of roots of different length, there is an arrow pointing from the larger to the shorter root. For $\mathfrak{sp}(2n,\mathbb{C})$ the diagram (usually denoted by $\mathscr{C}_n$) we may observe the Dynkin diagram in Figure 7.

There is also an easy way to label irreducible representations of semisimple Lie algebras by placing a (nonnegative) integer equal to the coefficient of the primitive weight ω_i over each node corresponding to the positive primitive root α_i, such that all other positive primitive roots are perpendicular to this primitive weight, i.e.

$$\frac{2\langle\omega_i,\alpha_j\rangle}{\langle\alpha_j,\alpha_j\rangle}=\delta_{ij}.$$

For example, the representation $\Gamma_{2,1}$ can be depicted (in increasing order of abstraction) using the weight diagram in Figure 8.

The Lie algebra $\mathfrak{sp}(4,\mathbb{C})$ is isomorphic to $\mathfrak{so}(5,\mathbb{C})$. The isomorphism may be constructed as follows: consider the dual of the standard representation V^* of $\mathfrak{sp}(4,\mathbb{C})$ on $\mathbb{C}^{4*}$. In the representation Λ^2V^* there is a one-dimensional

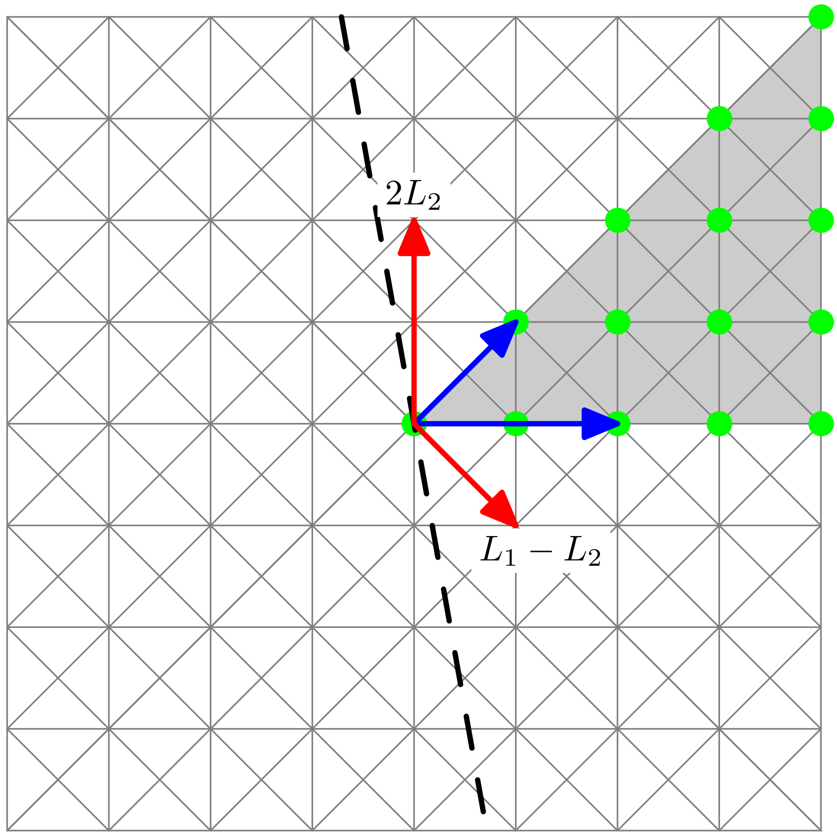


Figure 3: Choice of positive direction (black dashed), primitive roots (red), other positive roots (blue), Weyl chamber (shaded), possible highest weights (green)

invariant subspace given by the complex multiples of the canonical symplectic form ω, so the remaining irreducible representation W is five-dimensional by complete reducibility. There is an invariant non-degenarate scalar product on $\Lambda^2 V^*$ given on decomposable 2-vectors $a \wedge b$ $c \wedge d$ as $(a \wedge b, c \wedge d) = \omega(a, b)\omega(c, d)$ (this can be extended to the whole of $\Lambda^2 V^*$ by complex linearity). This inner product can then be restricted to W and W is invariant with respect to it by its very construction. The homomorphism constructed is an isomorphism by simplicity of both algebras.

Let us consider the real forms of $\mathfrak{sp}(4, \mathbb{C})$ using Satake diagrams (the compact

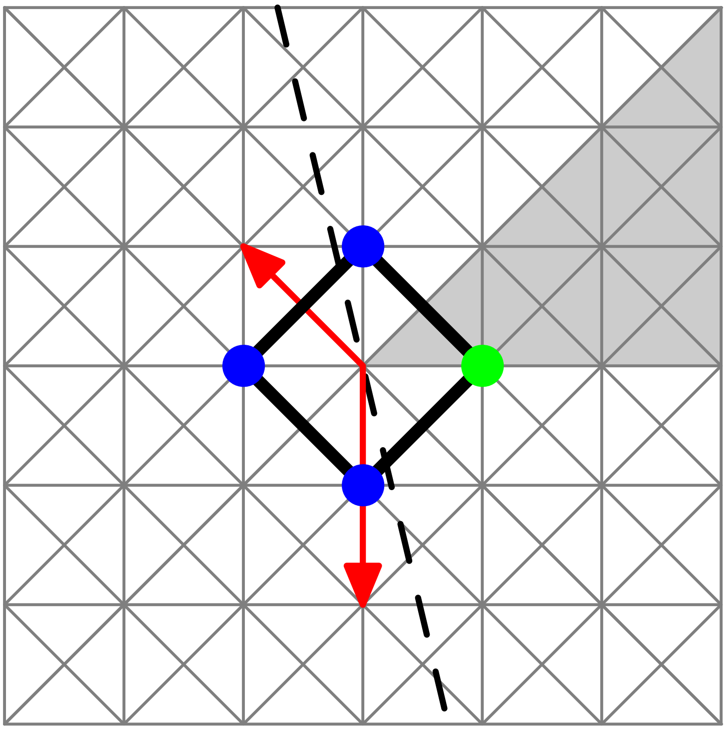


Figure 4: The highest weight representations $\Gamma_{1,0}$ the highest weight in green, other weights in blue, negative primitive roots in red

roots are denoted by black dots, the non-compact ones by hollow dots) and at the same time the question whether the representation in question is real or quaternionic which is decided by computing the Karpelevich index. The results are in the following Table 4.

2. The possible gradings on $\mathfrak{sp}(4, \mathbb{C})$

A $\mathbb{Z}$-grading on a Lie algebra $\mathfrak{g}$ is a vector space decomposition of the type

$$\cdots \oplus \mathfrak{g}_{-k} \oplus \cdots \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k \oplus \cdots ,$$

where $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$. A k-grading is characterized by the property that $\mathfrak{g}_i = \mathfrak{g}_{-i} = \{0\}$ for all $i > k$ and that such a k is the minimal possible. It may be

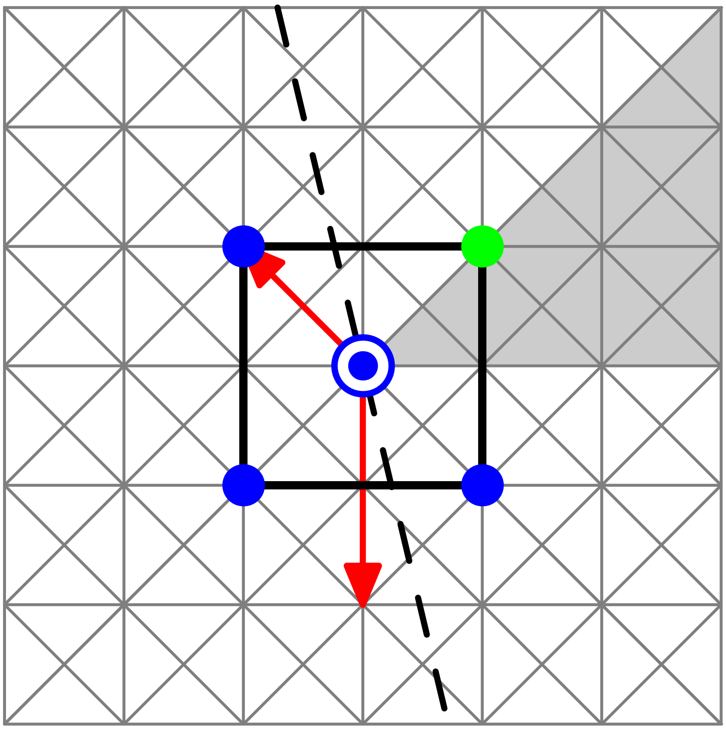


Figure 5: The highest weight representations $\Gamma_{0,1}$ the highest weight in green, other weights in blue, negative primitive roots in red

proved that the possible gradings on (semi)simple Lie algebras are in one to one correspondence to subsets of the set of simple roots Δ^0. Let

$$\mathfrak{p} = \bigoplus_{i=0}^{k} \mathfrak{g}_i$$

be the parabolic subalgebra. The subset is taken to be

$$\Sigma(\mathfrak{p}) = \{\alpha \in \Delta^0 | \mathfrak{g}_{-\alpha} \not\subseteq \mathfrak{p}\}.$$

So there are four possibilities in $\mathfrak{sp}(4, \mathbb{C})$ which can be seen in Table 5 where the subset taken is indicated by a cross in the Dynkin diagram.

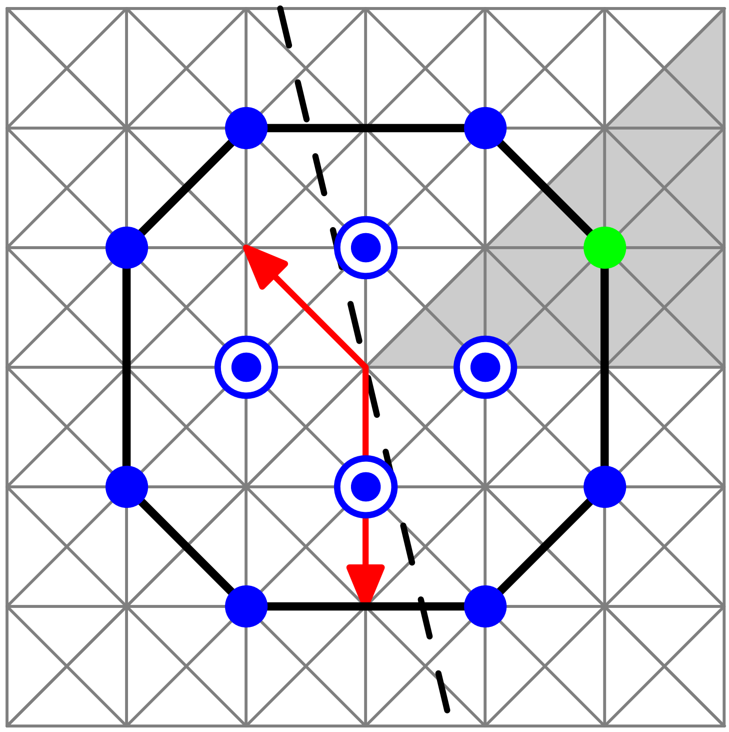


Figure 6: The highest weight representations $\Gamma_{1,1}$, the highest weight in green, other weights in blue, negative primitive roots in red

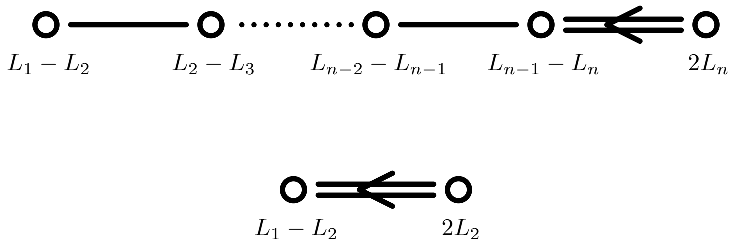


Figure 7: The Dynkin diagram of $\mathfrak{sp}(2n, \mathbb{C})$ and (more specifically) $\mathfrak{sp}(4, \mathbb{C})$

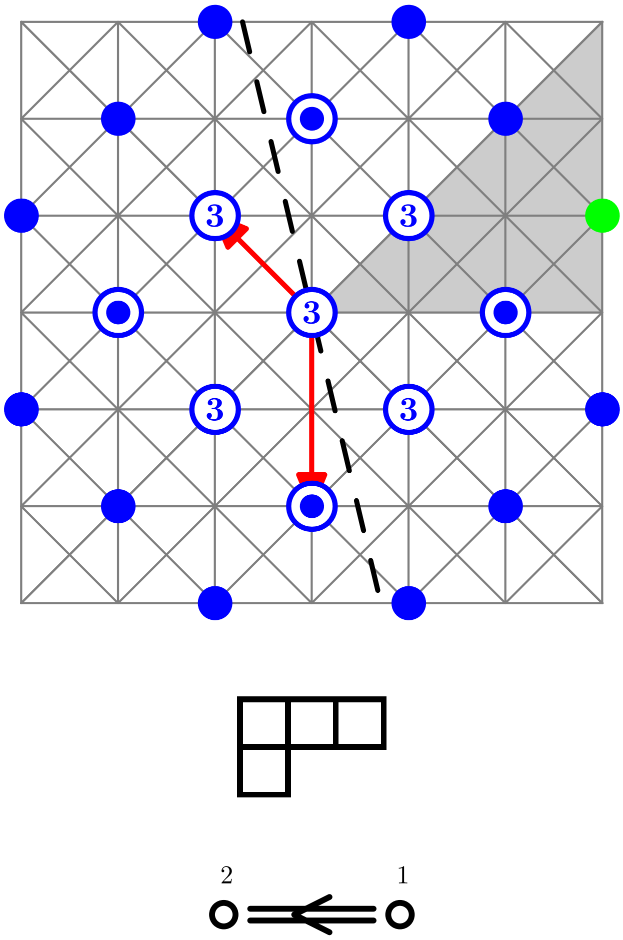


Figure 8: The weight diagram, Young diagram and Dynkin diagram of $\Gamma_{2,1}$

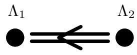
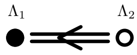
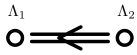
Real form	Dynkin diagram	Karpelevich index
$\mathfrak{sp}_2 \cong \mathfrak{so}_5$		$(-1)^{\Lambda_1}$
$\mathfrak{sp}_{1,1} \cong \mathfrak{so}_{1,4}$		$(-1)^{\Lambda_1}$
$\mathfrak{sp}_4(\mathbb{R}) \cong \mathfrak{so}_{2,3}$		$+1$

Table 4: The indices of representations of real forms of $\mathfrak{sp}(4, \mathbb{C})$

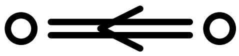
Crossed Dynkin diagram	Explicit decomposition
	$\mathfrak{sp}(4, \mathbb{C})$
	?
	$\langle V_2 \rangle \oplus \langle X_{1,2}, Z_{1,2} \rangle \oplus$ $\oplus \langle H_1, H_2, U_1, V_1 \rangle \oplus \langle X_{2,1}, Y_{1,2} \rangle \oplus \langle U_2 \rangle$
	$\langle X_{2,1}, Z_{1,2}, V_1, V_2 \rangle \oplus \langle H_1, H_2 \rangle \oplus$ $\oplus \langle X_{1,2}, Y_{1,2}, U_1, U_2 \rangle$

Table 5: Possible gradings on $\mathfrak{sp}(4, \mathbb{C})$