Spherically symmetric vacuum spacetimes

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1. Preliminaries. We consider a smooth pseudo-Riemannian manifold \((M, \langle \cdot, \cdot \rangle)\), with signature \((r, s)\). For simplicity, we suppose the manifold \(M\) is connected, the dimension of \(M\) is \(n\). First we need to consider the following question: We are given a smooth manifold \(M\). Under which conditions does there even exist a smooth nondegenerate metric field of signature \((r, s)\) on \(M\)?

**Lemma 1.** The following statements are equivalent:

1. There exist a smooth nondegenerate metric field \(\langle \cdot, \cdot \rangle\) of signature \((r, s)\) on \(M\).
2. There exists a smooth distribution \(V\) of constant rank \(r\) on \(M\).

**Proof.** (1) \(\Rightarrow\) (2). There exists a smooth Riemannian metric \((\cdot, \cdot)\) on \(M\) (see \([1]\)). Consider a point \(x \in M\), tangent vectors \(u, v \in T_x M\) and the vector subspace \(V_x = \{u \in T_x M | \langle u, v \rangle = (u,v), \forall v \in T_x M\}\). Then \(V = \bigsqcup_{x \in M} V_x\) is the sought distribution.

(2) \(\Rightarrow\) (1). We again use the existence of a Riemannian metric \((\cdot, \cdot)\) on \(M\). To any distribution \(V\) of rank \(r\) there exists a distribution \(V_\perp\) so that \((V, V_\perp) = 0\) and the rank of \(V_\perp\) is \(s = n-r\), \(n = \dim M\). We construct an involution \(\theta\) in the tangent space \(T_x M\) such that \(\theta(V) = \text{id}\) \(\theta(V_\perp) = -\text{id}\). Define \(\langle u, v \rangle_x = (u, \theta(v))_x\). Then \(\langle \cdot, \cdot \rangle\) is a semi-riemannian metric of signature \((r, s)\). \(\blacksquare\)

For a Lorentzian metric (of signature \((1, n-1)\)), this construction gives a distribution of rank 1. If we assume that \(M\) is orientable, this is equivalent to the existence of a vector field \(\xi\) which generates \(V\) at each point \(x \in M\) (in order for the distribution \(V\) to be of constant rank 1, the vector field \(\xi\) has to be everywhere non-zero).

2. Action of a compact Lie group on a semi-Riemannian manifold. Consider a compact Lie group \(G\) and a left action of \(G\) on \((M, \langle \cdot, \cdot \rangle)\), i.e. a smooth map

\[ G \times M \to M, \quad (g, x) \mapsto gx. \]

Let us denote by \(g_*\) the tangent map \(x \mapsto gx\) for a fixed \(g \in G\). The action is called isometric (with respect to \(\langle \cdot, \cdot \rangle\)) if \(\langle g_*\xi, g_*\eta \rangle = \langle \xi, \eta \rangle\) for all \(\xi, \eta \in \mathcal{X}(M)\).

Let \(\langle \cdot, \cdot \rangle\) be a Riemannian metric on \(M\). Then we have the following

**Lemma 2.** Let \(x \mapsto gx\) be an action of the compact Lie group \(G\) on \(M\). Then there exists a Riemannian metric \((\cdot, \cdot)\) on \(M\) with respect to which the action is isometric.

**Proof.** Let \((\cdot, \cdot)\)' be any Riemannian metric on \(M\). Construct

\[ \langle \xi, \eta \rangle = \frac{\int_G f(g_*\xi, g_*\eta) d\mu(g)}{\int_G d\mu(g)}, \]

where \(f \in C^\infty(M)\) with \(f(\xi, \eta) = \langle \xi, \eta \rangle\).

\[ \langle \xi, \eta \rangle = \int_G f(g_*\xi, g_*\eta) d\mu(g), \]

where \(f \in C^\infty_0(M)\) with \(f(\xi, \eta) = \langle \xi, \eta \rangle\).
where $d\mu$ is the Haar measure on $G$. This is invariant by construction and positive definite by inspection.

The preceding Lemma could have been proven without the assumption that $G$ is compact in which case one must assume the action to be proper. For the proof see [4].

**Lemma 3.** Let $x \mapsto gx$ be an action of the compact Lie group $G$ on $M$, isometric with respect to $\langle \cdot, \cdot \rangle$. Then the distribution $V$ from Lemma[1] can be chosen to be invariant, i.e. $g_*V = V$.

**Proof.** Use Lemma 2 to construct an invariant metric. The construction $(1) \Rightarrow (2)$ is now invariant with respect to the $G$-action. ■

3. **The homogeneous space $S^n$.** The group $O(n+1)$ acts on $\mathbb{R}^{n+1}$ by its defining representation

$$O(n+1) \to \text{GL}(\mathbb{R}^{n+1})$$

$$A \mapsto A.$$ (1)

The orbits of the defining representation are spheres $S^n$ (the zero vector in $\mathbb{R}^{n+1}$ is a singular orbit of dimension 0). Let us now restrict the defining representation to the subset $S^n \subset \mathbb{R}^{n+1}$, $S^n = \{(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1} | a_1^2 + \cdots + a_{n+1}^2 = 1\}$. This action is transitive. Let us denote by $s = (0, \ldots, 0, 1) \in S^n$ (the north pole). For each $x \in S^n$ there exists a $B \in O(n+1)$ so that $Bx = s$. If $x = s$ then $B$ can be f.e. the identity. If $x \neq s$ let us consider the orthonormal basis in $\mathbb{R}^{n+1}$ such that the last vector is $s$ and the second last vector lies in the plane given by $x$ and $s$. Let us further denote $\cos \varphi = \langle s, x \rangle$. Then

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix} \begin{pmatrix} 0 \\ \sin \varphi \\ \cos \varphi \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

and the action is transitive.

Consider the isotropy group at the point $s$,

$$\begin{pmatrix} A & v \\ w^t & a \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

so $v = 0$ and $a = 1$. For orthogonality to hold, we must have

$$\begin{pmatrix} A^t & w \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ w^t & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

so $w = 0$ and $A \in O(n)$. The isotropy subgroup at this point (and every other point by transitivity) is isomorphic to $O(n)$. So we can write

$$S^n = \frac{O(n+1)}{O(n)}.$$ (2)
There is an induced homogeneous metric on $S^n$ given up to a nonzero multiple. The tangent space at $s$ is $\mathbb{R}^n$, where we have the standard scalar product. Using the scalar product $\langle \cdot, \cdot \rangle$ at $s$ denoted by $\langle \cdot, \cdot \rangle_s$ we can define
\[
\langle u, v \rangle_s = \langle g_* g_s^{-1} u, g_* g_s^{-1} v \rangle_{g_*} = \alpha \langle g_s^{-1} u, g_s^{-1} v \rangle_s = \alpha \langle g^{-1} u, g^{-1} v \rangle_s,
\]
where $x = gs$ and such $g \in O(n + 1)$ exists by transitivity of the action and $g_*$ is $g$ by linearity of the action.

Let us describe the tangent space to $S^n$ more concretely. Choose a basis $e_{ij} = \delta_{ij} - \delta_{ji}$, $i < j$ in $\mathfrak{so}(n + 1)$. Then $[e_{ij}, e_{kl}] = -\delta_{ik} e_{jl} - \delta_{il} e_{jk} + \delta_{jk} e_{il} + \delta_{jl} e_{ik}$, where $e_{ij} = -e_{ji}$ if $i > j$. The Killing form is
\[
K(e_{ij}, e_{kl}) = \sum_{r<s} [e_{ij}, e_{rs}] [e_{kl}, e_{rs}]
\]
(3)

The group $O(n + 1)$ is compact, its Killing form is therefore negative definite and so is its restriction to every subspace of the Lie algebra or the factor space $\mathfrak{so}(n + 1)/\mathfrak{so}(n)$. In the $e_{ij}$ basis the Killing form is diagonal
\[
K(e_{ij}, e_{kl}) = -2n \delta_{ij,kl}.
\]
(4)

It may be proved (see [4]) that all $O(n + 1)$-invariant metrics on the sphere $S^n$ are constant nonzero multiples of the metric induced by the Killing form.

The structure of the tangent space of $S^n$ at the point $s$ is given as follows. The point $s = es$ corresponds to $e \in O(n + 1)$ and the tangent space at $e$ is given by matrices satisfying $X + X^t = 0$. The tangent space to the isotropy group at $s$ in $e$ is given by matrices
\[
\begin{pmatrix} Y & 0 \\ 0 & 0 \end{pmatrix},
\]
where $Y + Y^t = 0$, $Y$ is a matrix of order $n$. It holds
\[
T_s \frac{O(n + 1)}{O(n)} = T_e \frac{O(n + 1)}{O(n)} = \begin{pmatrix} 0 & v \\ -v^t & 0 \end{pmatrix}
\]
(5)

Pick a basis in this space
\[
(X_i) = \begin{pmatrix} 0 & e_i \\ -e_i^t & 0 \end{pmatrix},
\]
where $e_i$ is the standard basis in $\mathbb{R}^n$. We have the geodesic normal coordinates $(h_1, \ldots, h_n)$ of the point $x \in S^n$ in the neighborhood of $s$
\[
x = e^{h_1 X_1 + \cdots + h_n X_n} s.
\]
(6)

With the notation $h = \sqrt{h_1^2 + \cdots + h_n^2}$ we get
\[
x = \begin{pmatrix} h_1 \sin h \\ \vdots \\ h_n \sin h \\ h \cos h \end{pmatrix}.
\]
These coordinates are defined everywhere except at the south pole.

Using the Campbell-Baker-Hausdorff formula we also have

\[ x = e^{k_n X_n} \cdots e^{k_1 X_1} s \]

and \((k_1, \ldots, k_n)\) are the (generalized) spherical coordinates. The meaning of the preceding formula is that we get \(x \in S^n\) from \(s\) by successive rotations in the planes \([E_1, s]\), then \([E_2, s]\) and finally \([E_n, s]\), where

\[ E_i = \begin{pmatrix} e_i \\ 0 \end{pmatrix} \]

The computation gives

\[ x = \begin{pmatrix} \sin k_1 \\ \sin k_2 \cos k_1 \\ \sin k_3 \cos k_2 k_1 \\ \vdots \\ \sin k_n \cos k_{n-1} \cdots \cos k_1 \\ \cos k_n \cos k_{n-1} \cdots \cos k_1 \end{pmatrix}. \tag{7} \]

The \(k_i\) take values in \(-\pi/2 < k_1, \ldots, k_{n-1} < \pi/2, -\pi < k_n < \pi\). The spherical coordinates are orthogonal with respect to the metric on \(S^n\) by construction, the metric is

\[ G = \begin{pmatrix} 1 & \cos^2 k_1 & \cos^2 k_1 \cos^2 k_2 & \cdots \\ & \cos^2 k_1 \cos^2 k_2 & \cdots \\ & & \cdots & \cos^2 k_1 \ldots \cos^2 k_n \end{pmatrix}, \]

in geodesic normal coordinates the metric is not diagonal.

4. Smooth actions of compact groups on manifolds. Let \(\varphi : G \times M \to M\) be the left action of the Lie group \(G\) on a smooth manifold \(M\). Pick a point \(x \in M\) and consider the orbit \(Gx = \{y \in M | \exists g \in G : y = \varphi(g, x) = gx\}\) and the isotropy subgroup \(G_x = \{g \in G | x = \varphi(g, x) = gx\}\). The isotropy subgroups in two points \(x\) and \(y\) on the same orbit are isomorphic, the isomorphism is given by conjugation by such \(g\) that \(y = \varphi(g, x)\). \(G_y = gG_x g^{-1} \cong H\). The orbit \(Gx\) going through \(x\) is called of type \(G/H\).

Example 1. Consider the vector space \(M = \{X \in \mathfrak{gl}(3)|X = X^t, \text{Tr} X = 0\}\) with the action \(\varphi\) of \(\text{SO}(3)\) by conjugation \(\varphi : (g, X) \mapsto gXg^{-1}\). It is known from basic linear algebra that any symmetric matrix is diagonalizable by an orthogonal conjugation, the orbits can be parametrized by the three eigenvalues \(\lambda_1 \geq \lambda_2 \geq \lambda_3\) such that \(\lambda_1 + \lambda_2 + \lambda_3 = 0\). There are several orbit types on \(M\)

(i) \(\lambda_1 = \lambda_2 = \lambda_3 = 0\), the isotropy subgroup is the whole \(\text{SO}(3)\) and the orbit is a point.

(ii) \(\lambda_1 = \lambda_2 > \lambda_3\), the isotropy subgroup is

\[ \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}, \]

where \(A \in \text{O}(2)\), \(\det A = \pm 1\), the orbit is isomorphic to \(\mathbb{R}P^2 \cong \text{SO}(3)/(\text{O}(2) \times \mathbb{Z}_2)\).
Figure 1: Orbits on $M$

(iii) $\lambda_1 > \lambda_2 = \lambda_3$, here again

$$
\begin{pmatrix}
\pm 1 & 0 \\
0 & A
\end{pmatrix}
$$

and the orbit is isomorphic to $\mathbb{R}P^2 \cong \text{SO}(3)/(O(2) \ltimes \mathbb{Z}_2)$.

(iv) $\lambda_1 > \lambda_2 > \lambda_3$, the isotropy subgroup here is

$$
\begin{pmatrix}
\pm 1 & 0 & 0 \\
0 & \pm 1 & 0 \\
0 & 0 & \pm 1
\end{pmatrix}
$$

isomorphic to $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. The orbit is isomorphic to $\text{SO}(3)/(\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2)$.

Quite generally two orbits $Gx$, $Gy$ are of the same type $G/H$, if their isotropy subgroups are both isomorphic to $H$. When $H \subseteq G$ is a subgroup, we can partially order different subgroups using set inclusion $\subseteq$. We define

$$[H] \leq [H'] \iff \exists K \in [H], K' \in [H'] : K \subseteq K',$$

which is equivalent to

$$[H] \leq [H'] \iff \exists g \in G : g H g^{-1} \subseteq H'.$$

When $G$ is not compact, the relation need not be antisymmetric. We have

**Lemma 4.** Let $G$ be a compact Lie group, $H \subseteq G$ its closed subgroup. Then

$$g H g^{-1} \subseteq H \implies g H g^{-1} = H.$$

**Proof.** By iteration we have $g H g^{-1} \subseteq H \Rightarrow g^n H g^{-n} \subseteq H$ for all $n \in \mathbb{N}_0$. Let us analyze the set $A = \{g^n | n \in \mathbb{N}_0 \}$. We shall show that $g^{-1}$ lies in the closure $\overline{A}$. We need to distinguish two cases

(i) $e$ is a limit point in $A$. Then for each its neighborhood $U$, there must exist an index $n$ so that $g^n \in U$. It follows $g^n g^{-1} \in g^{-1} U \cap A$ and the set $g^{-1} U$ is a neighborhood of $g^{-1}$, all such $g^{-1} U$ are a local basis at $g^{-1} \in \overline{A}$. 
(ii) $e$ is a discrete point in $\bar{A}$. But $G$ is compact and $A$ is therefore a finite set, so $g^n = e$ for some $n \in \mathbb{N}$. We obtain $g^{-1} = g^{n-1} \in A$.

The conjugation $\text{conj}: (g, h) \mapsto ghg^{-1}$ is continuous as a map $G \times G \to G$ and $H$ is closed, so $\text{conj}(A, H) \subseteq H$, especially $g^{-1}Hg \subseteq H$. \hfill \blacksquare

Let $x \in M$ be a point and $Gx$ the orbit through it. The orbit is called **principal** if there exists an invariant neighborhood $U$ of the point $x \in M$ and for all $y \in U$ an equivariant map $Gx \to Gy$. Points which lie on principal orbits are called **regular**, other points are called **singular**. A subset $S \subset M$ is called a **slice** at $x$ if there exists a $G$-equivariant open neighborhood $U$ of the orbit $Gx$ and a smooth retraction $r: U \to Gx$ such that $S = r^{-1}(x)$.

**Example 2.** Consider the defining representation of $G = SO(3)$ on $M = \mathbb{R}^3$. Let $x = (0, 0, 1)$. The orbit is $Gx = S^2$. We shall show that this orbit is principal. Let $y$

$$U_{\epsilon} = \{(y_1, y_2, y_3) \in \mathbb{R}^3 | \epsilon^2 < y_1^2 + y_2^2 + y_3^2\},$$

where $\epsilon > 0$. The retraction $r: U_{\epsilon} \to Gx$ is defined as

$$r: (y_1, y_2, y_3) \mapsto \frac{(y_1, y_2, y_3)}{\sqrt{y_1^2 + y_2^2 + y_3^2}}.$$  

The point $O = (0, 0, 0)$ is a singular point of the action, $GO = SO(3)$, the orbit is the point $O$ itself. There are only regular points in any open neighborhood of the point $O$.

Orbits of singular points are themselves called **singular** (singular orbits are isomorphic to $G/K$, where dim $K >$ dim $H$). There is a third possibility: the orbit is of maximal dimension but is not isomorphic to the principal orbit. We call such orbits **exceptional**.

**Example 3.** Consider the left action $\psi: SO(3) \times SO(3) \to SO(3)$ of the group $G = SO(3)$ on itself by conjugation. $\psi: (g, h) \mapsto ghg^{-1}$. We know from linear algebra that there always exists an orthonormal basis with respect to which

$$h(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}.$$  

The orbit $Ge = Gh(0)$ is singular, the isotropy subgroup is the whole $G_e = SO(3)$. For $\varphi = \pi$ the orbit is exceptional $Gh(\pi) \cong \mathbb{R}P^2$. The remaining orbits $Gh(\varphi), 0 < \varphi < \pi$, are spheres $S^2$ and their isotropy subgroup is $SO(2)$. From this follows the model of the $SO(3)$ manifold as a closed ball of radius $\pi$, where we identify the antipodal points on the boundary. The center of the ball corresponds to the singular orbit.

5. **Warped products.** This part of the exposition follows [2]. Suppose $M = B \times F$, where $(B, \langle \cdot, \cdot \rangle_B)$ and $(F, \langle \cdot, \cdot \rangle_F)$ are (semi)riemannian manifolds, $f$ a positive function on $B$. We construct the (semi)riemannian metric on $M$: pick an arbitrary point $x = (a, b) \in M = B \times F$. Then the tangent space at this point is $T_xM = T_aB \oplus T_bF$ and each tangent vector $(x, \xi)$ can be unambiguously written as $(a, \alpha) + (b, \beta)$. The scalar product on $M$ is then defined by

$$\langle \xi, \xi' \rangle(x) := \langle \alpha, \alpha' \rangle_B(a) + f^2(a)\langle \beta, \beta' \rangle_F(b).$$  

(8)
If the metric signature on $B$ is $(r, s)$ and $(r', s')$ on $F$ then the metric signature on $M$ is obviously $(r + r', s + s')$. The whole construction is a generalization of a surface of revolution; in this case $B$ is a plane curve which does not intersect the axis of revolution, $f(a)$ gives the distance of the point $a$ from the axis, $F = S^1$. Warped products are denoted by $B \times_f F$.

On $p: B \times F \to B$ (and more generally on a Riemannian submersion $p: M \to B$) there exist special subbundles of the tangent bundle: the \textit{vertical} subbundle $VM = \ker p^\ast$, and the \textit{horizontal} subbundle $HM = VM^\perp$ (the definition of a Riemannian submersion demands that $H_xM \cong T_{p(x)}B$ for all $x \in M$). The sections of these subbundles are called \textit{vertical} resp. \textit{horizontal} vector fields. There is a special class of horizontal vector fields, called \textit{basic} defined as follows: Take any vector field $\eta$ on $B$. Then there exists a unique horizontal vector field $\xi$ such that $\xi p^\ast = p^\ast \eta$. The basic vector fields span $HM$ (for dimensional reasons).

We can compute the relevant tensor fields for warped products following [2]. Let $\xi, \eta$ be basic vector fields and $X, Y, Z$ vertical vector fields. Let Riemann$^F$ denote the Riemann curvature tensor field on the fiber $F$. We assume dim $M = 4$ and dim $F = 2$. For the Riemann curvature on $M$ we obtain

$$ Riemann_{XY}Z = Riemann_{XY}^F Z - \frac{\langle (d f)^\flat, (d f)^\flat \rangle_B}{f^2} \langle (X, Z)_F Y - \langle Y, Z \rangle_F X \rangle, $$

and defining the Hessian of the function $f$ by

$$ \text{Hessian}_f(\xi, \eta) = \langle \nabla_\xi (d f)^\flat, \eta \rangle_B = (\xi \eta - \nabla_\xi \eta) f, $$

which is a symmetric tensor field of type $(0, 2)$, we may write

$$ \langle Riemann_{\xi X \eta Y} \rangle = -\frac{\text{Hessian}_f(\xi, \eta)}{f} \langle X, Y \rangle_F, $$

for the Ricci curvature

$$ \text{Ricci}(\xi, \eta) = \text{Ricci}^B(\xi, \eta) - \frac{2}{f} \text{Hessian}_f(\xi, \eta) $$

$$ \text{Ricci}(\xi, X) = 0 $$

$$ \text{Ricci}(X, Y) = \text{Ricci}^F(X, Y) - \langle X, Y \rangle_F \left( \frac{\star \star d f}{f} + \frac{\langle (d f)^\flat, (d f)^\flat \rangle_B}{f^2} \right), $$

here $\star$ is the Hodge operator and $\nabla$ the Levi-Civita connection (both with respect to $\langle \cdot, \cdot \rangle_B$).

\textit{Example 4 (The Kruskal solution).} We take

$$ B = \{(v, u) \in \mathbb{R}^2 | u^2 - v^2 > 1 \}, $$

and

$$ f(u, v) = 1 + W \left( \frac{v^2 - u^2}{e} \right). $$

Then we define the metric on $B$ by

$$ \frac{4 e^{-f(u, v)}}{f(u, v)} (d u^2 - d v^2), $$

(12)
where \( z \mapsto W(z) \) is the principal branch of the Lambert W-function, the solution of \( z = W(z) e^{W(z)} \). The manifold \( F \) is the sphere \( S^2 \) with the standard negative definite metric induced by the Killing form. For the metric

\[
\frac{4e^{-f(u,v)}}{f(u,v)} \left( du^2 - dv^2 \right) + f^2(u,v) \gamma,
\]

where \( \gamma \) is the standard metric on \( S^2 \) given locally by \( dk_1^2 + \cos^2 k_1 dk_2^2 \), the following holds

\[
\text{Ricci} = 0, \quad R = 0, \quad \text{Einstein} = 0.
\] (13)

6. Centrally symmetric spacetimes. Let \( G = \text{SO}(3, \mathbb{R}) \) be the compact Lie group and \( (M, \langle \cdot, \cdot \rangle) \) a semi-Riemannian manifold of signature \((1, 3)\). We say that \( (M, \langle \cdot, \cdot \rangle) \) is centrally symmetric if there exists an isometric proper \( G \)-action \( \varphi \) all of whose orbits are spheres
$S^2 = \text{SO}(3)/\text{SO}(2)$.

$$\varphi: G \times M \to M$$
$$\varphi: (g, x) \mapsto \varphi(g, x) = gx.$$  

The action is **proper** if the preimages of compact sets by the map $(g, x) \mapsto (gx, x)$ are compact. The action is **isometric** if $\langle g_*\xi, g_*\eta \rangle = \langle \xi, \eta \rangle$ for all $g \in G$ and all vector fields $\xi, \eta \in \mathcal{X}(M)$. The orbit of the point $x \in M$ is denoted by $Gx$.

The sphere is viewed as the homogeneous space $G/Gx = \text{SO}(3, \mathbb{R})/\text{SO}(2, \mathbb{R})$, where $Gx$ is the stabilizer of $x \in M$. The Riemannian metric $\gamma$ on the sphere is constructed using the Maurer-Cartan form on $\text{SO}(3)$ corestricted from $\mathfrak{g} = \mathfrak{so}(3, \mathbb{R})$ to the factor vector space $\mathfrak{so}(3, \mathbb{R})/\mathfrak{so}(2, \mathbb{R})$, and the negative definite Killing form. This metric is unique up to a constant positive multiple (this corresponds to different sphere radii). Orbits of different points are therefore spheres $S^2$ with varying radii.

**Theorem 5.** Let $(M, \langle \cdot, \cdot \rangle)$ be a centrally symmetric spacetime. Then $M$ is the total space of a semi-Riemannian fibre bundle $(M, B, p, S^2)$, where $p: M \to B$ is a surjective submersion and the fibre is $p^{-1}(b) = S^2$. Moreover, the metric on this fibre bundle is a warped product

$$\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_B + f^2 \langle \cdot, \cdot \rangle_{S^2},$$

where $f$ is a positive function on $B$.

**Proof.** The space $B = M/G$ can be thought of as the space of orbits of the action by $G$ on $M$. This induces a topological and smooth structure on $B$ in the standard way such that the factor projection $p: M \to M/G$ is continuous and smooth.

We have to construct a local trivialization on some neighborhood of each point $x \in M$. We proceed as follows: $(M, \langle \cdot, \cdot \rangle)$ is a semi-Riemannian manifold of signature $(1, 3)$, therefore there exists a one-dimensional distribution $\xi$ which can be chosen invariant with respect to the $\text{SO}(3)$-action by Lemma 3 and the corresponding invariant Riemannian metric denoted by $\langle \cdot, \cdot \rangle$, see Lemma 2. We can now use the results from [4].

The orbit $Gx$ is a sphere $S^2$ embedded in $M$ by $\iota: S^2 \to M$. Consider the normal bundle $NS^2 := \{v \in TM | \langle v, w \rangle = 0 \text{ for all } w \in T\iota(S^2)\}$ and the exponential map applied to $0_x$ in a small enough ball $B_x(0_x)$ so that $\exp_x: T_xM \supset B_x(0_x) \to M$ is a diffeomorphism on its image $\exp_x(B_x(0_x)) \cap Gx$. $B_x(0_x)$ denotes a cylindrical neighborhood of $0_x$ in $NS^2$. The inverse is the sought local trivialization of $(M, B, p, S^2)$.

The inner product

$$\langle T_xp\xi_x, T_xp\eta_x \rangle_B = \langle \xi_x, \eta_x \rangle$$

is well defined for $\xi, \eta \in HM$. Therefore the metric on $M$ is a warped product. ■

**Lemma 6.** The one-dimensional distribution $\xi$ projects to $B$ via the map $Tp$ giving rise to a one-dimensional distribution on $B$.

**Proof.** $\xi$ can be chosen invariant, i.e. spanned by local horizontal vector fields. These fields correspond to basic vector fields by definition of a Riemannian submersion. ■
7. Birkhoff’s theorem. This section is almost entirely based on [5].

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