# Spherically symmetric vacuum spacetimes

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**1. Preliminaries.** We consider a smooth pseudo-Riemannian manifold $(M, \langle\cdot,\cdot\rangle)$, with signature $(r, s)$. For simplicity, we suppose the manifold $M$ is connected, the dimension of $M$ is $n$. First we need to consider the following question: We are given a smooth manifold $M$. Under which conditions does there even exist a smooth nondegenerate metric field of signature $(r, s)$ on $M$?

**Lemma 1.** *The following statements are equivalent:*

*(1) There exist a smooth nondegenerate metric field $\langle\cdot,\cdot\rangle$ of signature $(r, s)$ on $M$.*

*(2) There exists a smooth distribution $V$ of constant rank $r$ on $M$.*

*Proof.* (1) $\Rightarrow$ (2). There exists a smooth Riemannian metric $(\cdot,\cdot)$ on $M$ (see [1]). Consider a point $x \in M$, tangent vectors $u, v \in T_xM$ and the vector subspace $V_x = \{u \in T_xM | \langle u, v\rangle = (u, v), \forall v \in T_xM\}$. Then $V = \coprod_{x\in M} V_x$ is the sought distribution.

(2) $\Rightarrow$ (1). We again use the existence of a Riemannian metric $(\cdot,\cdot)$ on $M$. To any distribution $V$ of rank $r$ there exists a distribution $V_\perp$ so that $(V, V_\perp) = 0$ and the rank of $V_\perp$ is $s = n - r$, $n = \dim M$. We construct an involution $\theta$ in the tangent space $T_xM$ such that $\theta(V) = \mathrm{id}$ a $\theta(V_\perp) = -\mathrm{id}$. Define $\langle u, v\rangle_x = (u, \theta(v))_x$. Then $\langle\cdot,\cdot\rangle$ is a semi-riemannian metric of signature $(r, s)$. ■

For a Lorentzian metric (of signature $(1, n-1)$), this construction gives a distribution of rank 1. If we assume that $M$ is orientable, this is equivalent to the existence of a vector field $\xi$ which generates $V$ at each point $x \in M$ (in order for the distribution $V$ to be of constant rank 1, the vector field $\xi$ has to be everywhere non-zero).

**2. Action of a compact Lie group on a semi-Riemannian manifold.** Consider a compact Lie group $G$ and a left action of $G$ on $(M, \langle\cdot,\cdot\rangle)$, i.e. a smooth map

$$G \times M \to M, \quad (g, x) \mapsto gx.$$

Let us denote by $g_*$ the tangent map $x \mapsto gx$ for a fixed $g \in G$. The action is called isometric (with respect to $\langle\cdot,\cdot\rangle$) if $\langle g_*\xi, g_*\eta\rangle = \langle\xi, \eta\rangle$ for all $\xi, \eta \in \mathscr{X}(M)$.

Let $(\cdot,\cdot)$ be a Riemannian metric on $M$. Then we have the following

**Lemma 2.** *Let $x \mapsto gx$ be an action of the compact Lie group $G$ on $M$. Then there exists a Riemannian metric $(\cdot,\cdot)$ on $M$ with respect to which the action is isometric.*

*Proof.* Let $(\cdot,\cdot)'$ be any Riemannian metric on $M$. Construct

$$(\xi, \eta) = \frac{\int_G f(g_*\xi, g_*\eta)' \,\mathrm{d}\,\mu(g)}{\int_G \mathrm{d}\,\mu(g)},$$

where $\mathrm{d}\,\mu$ is the Haar measure on $G$. This is invariant by construction and positive definite by inspection. ■

The preceding Lemma could have been proven without the assumption that $G$ is compact in which case one must assume the action to be proper. For the proof see [4].

**Lemma 3.** *Let $x \mapsto gx$ be an action of the compact Lie group $G$ on $M$, isometric with respect to $\langle\cdot,\cdot\rangle$. Then the distribution $V$ from Lemma 1 can be chosen to be invariant, i.e. $g_* V = V$.*

*Proof.* Use Lemma 2 to construct an invariant metric. The construction (1) $\Rightarrow$ (2) is now invariant with respect to the $G$-action. ■

**3. The homogeneous space $\mathbf{S}^n$.** The group $\mathrm{O}(n+1)$ acts on $\mathbf{R}^{n+1}$ by its defining representation

$$\begin{aligned} \mathrm{O}(n+1) &\to \mathrm{GL}\left(\mathbf{R}^{n+1}\right) \\ A &\mapsto A. \end{aligned} \tag{1}$$

The orbits of the defining representation are spheres $\mathbf{S}^n$ (the zero vector in $\mathbf{R}^{n+1}$ is a singular orbit of dimension 0). Let us now restrict the defining representation to the subset $\mathbf{S}^n \subset \mathbf{R}^{n+1}$, $\mathbf{S}^n = \{(a_1,\ldots,a_{n+1}) \in \mathbf{R}^{n+1} | a_1^2 + \cdots + a_{n+1}^2 = 1\}$. This action is transitive. Let us denote by $s = (0,\ldots,0,1) \in \mathbf{S}^n$ (the north pole). For each $x \in \mathbf{S}^n$ there exists a $B \in \mathrm{O}(n+1)$ so that $Bx = s$. If $x = s$ then $B$ can be f.e. the identity. If $x \neq s$ let us consider the orthonormal basis in $\mathbf{R}^{n+1}$ such that the last vector is $s$ and the second last vector lies in the plane given by $x$ and $s$. Let us further denote $\cos\varphi = \langle s, x\rangle$. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} 0 \\ \sin\varphi \\ \cos\varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the action is transitive.

Consider the isotropy group at the point $s$,

$$\begin{pmatrix} A & v \\ w^{\mathrm{t}} & a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so $v = 0$ and $a = 1$. For orthogonality to hold, we must have

$$\begin{pmatrix} A^{\mathrm{t}} & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ w^{\mathrm{t}} & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $w = 0$ and $A \in \mathrm{O}(n)$. The isotropy subgroup at this point (and every other point by transitivity) is isomorphic to $\mathrm{O}(n)$. So we can write

$$\mathbf{S}^n = \frac{\mathrm{O}(n+1)}{\mathrm{O}(n)}. \tag{2}$$

There is a induced homogeneous metric on $\mathbf{S}^n$ given up to a nonzero multiple. The tangent space at $s$ is $\mathbf{R}^n$, where we have the standard scalar product. Using the scalar product $\langle\cdot,\cdot\rangle$ at $s$ denoted by $\langle\cdot,\cdot\rangle_s$ we can define

$$\langle u,v\rangle_x = \langle g_* g_*^{-1} u, g_* g_*^{-1} v\rangle_{gs} = \alpha\langle g_*^{-1} u, g_*^{-1} v\rangle_s = \alpha\langle g^{-1} u, g^{-1} v\rangle_s,$$

where $x = gs$ and such $g \in \mathrm{O}(n+1)$ exists by transitivity of the action and $g_* = g$ by linearity of the action.

Let us describe the tangent space to $\mathbf{S}^n$ more concretely. Choose a basis $e_{ij} = \delta_{ij} - \delta_{ji}$, $i < j$ in $\mathfrak{so}(n+1)$. Then $[e_{ij}, e_{kl}] = -\delta_{ik} e_{jl} - \delta_{il} e_{jk} + \delta_{jk} e_{il} + \delta_{jl} e_{ik}$, where $e_{ij} = -e_{ji}$ if $i > j$. The Killing form is

$$K(e_{ij}, e_{kl}) = \sum_{r<s} [e_{ij}, e_{rs}][e_{kl}, e_{rs}] \tag{3}$$

The group $\mathrm{O}(n+1)$ is compact, its Killing form is therefore negative definite and so is its restriction to every subspace of the Lie algebra or the factor space $\mathfrak{so}(n+1)/\mathfrak{so}(n)$. In the $e_{ij}$ basis the Killing form is diagonal

$$K(e_{ij}, e_{kl}) = -2n\delta_{ij,kl}. \tag{4}$$

It may be proved (see [3]) that all $\mathrm{O}(n+1)$-invariant metrics on the sphere $\mathbf{S}^n$ are constant nonzero multiples of the metric induced by the Killing form.

The structure of the tangent space of $\mathbf{S}^n$ at the point $s$ is given as follows. The point $s = es$ corrresponds to $e \in \mathrm{O}(n+1)$ and the tangent space at $e$ is given by matrices satisfying $X + X^{\mathrm{t}} = 0$. The tangent space to the isotropy group at $s$ in $e$ is given by matrices

$$\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix},$$

where $Y + Y^{\mathrm{t}} = 0$, $Y$ is a matrix of order $n$. It holds

$$T_s \frac{\mathrm{O}(n+1)}{\mathrm{O}(n)} = \frac{T_e\mathrm{O}(n+1)}{T_e\mathrm{O}(n)} = \begin{pmatrix} 0 & v \\ -v^{\mathrm{t}} & 0 \end{pmatrix} \tag{5}$$

Pick a basis in this space

$$(X_i) = \left(\begin{pmatrix} 0 & e_i \\ -e_i^{\mathrm{t}} & 0 \end{pmatrix}\right),$$

where $e_i$ is the standard basis in $\mathbf{R}^n$. We have the geodesic normal coordinates $(h_1, \dots, h_n)$ of the point $x \in \mathbf{S}^n$ in the neighborhood of $s$

$$x = \mathrm{e}^{h_1 X_1 + \cdots + h_n X_n}\, s. \tag{6}$$

With the notation $h = \sqrt{h_1^2 + \cdots + h_n^2}$ we get

$$x = \begin{pmatrix} \dfrac{h_1 \sin h}{h} \\ \vdots \\ \dfrac{h_n \sin h}{h} \\ \cos h \end{pmatrix}.$$

These coordinates are defined averywhere except at the south pole.

Using the Campbell-Baker-Hausdorff formula we also have

$$x = \mathrm{e}^{k_n X_n} \cdots \mathrm{e}^{k_1 X_1}\, s$$

and $(k_1, \ldots, k_n)$ are the (generalized) spherical coordinates. The meaning of the preceding formula is that we get $x \in \mathbf{S}^n$ from $s$ by successive rotations in the planes $[E_1, s]$, then $[E_2, s]$ and finally $[E_n, s]$, where

$$E_i = \begin{pmatrix} e_i \\ 0 \end{pmatrix}.$$

The computation gives

$$x = \begin{pmatrix} \sin k_1 \\ \sin k_2 \cos k_1 \\ \sin k_3 \cos k_2 \cos k_1 \\ \vdots \\ \sin k_n \cos k_{n-1} \cdots \cos k_1 \\ \cos k_n \cos k_{n-1} \cdots \cos k_1 \end{pmatrix}. \tag{7}$$

The $k_i$ take values in $-\pi/2 < k_1, \ldots, k_{n-1} < \pi/2$, $-\pi < k_n < \pi$. The spherical coordinates are orthogonal with respect to the metric on $\mathbf{S}^n$ by construction, the metric is

$$G = \begin{pmatrix} 1 & & & & \\ & \cos^2 k_1 & & & \\ & & \cos^2 k_1 \cos^2 k_2 & & \\ & & & \ddots & \\ & & & & \cos^2 k_1 \ldots \cos^2 k_n \end{pmatrix},$$

in geodesic normal coordinates the metric is not diagonal.

**4. Smooth actions of compact groups on manifolds.** Let $\varphi\colon G \times M \to M$ be the left action of the Lie group $G$ on a smooth manifold $M$. Pick a point $x \in M$ and consider the ***orbit*** $Gx = \{y \in M | \exists g \in G\colon y = \varphi(g, x) = gx\}$ and the ***isotropy subgroup*** $G_x = \{g \in G | x = \varphi(g, x) = gx\}$. The isotropy subgroups in two points $x$ and $y$ on the same orbit are isomorphic, the isomorphism is given by conjugation by such $g$ that $y = \varphi(g, x)$. $G_y = gG_xg^{-1} \cong H$. The orbit $Gx$ going through $x$ is called of ***type*** $G/H$ .

*Example* 1. Consider the vector space $M = \{X \in \mathfrak{gl}(3) | X = X^{\mathrm{t}}, \operatorname{Tr} X = 0\}$ with the action $\varphi$ of $\mathrm{SO}(3)$ by conjugation $\varphi\colon (g, X) \mapsto gXg^{-1}$. It is known from basic linear algebra that any symmetric matrix is diagonalizable by an orthogonal conjugation, the orbits can be parametrized by the three eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3$ such that $\lambda_1 + \lambda_2 + \lambda_3 = 0$. There are several orbit types on $M$

(i) $\lambda_1 = \lambda_2 = \lambda_3 = 0$, the isotropy subgroup is the whole $\mathrm{SO}(3)$ and the orbit is a point.

(ii) $\lambda_1 = \lambda_2 > \lambda_3$, the isotropy subgroup is

$$\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix},$$

where $A \in \mathrm{O}(2)$, $\det A = \pm 1$, the orbit is isomorphic to $\mathbf{R}P^2 \cong \mathrm{SO}(3)/(\mathrm{O}(2) \ltimes \mathbf{Z}_2)$.

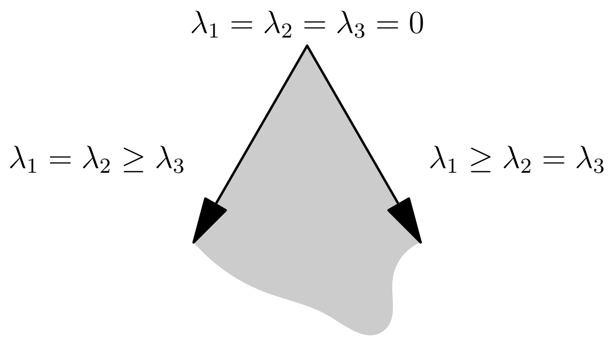


Figure 1: Orbits on $M$

(iii) $\lambda_1 > \lambda_2 = \lambda_3$, here again

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & A \end{pmatrix}$$

and the orbit is isomorphic to $\mathbf{R}P^2 \cong \mathrm{SO}(3)/(\mathrm{O}(2) \ltimes Z_2)$.

(iv) $\lambda_1 > \lambda_2 > \lambda_3$, the isotropy subgroup here is

$$\begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}$$

isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$. The orbit is isomorphic to $\mathrm{SO}(3)/(\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2)$.

Quite generally two orbits $Gx$, $Gy$ are of the same type $G/H$, if their isotropy subgroups are both isomorphic to $H$. When $H \subseteq G$ is a subgroup, we can partially order different subgroups using set inclusion $\subseteq$. We define

$$[H] \leq [H'] \iff \exists K \in [H], K' \in [H']\colon K \subseteq K',$$

which is equivalent to

$$[H] \leq [H'] \iff \exists g \in G\colon gHg^{-1} \subseteq H'.$$

When $G$ is not compact, the relation need not be antisymmetric. We have

**Lemma 4.** *Let $G$ be a compact Lie group, $H \subseteq G$ its closed subgroup. Then*

$$gHg^{-1} \subseteq H \implies gHg^{-1} = H.$$

*Proof.* By iteration we have $gHg^{-1} \subseteq H \Rightarrow g^n H g^{-n} \subseteq H$ for all $n \in \mathbf{N}_0$. Let us analyze the set $A = \{g^n | n \in \mathbf{N}_0\}$. We shall show that $g^{-1}$ lies in the closure $\bar{A}$. We need to distinguish two cases

(i) $e$ is a limit point in $A$. Then for each its neighborhood $U$, there must exist an index $n$ so that $g^n \in U$. It follows $g^{n-1} \in g^{-1}U \cap A$ and the set $g^{-1}U$ is a neighborhood of $g^{-1}$, all such $g^{-1}U$ are a local basis at $g^{-1} \in \bar{A}$.

(ii) $e$ is a discrete point in $\bar{A}$. But $G$ is compact and $A$ is therefore a finite set, so $g^n = e$ for some $n \in \mathbf{N}$. We obtain $g^{-1} = g^{n-1} \in A$.

The conjugation conj: $(g, h) \mapsto ghg^{-1}$ is continuous as a map $G \times G \to G$ and $H$ is closed, so $\operatorname{conj}(\bar{A}, H) \subseteq H$, especially $g^{-1}Hg \subseteq H$. ■

Let $x \in M$ be a point and $Gx$ the orbit through it. The orbit is called ***principal*** if there exists an invariant neighborhood $U$ of the point $x \in M$ and for all $y \in U$ an equivariant map $Gx \to Gy$. Points which lie on principal orbits are called ***regular***, other points are called ***singular***. A subset $S \subset M$ is called a ***slice*** at $x$ if there exists a $G$-equivariant open neighborhood $U$ of the orbit $Gx$ and a smooth retraction $r: U \to Gx$ such that $S = r^{-1}(x)$.

*Example* 2. Consider the defining representation of $G = \mathrm{SO}(3)$ on $M = \mathbf{R}^3$. Let $x = (0, 0, 1)$. The orbit is $Gx = \mathbf{S}^2$. We shall show that this orbit is principal Let $y$

$$U_\epsilon = \{(y_1, y_2, y_3) \in \mathbf{R}^3 | \epsilon^2 < y_1^2 + y_2^2 + y_3^2\},$$

where $\epsilon > 0$. The retraction $r: U_\epsilon \to Gx$ is defined as

$$r: (y_1, y_2, y_3) \mapsto \frac{(y_1, y_2, y_3)}{\sqrt{y_1^2 + y_2^2 + y_3^2}}.$$

The point $O = (0, 0, 0)$ is a singular point of the action, $G_O = \mathrm{SO}(3)$, the orbit is the point $O$ itself. There are only regular points in any open neighborhood of the point $O$.

Orbits of singular points are themselves called ***singular*** (singular orbits are isomorphic to $G/K$, where $\dim K > \dim H$). There is a third possibillity: the orbit is of maximal dimension but is not isomorphic to the principal orbit. We call such orbits ***exceptional***.

*Example* 3. Consider the left action $\psi: \mathrm{SO}(3) \times \mathrm{SO}(3) \to \mathrm{SO}(3)$ of the group $G = \mathrm{SO}(3)$ on itself by conjugation. $\psi: (g, h) \mapsto ghg^{-1}$. We know from linear algebra that there always exists an orthonormal basis with respect to which

$$h(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

The orbit $Ge = Gh(0)$ is singular, the isotropy subgroup is the whole $G_e = \mathrm{SO}(3)$. For $\varphi = \pi$ the orbit is exceptional $Gh(\pi) \cong \mathbf{R}P^2$. The remaining orbits $Gh(\varphi)$, $0 < \varphi < \pi$, are spheres $\mathbf{S}^2$ and their isotropy subgroup is $\mathrm{SO}(2)$. From this follows the model of the $\mathrm{SO}(3)$ manifold as a closed ball of radius $\pi$, where we identify the antipodal points on the boundary. The center of the ball corresponds to the singular orbit.

**5. Warped products.** This part of the exposition follows [2]. Suppose $M = B \times F$, where $(B, \langle\cdot,\cdot\rangle_B)$ and $(F, \langle\cdot,\cdot\rangle_F)$ are (semi)riemannian manifolds, $f$ a positive function on $B$. We construct the (semi)riemannian metric on $M$: pick an arbitrary point $x = (a, b) \in M = B \times F$. Then the tangent space at this point is $T_xM = T_aB \oplus T_bF$ and each tangent vector $(x, \xi)$ can be unambiguously written as $(a, \alpha) + (b, \beta)$. The scalar product on $M$ is then defined by

$$\langle \xi, \xi' \rangle(x) := \langle \alpha, \alpha' \rangle_B(a) + f^2(a) \langle \beta, \beta' \rangle_F(b). \tag{8}$$

If the metric signature on $B$ is $(r, s)$ and $(r', s')$ on $F$ then the metric signature on $M$ is obviously $(r + r', s + s')$. The whole construction is a generalization of a surface of revolution; in this case $B$ is a plane curve which does not intersect the axis of revolution, $f(a)$ gives the distance of the point $a$ from the axis, $F = \mathbf{S}^1$. Warped products are denoted by $B \times_f F$.

On $p\colon B \times F \to B$ (and more generally on a Riemannian submersion $p\colon M \to B$) there exist special subbundles of the tangent bundle: the ***vertical*** subbundle $VM = \ker p_*$ and the ***horizontal*** subbundle $HM = VM^\perp$ (the definition of a Riemannian submersion demands that $H_x M \cong T_{p(x)} B$ for all $x \in M$). The sections of these subbundles are called ***vertical*** resp. ***horizontal*** vector fields. There is a special class of horizontal vector fields, called ***basic*** defined as follows: Take any vector field $\eta$ on $B$. Then there exists a unique horizontal vector field $\xi$ such that $\xi p^* = p^* \eta$. The basic vector fields span $HM$ (for dimensional reasons).

We can compute the relevant tensor fields for warped products following [2]. Let $\xi$, $\eta$ be basic vector fields and $X$, $Y$, $Z$ vertical vector fields. Let $\mathrm{Riemann}^F$ denote the Riemann curvature tensor field on the fiber $F$. We assume $\dim M = 4$ and $\dim F = 2$. For the Riemann curvature on $M$ we obtain

$$\mathrm{Riemann}_{XY} Z = \mathrm{Riemann}^F_{XY} Z - \frac{\langle (\mathrm{d}\, f)^\#, (\mathrm{d}\, f)^\# \rangle_B}{f^2} \left( \langle X, Z \rangle_F Y - \langle Y, Z \rangle_F X \right),$$

and defining the Hessian of the function $f$ by

$$\mathrm{Hessian}_f(\xi, \eta) = \langle [\nabla_\xi (\mathrm{d}\, f)^\#, \eta \rangle_B = (\xi \eta - \nabla_\xi \eta) f,$$

which is a symmetric tensor field of type $(0, 2)$, we may write

$$\langle \mathrm{Riemann}_{\xi X} \eta, Y \rangle = -\frac{\mathrm{Hessian}_f(\xi, \eta)}{f} \langle X, Y \rangle_F,$$

for the Ricci curvature

$$\mathrm{Ricci}(\xi, \eta) = \mathrm{Ricci}^B(\xi, \eta) - \frac{2}{f} \mathrm{Hessian}_f(\xi, \eta) \tag{9}$$

$$\mathrm{Ricci}(\xi, X) = 0 \tag{10}$$

$$\mathrm{Ricci}(X, Y) = \mathrm{Ricci}^F(X, Y) - \langle X, Y \rangle_F \left( \frac{\star\, \mathrm{d} \star \mathrm{d}\, f}{f} + \frac{\langle (\mathrm{d}\, f)^\#, (\mathrm{d}\, f)^\# \rangle_B}{f^2} \right), \tag{11}$$

here $\star$ is the Hodge operator and $\nabla$ the Levi-Civita connection (both with respect to $\langle \cdot, \cdot \rangle_B$).

*Example* 4 (The Kruskal solution). We take

$$B = \{ (v, u) \in \mathbf{R}^2 | u^2 - v^2 > 1 \},$$

and

$$f(u, v) = 1 + \mathrm{W} \left( \frac{v^2 - u^2}{\mathrm{e}} \right).$$

Then we define the metric on $B$ by

$$\frac{4\, \mathrm{e}^{-f(u,v)}}{f(u, v)} \left( \mathrm{d}\, u^2 - \mathrm{d}\, v^2 \right), \tag{12}$$

where $z \mapsto \mathrm{W}(z)$ is the principal branch of the Lambert W-function, the solution of $z = \mathrm{W}(z)\,\mathrm{e}^{\mathrm{W}(z)}$. The manifold $F$ is the sphere $\mathbf{S}^2$ with the standard negative definite metric induced by the Killing form. For the metric

$$\frac{4\,\mathrm{e}^{-f(u,v)}}{f(u,v)}\left(\mathrm{d}\,u^2 - \mathrm{d}\,v^2\right) + f^2(u,v)\gamma,$$

where $\gamma$ is the standard metric on $\mathbf{S}^2$ given locally by $\mathrm{d}\,k_1^2 + \cos^2 k_1\,\mathrm{d}\,k_2^2$, the following holds

$$\mathrm{Ricci} = 0, \quad \mathrm{R} = 0, \quad \mathrm{Einstein} = 0. \tag{13}$$

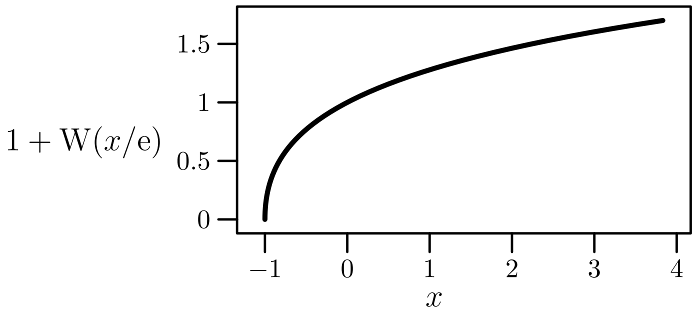


Figure 2: The Function $1 + \mathrm{W}(x/\mathrm{e})$

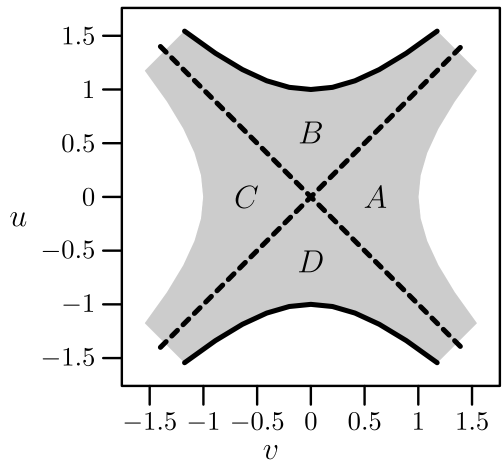


Figure 3: Hyperbolic plane $u^2 - v^2 > 1$

**6. Centrally symmetric spacetimes.** Let $G = \mathrm{SO}(3, \mathbf{R})$ be the compact Lie group and $(M, \langle\cdot,\cdot\rangle)$ a semi-Riemannian manifold of signature $(1, 3)$. We say that $(M, \langle\cdot,\cdot\rangle)$ is ***centrally symmetric*** if there exists an isometric proper $G$-action $\varphi$ all of whose orbits are spheres

$\mathbf{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$.

$$\varphi\colon G \times M \to M$$
$$\varphi\colon (g,x) \mapsto \varphi(g,x) = gx.$$

The action is ***proper*** if the preimages of compact sets by the map $(g,x) \mapsto (gx,x)$ are compact. The action is ***isometric*** if $\langle g_*\xi, g_*\eta\rangle = \langle \xi, \eta\rangle$ for all $g \in G$ and all vector fields $\xi, \eta \in \mathscr{X}(M)$. The orbit of the point $x \in M$ is denoted by $Gx$.

The sphere is viewed as the homogeneous space $G/G_x = \mathrm{SO}(3,\mathbf{R})/\mathrm{SO}(2,\mathbf{R})$, where $G_x$ is the stabilizer of $x \in M$. The Riemannian metric $\gamma$ on the sphere is constructed using the Maurer-Cartan form on SO(3) corestricted from $\mathfrak{g} = \mathfrak{so}(3,\mathbf{R})$ to the factor vector space $\mathfrak{so}(3,\mathbf{R})/\mathfrak{so}(2,\mathbf{R})$ and the negative definite Killing form. This metric is unique up to a constant positive multiple (this corresponds to different sphere radii). Orbits of different points are therefore spheres $\mathbf{S}^2$ with varying radii.

**Theorem 5.** *Let $(M, \langle\cdot,\cdot\rangle)$ be a centrally symmetric spacetime. Then $M$ is the total space of a semi-Riemannian fibre bundle $(M, B, p, \mathbf{S}^2)$, where $p\colon M \to B$ is a surjective submersion and the fibre is $p^{-1}(b) = \mathbf{S}^2$. Moreover, the metric on this fibre bundle is a warped product*

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_B + f^2\langle\cdot,\cdot\rangle_{\mathbf{S}^2},$$

*where $f$ is a positive function on $B$.*

*Proof.* The space $B = M/G$ can be thought of as the space of orbits of the action by $G$ on $M$. This induces a topological and smooth structure on $B$ in the standard way such that the factor projection $p\colon M \to M/G$ is continuous and smooth.

We have to construct a local trivialization on some neighborhood of each point $x \in M$. We proceed as follows: $(M, \langle\cdot,\cdot\rangle)$ is a semi-Riemannian manifold of signature $(1,3)$, therefore there exists a one-dimensional distribution $\xi$ which can be chosen invariant with respect to the SO(3)-action by Lemma 3 and the corresponding invariant Riemannian metric denoted by $(\cdot,\cdot)$, see Lemma 2. We can now use the results from [4].

The orbit $Gx$ is a sphere $\mathbf{S}^2$ embedded in $M$ by $\iota\colon \mathbf{S}^2 \to M$. Consider the normal bundle $N\mathbf{S}^2 := \{v \in TM | (v,w) = 0 \text{ for all } w \in T\iota T\mathbf{S}^2\}$ and the exponential map applied to $0_x$ in a small enough ball $B_r(0_x)$ so that $\exp_x\colon T_xM \supset B_r(0_x) \to M$ is a diffeomorphism on its image $\exp_x(B_r(0_x)) \cap Gx$. $B_r(0_x)$ denotes a cylindrical neighborhood of $0_x$ in $N\mathbf{S}^2$. The inverse is the sought local trivialization of $(M, B, p, \mathbf{S}^2)$.

The inner product

$$\langle T_xp\xi_x, T_xp\eta_x\rangle_B = \langle \xi_x, \eta_x\rangle$$

is well defined for $\xi, \eta \in HM$. Therefore the metric on $M$ is a warped product. ∎

**Lemma 6.** *The one-dimensional distribution $\xi$ projects to $B$ via the map $Tp$ giving rise to a one-dimensional distribution on $B$.*

*Proof.* $\xi$ can be chosen invariant, i.e. spanned by local horizontal vector fields. These fields correspond to basic vector fields by definition of a Riemannian submersion. ∎

**7. Birkhoff's theorem.** This section is almost entirely based on [5].

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