Representation of the Variational Sequence*

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Introduction

The r-th order variational sequence, i. e. on the r-jet prolongation of a fibered manifold $\pi : Y \to X$, is the quotient sequence of the de Rham sequence with respect to its subsequence of contact differential forms. We give the complete representation of the r-th order variational sequence by means of differential forms generally defined on the $(2r+1)$-st jet prolongation of the fibered manifold Y. There is a close relationship between the representatives and their induced mappings on one hand and the standard objects of the calculus of variations, such as Lagrangians, dynamical forms and the Euler-Lagrange and Helmholtz-Sonin mapping on the other hand.

Main References

[1] I. Anderson, *The Variational Bicomplex*, Preprint, Utah State University, 1989.
[2] D.R. Grigore, The variational sequence on finite jet bundle extensions and the Lagrange formalism, Diff. Geom. Appl. 10 (1999) 43–77.
[3] J. Kašparová, Variational sequences in field theory, Ph. D. thesis, Mathematical Institute, Silesian University, Opava, 2000, pp. 71.
[4] M. Krbek, J. Musilová, J. Kašparová, The variational sequence: Local and global properites. In Proceedings of the Seminar on Differential Geometry, Silesian university, Opava, 2001, 15-38, Mathematical Publications, ISBN 80-7248-104-5.
[5] M. Krbek, J. Musilová, J. Kašparová, Representation of the variational sequence in field theory. In Proc. Colloq. Diff. Geometry., Debrecen, Hungary, University of Debrecen, 2001, 147-306.
[6] D. Krupka, Variational sequences on finite order jet spaces, in: *Differential Geometry and its Applications*, Proc. Conf., J. Janyška and D. Krupka, ed., Brno Czechoslovakia, 1989 (World Scientific, Singapore, 1990) 236–254.
[7] D. Krupka, *The Geometry of Lagrange Structures*, preprint series in Global Analysis GA 7/1997, Silesian University, Opava 1997, pp. 82.
[8] Šeděnková J.: On the invariant variational sequences in mechanics. In: Rend. Cont. Mat. Palermo. Proc. 22nd Winter School Geometry and Physics. Srní, Czech Republic, to appear.

Basic notation

– The fibered manifold $\pi : Y \to X$, $\dim Y = m + n$, $\dim X = n$
– The r-th prolongation J^rY, $\dim J^rY = n + m\binom{n+r}{n}$
– The canonical projections $\pi^r : J^rY \to X$ and $\pi^{r,s} : J^rY \to J^sY$, where $r \geq s \geq 0$
– Sections of the fibered manifold Y denoted by γ, their prolongations to J^rY denoted by $J^r\gamma$
– Fibered charts (W, ψ), where $\psi = (x^i, y^\sigma)$, $1 \leq i \leq n$, $1 \leq \sigma \leq m$ on Y
– Associated charts on (U, ϕ), where $\phi = (x^i)$ $1 \leq i \leq n$ on X and (W^r, ψ_r), where $W^r = (\pi^{r,0})^{-1}(W)$ and $\psi_r = (x^i, y^\sigma, y^\sigma_{j_1}, \ldots, y^\sigma_{j_1 \ldots j_r})$, we shall also use multiindices $J = (j_1 \ldots j_s)$, where $|J| = s$.

Contact forms

Let $W \subset Y$ be an open set. We denote $\Omega^r_0 W$ the ring of smooth functions on $W^r = (\pi^{r,0})^{-1}(W)$ and $\Omega^r_q W$ the $\Omega^r_0 W$-module of smooth differential q-forms on W^r. The submodule of contact, for $1 \leq q \leq n$ (resp. strongly contact, for $n + 1 \leq q$) q-forms will be denoted $\Omega^r_{q,\mathrm{c}} W = \ker \mathrm{h} = \ker \mathrm{p}_0$ (resp.$\Omega^r_{q,\mathrm{c}} W = \ker \mathrm{p}_{q-n}$). The contact ideal will be denoted $\Theta^r_q W = \Omega^r_q W + \mathrm{d}\, \Theta^r_{q-1,\mathrm{c}} W$. We shall sometimes omit W for brevity. We denote $\omega^\sigma_J = \mathrm{d}\, y^\sigma_J - y^\sigma_{Ji} \,\mathrm{d}\, x^i$, the contact 1-forms, $\omega_i = (-1)^{i-1} \,\mathrm{d}\, x^1 \wedge \ldots \wedge \mathrm{d}\,\hat{x}^i \wedge \ldots \mathrm{d}\, x^n$ and $\omega_0 = \mathrm{d}\, x^1 \wedge \ldots \wedge \mathrm{d}\, x^n$ frequently used horizontal forms. In our calculations we use the following identities

$$\mathrm{d}\, x^i \wedge \omega_i = \omega_0 \qquad \text{and} \qquad \mathrm{d}\, \omega^\sigma_J \wedge \omega_i = -\omega^\sigma_{Ji} \wedge \omega_0.$$

The coordinate expression of a q-form $\eta \in \Omega^r_q W$ is

$$\eta = \sum_{s=0}^{q} P^{J_1 \ldots J_s}_{\sigma_1 \ldots \sigma_s \; i_{s+1} \ldots i_q} \,\mathrm{d}\, y^{\sigma_1}_{J_1} \wedge \ldots \mathrm{d}\, y^{\sigma_s}_{J_s} \wedge \mathrm{d}\, x^{i_{s+1}} \wedge \ldots \wedge \mathrm{d}\, x^{i_q}. \tag{1}$$

The summation extends over all values of the multiindices J_k, such that $|J_k| \leq r$.

The pulled-back form η can be uniquely decomposed as

$$(\pi^{r+1,r})^* \eta = \sum_{k=0}^{q} \mathrm{p}_k\, \eta, \qquad \text{where} \tag{2}$$

$$\mathrm{p}_k\, \eta = \sum_{s=k}^{q} \tbinom{s}{k} P^{J_1 \ldots J_s}_{\sigma_1 \ldots \sigma_s \; i_{s+1} \ldots i_q} y^{\sigma_{k+1}}_{J_{k+1} i_{k+1}} \cdots y^{\sigma_s}_{J_s i_s} \omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_k}_{J_k} \wedge \mathrm{d}\, x^{i_{k+1}} \wedge \ldots \wedge \mathrm{d}\, x^{i_q},$$

is the k-contact component of η.

Formal derivative of differential forms

We shall define the formal derivative of differential forms by first defining it for q-contact q-forms and then extending the definition by straightforward generalization.

Definition 1: Let $\eta \in \Omega^r_q W$ be a q-contact q-form. We define the formal derivative of η, also q-contact by

$$\mathrm{p}_q \,\mathrm{d}\,\eta = (-1)^q \,\mathrm{D}_i\,\eta \wedge \mathrm{d}\,x^i. \tag{3}$$

We must ensure that the formal derivative is defined uniquely by (3). To this aim we calculate the formal derivative in coordinates.

$$\eta = A^{J_1\dots J_q}_{\sigma_1\dots\sigma_q}\omega^{\sigma_1}_{J_1}\wedge\ldots\wedge\omega^{\sigma_q}_{J_q}$$

$$(-1)^q\,\mathrm{p}_q\,\mathrm{d}\,\eta = \mathrm{D}_i\,A^{J_1\dots J_q}_{\sigma_1\dots\sigma_q}\wedge\omega^{\sigma_1}_{J_1}\wedge\ldots\wedge\omega^{\sigma_q}_{J_q}\wedge\mathrm{d}\,x^i + q A^{J_1\dots J_q}_{\sigma_1\dots\sigma_q}\omega^{\sigma_1}_{J_1 i}\wedge\omega^{\sigma_2}_{J_2}\wedge\ldots\wedge\omega^{\sigma_q}_{J_q}\wedge\mathrm{d}\,x^i$$

$$\mathrm{D}_i\,\eta = \mathrm{D}_i\,A^{J_1\dots J_q}_{\sigma_1\dots\sigma_q}\wedge\omega^{\sigma_1}_{J_1}\wedge\ldots\wedge\omega^{\sigma_q}_{J_q} + q A^{J_1\dots J_q}_{\sigma_1\dots\sigma_q}\omega^{\sigma_1}_{J_1 i}\wedge\omega^{\sigma_2}_{J_2}\wedge\ldots\wedge\omega^{\sigma_q}_{J_q} \tag{4}$$

Theorem 1: *(Properties of formal derivatives) Let (W,ψ), $\psi=(x^i,y^\sigma)$ and $(\bar W,\bar\psi)$, $\bar\psi=(\bar x^i,\bar y^\sigma)$ be fibered charts on Y such that $W\cap\bar W\neq\emptyset$, (U,ϕ), $\phi=(x^i)$ and $(\bar U,\bar\phi)$, $\bar\phi=(\bar x^i)$ the associated charts on X. Let $\eta\in\Omega^r_q W$ be a q-contact q-form. Then*

$$\bar{\mathrm{D}}_i\eta = \frac{\partial x^j}{\partial\bar x^i}\,\mathrm{D}_j\,\eta. \tag{5}$$

Let $\rho\in\Omega^r_s W$ be another s-contact s-form. Then

$$\mathrm{D}_i(\eta\wedge\rho) = \mathrm{D}_i\,\eta\wedge\rho + \eta\wedge\mathrm{D}_i\,\rho. \tag{6}$$

$$\mathrm{D}_i\,\mathrm{D}_j\,\eta = \mathrm{D}_j\,\mathrm{D}_i\,\eta, \qquad \textit{enables to write } \mathrm{D}_J = \mathrm{D}_{j_1\dots j_s}. \tag{7}$$

We shall now extend the definition of the formal derivative to arbitrary forms in such a way that the property (6) remains valid.

Definition 2: We define

$$\mathrm{D}_i\,\mathrm{d}\,x^j = 0, \qquad \text{for } 1\le i,j\le n. \tag{8}$$

Now, let $\eta\in\Omega^r_q W$ be a q-form and $\mathrm{p}_k\,\eta\in\Omega^{r+1}_q W$ its k-contact component in the decomposition (2). We define $\mathrm{D}_i\,\mathrm{p}_k\,\eta$ by the direct use of the formulae (8) and (6), taking into account the formula (4). Finally, we define

$$\mathrm{D}_i\,\eta = \sum_{k=0}^{q}\mathrm{D}_i\,\mathrm{p}_k\,\eta. \tag{9}$$

Notice that for $\eta \in \Omega^r_q W$ arbitrary, it still holds that

$$\mathrm{D}_i \, \mathrm{p}_k \, \eta \wedge \mathrm{d}\, x^i = (-1)^q \, \mathrm{p}_k \, \mathrm{d} \, \mathrm{p}_k \, \eta,$$

although this property fails to define D_i uniquely. The transformation properties (5) of formal derivatives therefore remain unchanged. The commutation rule (7) also remains valid.

Example: Let us describe the effect of D_i on functions and contact 1-forms. Functions are seen as 0-contact 0-forms so that for a function $f \in \Omega^r_0 W$ we get the usual

$$\mathrm{D}_i \, f = \frac{\partial f}{\partial x^i} + \sum_{|K|=0}^{r} \frac{\partial f}{\partial y^\sigma_J} y^\sigma_{Ji}. \tag{10}$$

For the contact 1-forms we obtain

$$\begin{aligned} \mathrm{p}_1 \, \mathrm{d} \, \omega^\sigma_J &= \mathrm{p}_1 \, \mathrm{d} \left(\mathrm{d} \, y^\sigma_J - y^\sigma_{Ji} \, \mathrm{d} \, x^i \right) = -\omega^\sigma_{Ji} \wedge \mathrm{d} \, x^i \\ \mathrm{D}_i \, \omega^\sigma_J &= \omega^\sigma_{Ji}. \end{aligned} \tag{11}$$

Formal differential operators

Let us consider Ω^r_q with the structure of afibered bundle and let $V_r \subset TJ^rY$ be the subbundle of π^r-vertical vector fields on J^rW (that is $V_r = \ker T\pi^r$) and let $\Xi \in V_0$.

Definition 3: The mapping $P : V_0 \to \Omega^r_q$ is called a *formal differential operator*, if there exists a linear bundle map $L : V_r \to \Omega^s_q$, $r \le s$ such that the diagram

$$\begin{tikzcd} V_r \arrow[r, "L"] \arrow[d] & \Omega^s_q \arrow[d] \\ J^rY & J^sY \arrow[l, "\pi^{s,r}"'] \end{tikzcd}$$

commutes (the unlabeled arrows denote bundle projections) and

$$P(\Xi) = L(J^r \Xi). \tag{12}$$

The integer r is called the *order* of the formal differential operator P.

Examples: The following expressions define formal differential operators.

- Let $\rho \in \Omega^r_{q+1}$ be a given $(q+1)$-form. Then

  (13) $$P_\rho(\Xi) = J^r\Xi \neg \rho$$

  is a formal differential operator, $P_\rho : V_0 \to \Omega^r_q$ associated with ρ.

- Let $\eta \in \Omega^r_q$ be a given q-form. Then

  (14) $$P_\eta(\Xi) = \mathcal{L}_{J^r\Xi}\eta$$

  is a formal differential operator $P_\eta : V_0 \to \Omega^r_q$ associated with η.

The coordinate expression for any formal differential operator is

(15) $$P(\Xi) = \sum_{|I|=0}^{r} \mathrm{D}_I(\Xi^\sigma) P^I_\sigma,$$

where the coefficients P^I_σ are differential forms in Ω^r_q.
Lemma 3: *Any formal differential operator P can be uniquely rewritten as*

(16) $$P(\Xi) = \sum_{|J|=0}^{r} \mathrm{D}_J\left(\Xi^\sigma Q^J_\sigma\right), \qquad \textit{where}$$

(17) $$Q^J_\sigma = \sum_{|K|=0}^{r-|J|} \binom{|J|+|K|}{|J|} (-1)^{|K|} \mathrm{D}_K\, P^{JK}_\sigma.$$

Proof-comments: We show uniqueness by proving that if Q^J_σ is zero, then also P^J_σ is zero for $1 \le \sigma \le m$ and $0 \le |J| \le r$ by using the formula (6) on (16). Then we verify that the solution (17) satisfies the equation (16) identically. □

The question of the meaning and transformation properties of the coefficients Q^J_σ arises. To provide the answer to this question we first prove the following lemma.
Lemma 4: *Let Ξ be a π-vertical vector field on Y and $\eta \in \Omega^r_q$ a differential q-form. Then the following holds:*
(a)

(18) $$J^{r+2}\Xi \neg \mathrm{p}_k\, \mathrm{d}\, \mathrm{p}_k\, \eta = -\, \mathrm{p}_{k-1}\, \mathrm{d}\left(J^{r+1}\Xi \neg \mathrm{p}_k\, \eta\right)$$

(b)

$$\mathcal{L}_{J^{r+2}\Xi}\left(\pi^{r+2,r+1}\right)^* \mathrm{p}_k\, \eta = J^{r+2}\Xi \neg \mathrm{p}_{k+1}\, \mathrm{d}\, \mathrm{p}_k\, \eta + \mathrm{p}_k\, \mathrm{d}\left(J^{r+1}\Xi \neg \mathrm{p}_k\, \eta\right)$$

Proof-comments: The proof is the consequence of the formula

$$\mathcal{L}_\xi \eta = \xi \neg \mathrm{d}\, \eta + \mathrm{d}\, \xi \neg \eta$$

and the fact that the Lie derivative by the prolongation of a π-vertical vector field does not change the order of contactness of a differential form. □

Now we turn to the question of the nature of the locally defined coefficients Q^J_σ and determine if they have some geometric meaning.

Theorem 5: *Let $P : V_0 \to \mathrm{p}_k\, \Omega^r_{n+k}$ be a formal differential operator. Then there exists a unique formal differential operator*

$$Q : V_0 \to \mathrm{p}_k\, \Omega^{2r+1}_{n+k}, \qquad \textit{such that}$$

$$P(\Xi) = Q(\Xi) + \mathrm{p}_k\, \mathrm{d}\, \mathrm{p}_k\, R(\Xi), \tag{19}$$

where $R : V_0 \to \mathrm{p}_k\, \Omega^{2r}_{n+k-1}$ is a local formal differential operator and

$$Q(\Xi) = Q_\sigma \Xi^\sigma, \tag{20}$$

where Q_σ are defined in terms of P^J_σ as in (17). Q is called the Euler operator *corresponding to P.*

Proof: Local existence of the decomposition (19). As in (16), we can write

$$P(\Xi) = Q_\sigma \Xi^\sigma + \sum_{|J|=1}^{r} \mathrm{D}_J(\Xi^\sigma Q^J_\sigma) = Q_\sigma \Xi^\sigma + \mathrm{D}_i \sum_{|J|=0}^{r-1} \mathrm{D}_J(\Xi^\sigma Q^{Ji}_\sigma). \tag{21}$$

The coefficients Q^J_σ are k-contact differential $(k+n)$-forms

$$Q^J_\sigma = B^J_\sigma \wedge \omega_0,$$

where B^J_σ are k-contact k-forms. Plugging into (21) we get

$$P(\Xi) = Q_\sigma \Xi^\sigma + \mathrm{D}_i \sum_{|J|=0}^{r-1} \mathrm{D}_J \left(\Xi^\sigma B^{Ji}_\sigma \wedge \omega_0\right) = Q_\sigma \Xi^\sigma + \mathrm{D}_i \sum_{|J|=0}^{r-1} \mathrm{D}_J \left[\Xi^\sigma B^{Jl}_\sigma \delta^i_l \wedge \omega_0\right]$$

$$P(\Xi) = Q_\sigma \Xi^\sigma + \mathrm{p}_k\, \mathrm{d}\, \mathrm{p}_k (-1)^{n-1} \sum_{|J|=0}^{r-1} \mathrm{D}_J \left[\Xi^\sigma B^{Jl}_\sigma \wedge \omega_l\right]$$

the needed local decomposition.

Uniqueness. Suppose that Q is not given uniquely and

$$P(\Xi) = Q(\Xi) + \mathrm{p}_k\, \mathrm{d}\, \mathrm{p}_k\, R(\Xi) = \tilde{Q}(\Xi) + \mathrm{p}_k\, \mathrm{d}\, \mathrm{p}_k\, \tilde{R}(\Xi).$$

Subtracting the equations we obtain

$$\Xi^\sigma M_\sigma = Q(\Xi) - \tilde{Q}(\Xi) = \mathrm{p}_k \,\mathrm{d}\, \mathrm{p}_k[\tilde{R}(\Xi) - R(\Xi)] = \mathrm{p}_k \,\mathrm{d}\, \mathrm{p}_k(R - \tilde{R})(\Xi).$$

Let Ξ_j, $j \in \{1, \ldots, k\}$ be arbitrary π-vertical vector fields on Y and let us form the expression

$$J^{r+2}\Xi_k \neg \ldots \neg J^{r+2}\Xi_1 \neg \mathrm{p}_k \,\mathrm{d}\, \mathrm{p}_k(R - \tilde{R})(\Xi)$$

We apply (18) and arrive at

$$\Xi^\sigma N_\sigma = (-1)^k \,\mathrm{h}\,\mathrm{d}[J^{r+1}\Xi_k \neg \ldots \neg J^{r+1}\Xi_1 \neg \mathrm{p}_k(R - \tilde{R})(\Xi)]$$

with $N_\sigma = J^{r+2}\Xi_k \neg \ldots \neg J^{r+2}\Xi_1 \neg M_\sigma$. Choose an arbitrary piece $\Omega \subset U \subset X$ in the coordinate neighborhood and pick Ξ_j so that $\operatorname{supp} \Xi_j \subset \Omega$, and a local section γ of Y. Form the integral

$$\int_\Omega (J^{r+2}\gamma)^* \Xi^\sigma N_\sigma.$$

Due to Stokes' theorem this integral is zero. The section γ was arbitrary so $N_\sigma = 0$. Finally, the vector fields $\Xi_1, \ldots, \Xi_k$ were arbitrary so $M_\sigma = 0$ and $Q(\Xi) = \tilde{Q}(\Xi)$.

Uniqueness and local existence suffice to imply that the operator Q is globally well defined. $\square$

The Representation Mapping

For differential q-forms $\rho \in \Omega^r_q$, where $1 \leq q \leq n$ the representation mapping is given by the horizontalization $\mathrm{h} = \mathrm{p}_0$.

The representation mapping I for $q > n$ shall be defined by means of the Euler operator corresponding to the formal differential operator defined by $J^r\Xi \neg \mathrm{p}_{k+1}\, \rho$, where $\rho \in \Omega^r_{k+n+1}$ is a given differential form.

Definition 4: Let $\rho \in \Omega^r_{k+n+1}$ a $(k+n+1)$-form. The *interior Euler mapping* $I : \Omega^r_{n+k+1} \to \Omega^{2r+1}_{n+k+1}$, $k \geq 0$ is defined as follows:

$$I(\rho) := \tfrac{1}{k+1}\omega^\sigma \wedge \sum_{|J|=0}^{r} (-1)^{|J|}\, \mathrm{D}_J \left(\partial^J_\sigma \neg \mathrm{p}_{k+1}\, \rho\right). \tag{22}$$

The mapping I is connected to the Euler operator corresponding to

$$P(\Xi) = J^r\Xi \neg \rho = \sum_{|J|=0}^{r} (D_J\Xi^\sigma)(\partial^J_\sigma \neg \mathrm{p}_{k+1}\, \rho)$$

From Theorem 5 we have $Q(\Xi) = Q_\sigma \Xi^\sigma$ and the expressions $I = \frac{1}{k+1}\omega^\sigma \wedge Q_\sigma$ define a differential form.

Lemma 6: *For any $\eta \in \Omega^r_{k+n}$, it holds that $I \circ \mathrm{p}_{k+1} \circ \mathrm{d} \circ \mathrm{p}_{k+1}\, \eta = 0$.*

Proof: Due to the equation (18) one has

$$J^{r+2}\Xi \neg \mathrm{D}_j\, \mathrm{p}_{k+1}\, \eta \wedge \mathrm{d}\, x^j = -\, \mathrm{D}_j \left(J^{r+1}\Xi \neg \mathrm{p}_{k+1}\, \eta\right) \wedge \mathrm{d}\, x^j.$$

Using the equation (15) on both sides of the previous equation we obtain

$$\begin{aligned}\sum_{|J|=0}^{r+2} \mathrm{D}_J \left[\Xi^\sigma Q^J_\sigma(\mathrm{D}_j\, \mathrm{p}_{k+1}\, \eta)\right] &= -\, \mathrm{D}_j \left\{ \sum_{|J|=0}^{r+1} \mathrm{D}_J \left[\Xi^\sigma Q^J_\sigma(\mathrm{p}_{k+1}\, \eta)\right] \right\} \\ &= -\sum_{|J|=1}^{r+2} \mathrm{D}_J \left[\Xi^\sigma \delta^{(i}_j Q^{K)}_\sigma(\mathrm{p}_{k+1}\, \eta)\right],\end{aligned}$$

where $iK = J$. From this equation it follows that $Q^J_\sigma(\mathrm{D}_i\, \eta) = -\delta^{(j}_i Q^{K)}_\sigma(\eta)$, in particular $Q_\sigma(\mathrm{p}_{k+1}\, \mathrm{d}\, \mathrm{p}_{k+1}\, \eta) = 0$ which implies the Lemma 6.

Theorem 7: *The interior Euler mapping* $I : \Omega^r_{n+k+1} \to \Omega^{2r+1}_{n+k+1}$ *has the following properties*

(a) The relation $I(\mathrm{p}_{k+1}\,\rho) \sim \mathrm{p}_{k+1}\,\rho$ *is fulfilled in the generalized sense, meaning that*

$$\left(\pi^{2r+1,r+1}\right)^* \mathrm{p}_{k+1}\,\rho - I(\mathrm{p}_{k+1}\,\rho) \in \Theta^{2r+1}_{k+1}. \tag{23}$$

(b) The operator I *is a projection in the following sense*

$$I^2 = I \circ \left(\pi^{2r+1,r}\right)^*. \tag{24}$$

(c) The operator I *is the representation of the variational sequence by forms, more precisely*

$$I\,\mathrm{p}_{k+3}\,\mathrm{d}\,I\,\mathrm{p}_{k+2}\,\mathrm{d}\,\rho = 0. \tag{25}$$

Proof: (a) The consequence of theorem 5.
(b) We apply the operator I to the property (a) and use lemma 6.
(c) Let $\eta = \mathrm{p}_{k+2}\,\mathrm{d}\,\rho$. Then from theorem 5 we have

$$\mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\rho + \mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+2}\,\rho = \mathrm{p}_{k+2}\,\mathrm{d}\,\rho = I(\mathrm{p}_{k+2}\,d\rho) + \mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+2}\,\zeta|_W\,.$$

To this equation we apply first the operator $\mathrm{p}_{k+3}\,\mathrm{d}$ and obtain

$$0 = \mathrm{p}_{k+3}\,\mathrm{d}\,\mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+1}\,\rho = \mathrm{p}_{k+3}\,\mathrm{d}\,I\,\mathrm{p}_{k+2}\,\mathrm{d}\,\rho + \mathrm{p}_{k+3}\,\mathrm{d}\,\mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+2}\,(\zeta|_W - \rho)\,.$$

The left hand side of this equation vanishes identically and on the right hand side we obtain

$$\mathrm{p}_{k+3}\,\mathrm{d}\,\mathrm{p}_{k+2}\,\mathrm{d}\,\mathrm{p}_{k+2}\,(\zeta|_W - \rho) = -\,\mathrm{p}_{k+3}\,\mathrm{d}\,\mathrm{p}_{k+3}\,\mathrm{d}\,\mathrm{p}_{k+2}\,(\zeta|_W - \rho)$$

and get

$$\mathrm{p}_{k+3}\,\mathrm{d}\,I\,\mathrm{p}_{k+2}\,\mathrm{d}\,\rho = \mathrm{p}_{k+3}\,\mathrm{d}\,\mathrm{p}_{k+3}\,\mathrm{d}\,\mathrm{p}_{k+2}\,\zeta.$$

Now we once more apply I, use the lemma 6 and obtain

$$I\,\mathrm{p}_{k+3}\,\mathrm{d}\,I\,\mathrm{p}_{k+2}\,\mathrm{d}\,\rho = 0.$$

Examples of Representatives

Let $\rho \in \Omega^r_q$ be a differential q-form given in a fibered chart by (1).

- Let $\rho \in \Omega^r_n$ be a differential n-form. Then $[\rho]$, its class, is represented by the horizontal form $\mathrm{h}\,\rho \in \Omega^{r+1}_n$ which is polynomial in the coordinates y^σ_J, $|J| = r+1$, as seen from (2). These representatives are customarily called Lagrangians and in local coordinates we have

$$L_\rho = \mathrm{h}\,\rho = \varepsilon^{i_1\dots i_n} \sum_{s=0}^{n} P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\; i_{s+1}\dots i_n} y^{\sigma_1}_{J_1 i_1} \cdots y^{\sigma_s}_{J_s i_s} \omega_0. \tag{26}$$

- Let $\rho \in \Omega^r_{n+1}$ be a differential $(n+1)$-form. Then $[\rho]$, its class, is represented by the horizontal form $I\rho \in \Omega^{2r+1}_{n+1}$ which is polynomial in the coordinates y^σ_J, $|J| \geq r+1$, as can be seen from (2) and the formulas (4) and (10) for formal derivatives. These representatives are called dynamical forms and their local expressions are

$$E_\rho = I\rho = \sum_{|J_1|=0}^{r} (-1)^{|J_1|}\, \mathrm{D}_{J_1} \left(\varepsilon^{i_2\dots i_{n+1}} \sum_{s=1}^{q} s P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\; i_{s+1}\dots i_q} y^{\sigma_2}_{J_2 i_2} \cdots y^{\sigma_s}_{J_s i_s} \right) \omega^{\sigma_1} \wedge \omega_0. \tag{27}$$

- Let $\rho \in \Omega^r_{n+2}$ be a differential $(n+2)$-form. Then $[\rho]$, its class, is represented by the horizontal form $I\rho \in \Omega^{2r+1}_{n+2}$ which is polynomial in the coordinates y^σ_J, $|J| \geq r+1$, as can be seen from (2) and the formulas (4) and (10) for formal derivatives. These representatives are called Helmholtz-Sonin forms and their local expressions are

$$\begin{aligned} H_\rho = I\rho = \tfrac{1}{2} \sum_{a=0}^{2r} \Bigg[\sum_{b=0}^{a} \sum_{c=a-b}^{r} (-1)^c \binom{c}{a-b} \mathrm{D}_{j_{a+1}\dots j_{b+c}} \\ \left(\varepsilon^{i_3\dots i_{n+2}} \sum_{s=2}^{q} \binom{s}{2} P^{(j_1\dots j_b)\;(j_{b+1}\dots j_{b+c}) J_3\dots J_s}_{\sigma_1\;\;\;\;\;\;\;\sigma_2\;\;\;\;\;\;\;\;\;\;\;\;\;\sigma_3\dots\sigma_s\; i_{s+1}\dots i_q} y^{\sigma_3}_{J_3 i_3} \cdots y^{\sigma_s}_{J_s i_s} \right) \Bigg] \omega^{\sigma_1}_{(j_1\dots j_a)} \wedge \omega^{\sigma_2} \wedge \omega_0. \end{aligned} \tag{28}$$

Examples of Induced Mappings

– Let $\rho \in \Omega^r_n$ be an n-form. We compute the representative of the class $[\mathrm{d}\,\rho]$ given by $I\,\mathrm{d}\,\mathrm{h}\,\rho$. In local coordinates we obtain by using (26) and (27)

$$(29)\quad I\,\mathrm{d}\,\mathrm{h}\,\rho = \sum_{|J|=0}^{r} (-1)^{|J|}\,\mathrm{D}_J\,\frac{\partial}{\partial y^\nu_J}\left[\varepsilon^{i_1\dots i_n}\sum_{s=0}^{n} P^{J_1\dots J_s}_{\sigma_1\dots\sigma_s\; i_{s+1}\dots i_n} y^{\sigma_1}_{J_1 i_1}\cdots y^{\sigma_s}_{J_s i_s}\right]\omega^\nu \wedge \omega_0.$$

We see that for a horizontal form $\lambda = \mathfrak{L}\omega_0$ we recover the Euler-Lagrange expressions associated with λ

$$(30)\qquad I\,\mathrm{d}\,\lambda = \sum_{|J|=0}^{r} (-1)^{|J|}\,\mathrm{D}_J\,\frac{\partial \mathfrak{L}}{\partial y^\nu_J}\omega^\nu \wedge \omega_0.$$

– Let $\rho \in \Omega^r_{n+1}$ be an $(n+1)$-form. The representative of the class $[\mathrm{d}\,\rho]$ is given by $I\,\mathrm{d}\,\rho$.

$$(31)\quad I\,\mathrm{d}\,I\rho = \tfrac{1}{2}\sum_{a=0}^{2r}\left[\left(\frac{\partial \mathfrak{E}_\nu}{\partial y^\sigma_{j_1\dots j_a}} - (-1)^a\frac{\partial \mathfrak{E}_\sigma}{\partial y^\nu_{j_1\dots j_a}} - \right.\right.$$
$$\left.\left. - \sum_{b=a+1}^{r} (-1)^b \binom{b}{a}\,\mathrm{D}_{j_{a+1}\dots j_b}\,\frac{\partial \mathfrak{E}_\sigma}{\partial y^\nu_{j_1\dots j_b}}\right)\omega^\sigma_{j_1\dots j_a}\right]\wedge\omega^\nu\wedge\omega_0,$$

where the coefficients $\mathfrak{E}^\sigma$ are the components of E_ρ in the formula (27)

$$(32)\qquad E_\rho = \mathfrak{E}_\sigma\omega^\sigma \wedge \omega_0.$$

We can also consider an arbitrary dynamical form (32) as a special case and obtain formally identical representatives to (31).