Exercises

1. Suppose a and b are real numbers, not both 0. Find real numbers c and d such that
$$1/(a+bi) = c + di.$$

2. Show that
$$\frac{-1+\sqrt{3}i}{2}$$
is a cube root of 1 (meaning that its cube equals 1).

3. Prove that $-(-v) = v$ for every $v \in V$.

4. Prove that if $a \in \mathbf{F}$, $v \in V$, and $av = 0$, then $a = 0$ or $v = 0$.

5. For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$:

   (a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$;

   (b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$;

   (c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$;

   (d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$.

6. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a subspace of $\mathbf{R}^2$.

7. Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$.

8. Prove that the intersection of any collection of subspaces of V is a subspace of V.

9. Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

10. Suppose that U is a subspace of V. What is $U + U$?

11. Is the operation of addition on the subspaces of V commutative? Associative? (In other words, if U_1, U_2, U_3 are subspaces of V, is $U_1 + U_2 = U_2 + U_1$? Is $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$?)

12. Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

13. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
$$U_1 + W = U_2 + W,$$
then $U_1 = U_2$.

14. Suppose U is the subspace of $\mathcal{P}(\mathbf{F})$ consisting of all polynomials p of the form
$$p(z) = az^2 + bz^5,$$
where $a, b \in \mathbf{F}$. Find a subspace W of $\mathcal{P}(\mathbf{F})$ such that $\mathcal{P}(\mathbf{F}) = U \oplus W$.

15. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that
$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W,$$
then $U_1 = U_2$.

Exercises

1. Prove that if $(v_1, \ldots, v_n)$ spans V, then so does the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

2. Prove that if $(v_1, \ldots, v_n)$ is linearly independent in V, then so is the list

$$(v_1 - v_2, v_2 - v_3, \ldots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

3. Suppose $(v_1, \ldots, v_n)$ is linearly independent in V and $w \in V$. Prove that if $(v_1 + w, \ldots, v_n + w)$ is linearly dependent, then $w \in \operatorname{span}(v_1, \ldots, v_n)$.

4. Suppose m is a positive integer. Is the set consisting of 0 and all polynomials with coefficients in $\mathbf{F}$ and with degree equal to m a subspace of $\mathcal{P}(\mathbf{F})$?

5. Prove that $\mathbf{F}^\infty$ is infinite dimensional.

6. Prove that the real vector space consisting of all continuous real-valued functions on the interval $[0, 1]$ is infinite dimensional.

7. Prove that V is infinite dimensional if and only if there is a sequence $v_1, v_2, \ldots$ of vectors in V such that $(v_1, \ldots, v_n)$ is linearly independent for every positive integer n.

8. Let U be the subspace of $\mathbf{R}^5$ defined by

$$U = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{R}^5 : x_1 = 3x_2 \text{ and } x_3 = 7x_4\}.$$

Find a basis of U.

9. Prove or disprove: there exists a basis (p_0, p_1, p_2, p_3) of $\mathcal{P}_3(\mathbf{F})$ such that none of the polynomials p_0, p_1, p_2, p_3 has degree 2.

10. Suppose that V is finite dimensional, with $\dim V = n$. Prove that there exist one-dimensional subspaces $U_1, \ldots, U_n$ of V such that

$$V = U_1 \oplus \cdots \oplus U_n.$$

11. Suppose that V is finite dimensional and U is a subspace of V such that $\dim U = \dim V$. Prove that $U = V$.

12. Suppose that $p_0, p_1, \dots, p_m$ are polynomials in $\mathcal{P}_m(\mathbf{F})$ such that $p_j(2) = 0$ for each j. Prove that $(p_0, p_1, \dots, p_m)$ is not linearly independent in $\mathcal{P}_m(\mathbf{F})$.

13. Suppose U and W are subspaces of $\mathbf{R}^8$ such that $\dim U = 3$, $\dim W = 5$, and $U + W = \mathbf{R}^8$. Prove that $U \cap W = \{0\}$.

14. Suppose that U and W are both five-dimensional subspaces of $\mathbf{R}^9$. Prove that $U \cap W \neq \{0\}$.

15. You might guess, by analogy with the formula for the number of elements in the union of three subsets of a finite set, that if U_1, U_2, U_3 are subspaces of a finite-dimensional vector space, then

$$\begin{aligned}\dim(U_1 + U_2 + U_3)& \\ = \dim U_1 &+ \dim U_2 + \dim U_3 \\ &- \dim(U_1 \cap U_2) - \dim(U_1 \cap U_3) - \dim(U_2 \cap U_3) \\ &+ \dim(U_1 \cap U_2 \cap U_3).\end{aligned}$$

Prove this or give a counterexample.

16. Prove that if V is finite dimensional and $U_1, \dots, U_m$ are subspaces of V, then

$$\dim(U_1 + \cdots + U_m) \leq \dim U_1 + \cdots + \dim U_m.$$

17. Suppose V is finite dimensional. Prove that if $U_1, \dots, U_m$ are subspaces of V such that $V = U_1 \oplus \cdots \oplus U_m$, then

$$\dim V = \dim U_1 + \cdots + \dim U_m.$$

This exercise deepens the analogy between direct sums of subspaces and disjoint unions of subsets. Specifically, compare this exercise to the following obvious statement: if a finite set is written as a disjoint union of subsets, then the number of elements in the set equals the sum of the number of elements in the disjoint subsets.

Exercises

1. Show that every linear map from a one-dimensional vector space to itself is multiplication by some scalar. More precisely, prove that if $\dim V = 1$ and $T \in \mathcal{L}(V, V)$, then there exists $a \in \mathbf{F}$ such that $Tv = av$ for all $v \in V$.

2. Give an example of a function $f : \mathbf{R}^2 \to \mathbf{R}$ such that
$$f(av) = af(v)$$
for all $a \in \mathbf{R}$ and all $v \in \mathbf{R}^2$ but f is not linear.

Exercise 2 shows that homogeneity alone is not enough to imply that a function is a linear map. Additivity alone is also not enough to imply that a function is a linear map, although the proof of this involves advanced tools that are beyond the scope of this book.

3. Suppose that V is finite dimensional. Prove that any linear map on a subspace of V can be extended to a linear map on V. In other words, show that if U is a subspace of V and $S \in \mathcal{L}(U, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

4. Suppose that T is a linear map from V to $\mathbf{F}$. Prove that if $u \in V$ is not in $\operatorname{null} T$, then
$$V = \operatorname{null} T \oplus \{au : a \in \mathbf{F}\}.$$

5. Suppose that $T \in \mathcal{L}(V, W)$ is injective and $(v_1, \ldots, v_n)$ is linearly independent in V. Prove that $(Tv_1, \ldots, Tv_n)$ is linearly independent in W.

6. Prove that if $S_1, \ldots, S_n$ are injective linear maps such that $S_1 \ldots S_n$ makes sense, then $S_1 \ldots S_n$ is injective.

7. Prove that if $(v_1, \ldots, v_n)$ spans V and $T \in \mathcal{L}(V, W)$ is surjective, then $(Tv_1, \ldots, Tv_n)$ spans W.

8. Suppose that V is finite dimensional and that $T \in \mathcal{L}(V, W)$. Prove that there exists a subspace U of V such that $U \cap \operatorname{null} T = \{0\}$ and $\operatorname{range} T = \{Tu : u \in U\}$.

9. Prove that if T is a linear map from $\mathbf{F}^4$ to $\mathbf{F}^2$ such that
$$\operatorname{null} T = \{(x_1, x_2, x_3, x_4) \in \mathbf{F}^4 : x_1 = 5x_2 \text{ and } x_3 = 7x_4\},$$
then T is surjective.

10. Prove that there does not exist a linear map from $\mathbf{F}^5$ to $\mathbf{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbf{F}^5 : x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

11. Prove that if there exists a linear map on V whose null space and range are both finite dimensional, then V is finite dimensional.

12. Suppose that V and W are both finite dimensional. Prove that there exists a surjective linear map from V onto W if and only if $\dim W \leq \dim V$.

13. Suppose that V and W are finite dimensional and that U is a subspace of V. Prove that there exists $T \in \mathcal{L}(V, W)$ such that $\operatorname{null} T = U$ if and only if $\dim U \geq \dim V - \dim W$.

14. Suppose that W is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is injective if and only if there exists $S \in \mathcal{L}(W, V)$ such that ST is the identity map on V.

15. Suppose that V is finite dimensional and $T \in \mathcal{L}(V, W)$. Prove that T is surjective if and only if there exists $S \in \mathcal{L}(W, V)$ such that TS is the identity map on W.

16. Suppose that U and V are finite-dimensional vector spaces and that $S \in \mathcal{L}(V, W)$, $T \in \mathcal{L}(U, V)$. Prove that

$$\dim \operatorname{null} ST \leq \dim \operatorname{null} S + \dim \operatorname{null} T.$$

17. Prove that the distributive property holds for matrix addition and matrix multiplication. In other words, suppose A, B, and C are matrices whose sizes are such that $A(B + C)$ makes sense. Prove that $AB + AC$ makes sense and that $A(B + C) = AB + AC$.

18. Prove that matrix multiplication is associative. In other words, suppose A, B, and C are matrices whose sizes are such that $(AB)C$ makes sense. Prove that $A(BC)$ makes sense and that $(AB)C = A(BC)$.

19. Suppose $T \in \mathcal{L}(\mathbf{F}^n, \mathbf{F}^m)$ and that

This exercise shows that T has the form promised on page 39.

$$\mathcal{M}(T) = \begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{m,1} & \dots & a_{m,n} \end{bmatrix},$$

where we are using the standard bases. Prove that

$$T(x_1, \dots, x_n) = (a_{1,1}x_1 + \cdots + a_{1,n}x_n, \dots, a_{m,1}x_1 + \cdots + a_{m,n}x_n)$$

for every $(x_1, \dots, x_n) \in \mathbf{F}^n$.

20. Suppose $(v_1, \dots, v_n)$ is a basis of V. Prove that the function $T\colon V \to \mathrm{Mat}(n, 1, \mathbf{F})$ defined by

$$Tv = \mathcal{M}(v)$$

is an invertible linear map of V onto $\mathrm{Mat}(n, 1, \mathbf{F})$; here $\mathcal{M}(v)$ is the matrix of $v \in V$ with respect to the basis $(v_1, \dots, v_n)$.

21. Prove that every linear map from $\mathrm{Mat}(n, 1, \mathbf{F})$ to $\mathrm{Mat}(m, 1, \mathbf{F})$ is given by a matrix multiplication. In other words, prove that if $T \in \mathcal{L}(\mathrm{Mat}(n, 1, \mathbf{F}), \mathrm{Mat}(m, 1, \mathbf{F}))$, then there exists an m-by-n matrix A such that $TB = AB$ for every $B \in \mathrm{Mat}(n, 1, \mathbf{F})$.

22. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that ST is invertible if and only if both S and T are invertible.

23. Suppose that V is finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

24. Suppose that V is finite dimensional and $T \in \mathcal{L}(V)$. Prove that T is a scalar multiple of the identity if and only if $ST = TS$ for every $S \in \mathcal{L}(V)$.

25. Prove that if V is finite dimensional with $\dim V > 1$, then the set of noninvertible operators on V is not a subspace of $\mathcal{L}(V)$.

26. Suppose n is a positive integer and $a_{i,j} \in \mathbf{F}$ for $i, j = 1, \dots, n$. Prove that the following are equivalent:

    (a) The trivial solution $x_1 = \cdots = x_n = 0$ is the only solution to the homogeneous system of equations

$$\sum_{k=1}^{n} a_{1,k}x_k = 0$$

$$\vdots$$

$$\sum_{k=1}^{n} a_{n,k}x_k = 0.$$

    (b) For every $c_1, \dots, c_n \in \mathbf{F}$, there exists a solution to the system of equations

$$\sum_{k=1}^{n} a_{1,k}x_k = c_1$$

$$\vdots$$

$$\sum_{k=1}^{n} a_{n,k}x_k = c_n.$$

    Note that here we have the same number of equations as variables.

Exercises

1. Suppose m and n are positive integers with $m \leq n$. Prove that there exists a polynomial $p \in \mathcal{P}_n(\mathbf{F})$ with exactly m distinct roots.

2. Suppose that $z_1, \ldots, z_{m+1}$ are distinct elements of $\mathbf{F}$ and that $w_1, \ldots, w_{m+1} \in \mathbf{F}$. Prove that there exists a unique polynomial $p \in \mathcal{P}_m(\mathbf{F})$ such that
$$p(z_j) = w_j$$
for $j = 1, \ldots, m + 1$.

3. Prove that if $p, q \in \mathcal{P}(\mathbf{F})$, with $p \neq 0$, then there exist unique polynomials $s, r \in \mathcal{P}(\mathbf{F})$ such that
$$q = sp + r$$
and $\deg r < \deg p$. In other words, add a uniqueness statement to the division algorithm (4.5).

4. Suppose $p \in \mathcal{P}(\mathbf{C})$ has degree m. Prove that p has m distinct roots if and only if p and its derivative p' have no roots in common.

5. Prove that every polynomial with odd degree and real coefficients has a real root.

Exercises

1. Suppose $T \in \mathcal{L}(V)$. Prove that if $U_1, \dots, U_m$ are subspaces of V invariant under T, then $U_1 + \cdots + U_m$ is invariant under T.

2. Suppose $T \in \mathcal{L}(V)$. Prove that the intersection of any collection of subspaces of V invariant under T is invariant under T.

3. Prove or give a counterexample: if U is a subspace of V that is invariant under every operator on V, then $U = \{0\}$ or $U = V$.

4. Suppose that $S, T \in \mathcal{L}(V)$ are such that $ST = TS$. Prove that $\operatorname{null}(T - \lambda I)$ is invariant under S for every $\lambda \in \mathbf{F}$.

5. Define $T \in \mathcal{L}(\mathbf{F}^2)$ by
$$T(w, z) = (z, w).$$
Find all eigenvalues and eigenvectors of T.

6. Define $T \in \mathcal{L}(\mathbf{F}^3)$ by
$$T(z_1, z_2, z_3) = (2z_2, 0, 5z_3).$$
Find all eigenvalues and eigenvectors of T.

7. Suppose n is a positive integer and $T \in \mathcal{L}(\mathbf{F}^n)$ is defined by
$$T(x_1, \dots, x_n) = (x_1 + \cdots + x_n, \dots, x_1 + \cdots + x_n);$$
in other words, T is the operator whose matrix (with respect to the standard basis) consists of all 1's. Find all eigenvalues and eigenvectors of T.

8. Find all eigenvalues and eigenvectors of the backward shift operator $T \in \mathcal{L}(\mathbf{F}^\infty)$ defined by
$$T(z_1, z_2, z_3, \dots) = (z_2, z_3, \dots).$$

9. Suppose $T \in \mathcal{L}(V)$ and $\dim \operatorname{range} T = k$. Prove that T has at most $k + 1$ distinct eigenvalues.

10. Suppose $T \in \mathcal{L}(V)$ is invertible and $\lambda \in \mathbf{F} \setminus \{0\}$. Prove that λ is an eigenvalue of T if and only if $\frac{1}{\lambda}$ is an eigenvalue of T^{-1}.

11. Suppose $S, T \in \mathcal{L}(V)$. Prove that ST and TS have the same eigenvalues.

12. Suppose $T \in \mathcal{L}(V)$ is such that every vector in V is an eigenvector of T. Prove that T is a scalar multiple of the identity operator.

13. Suppose $T \in \mathcal{L}(V)$ is such that every subspace of V with dimension $\dim V - 1$ is invariant under T. Prove that T is a scalar multiple of the identity operator.

14. Suppose $S, T \in \mathcal{L}(V)$ and S is invertible. Prove that if $p \in \mathcal{P}(\mathbf{F})$ is a polynomial, then

$$p(STS^{-1}) = Sp(T)S^{-1}.$$

15. Suppose $\mathbf{F} = \mathbf{C}$, $T \in \mathcal{L}(V)$, $p \in \mathcal{P}(\mathbf{C})$, and $a \in \mathbf{C}$. Prove that a is an eigenvalue of $p(T)$ if and only if $a = p(\lambda)$ for some eigenvalue λ of T.

16. Show that the result in the previous exercise does not hold if $\mathbf{C}$ is replaced with $\mathbf{R}$.

17. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that T has an invariant subspace of dimension j for each $j = 1, \dots, \dim V$.

18. Give an example of an operator whose matrix with respect to some basis contains only 0's on the diagonal, but the operator is invertible.

19. Give an example of an operator whose matrix with respect to some basis contains only nonzero numbers on the diagonal, but the operator is not invertible.

These two exercises show that 5.16 fails without the hypothesis that an upper-triangular matrix is under consideration.

20. Suppose that $T \in \mathcal{L}(V)$ has $\dim V$ distinct eigenvalues and that $S \in \mathcal{L}(V)$ has the same eigenvectors as T (not necessarily with the same eigenvalues). Prove that $ST = TS$.

21. Suppose $P \in \mathcal{L}(V)$ and $P^2 = P$. Prove that $V = \operatorname{null} P \oplus \operatorname{range} P$.

22. Suppose $V = U \oplus W$, where U and W are nonzero subspaces of V. Find all eigenvalues and eigenvectors of $P_{U,W}$.

23. Give an example of an operator $T \in \mathcal{L}(\mathbf{R}^4)$ such that T has no (real) eigenvalues.

24. Suppose V is a real vector space and $T \in \mathcal{L}(V)$ has no eigenvalues. Prove that every subspace of V invariant under T has even dimension.

Exercises

1. Prove that if x, y are nonzero vectors in $\mathbf{R}^2$, then
$$\langle x, y\rangle = \|x\|\,\|y\| \cos\theta,$$
where θ is the angle between x and y (thinking of x and y as arrows with initial point at the origin). *Hint:* draw the triangle formed by x, y, and $x - y$; then use the law of cosines.

2. Suppose $u, v \in V$. Prove that $\langle u, v\rangle = 0$ if and only if
$$\|u\| \le \|u + av\|$$
for all $a \in \mathbf{F}$.

3. Prove that
$$\Big(\sum_{j=1}^{n} a_j b_j\Big)^2 \le \Big(\sum_{j=1}^{n} j a_j{}^2\Big)\Big(\sum_{j=1}^{n} \frac{b_j{}^2}{j}\Big)$$
for all real numbers $a_1, \dots, a_n$ and $b_1, \dots, b_n$.

4. Suppose $u, v \in V$ are such that
$$\|u\| = 3, \quad \|u + v\| = 4, \quad \|u - v\| = 6.$$
What number must $\|v\|$ equal?

5. Prove or disprove: there is an inner product on $\mathbf{R}^2$ such that the associated norm is given by
$$\|(x_1, x_2)\| = |x_1| + |x_2|$$
for all $(x_1, x_2) \in \mathbf{R}^2$.

6. Prove that if V is a real inner-product space, then
$$\langle u, v\rangle = \frac{\|u + v\|^2 - \|u - v\|^2}{4}$$
for all $u, v \in V$.

7. Prove that if V is a complex inner-product space, then
$$\langle u, v\rangle = \frac{\|u + v\|^2 - \|u - v\|^2 + \|u + iv\|^2 i - \|u - iv\|^2 i}{4}$$
for all $u, v \in V$.

8. A norm on a vector space U is a function $\| \ \| : U \to [0, \infty)$ such that $\|u\| = 0$ if and only if $u = 0$, $\|\alpha u\| = |\alpha| \|u\|$ for all $\alpha \in \mathbf{F}$ and all $u \in U$, and $\|u + v\| \le \|u\| + \|v\|$ for all $u, v \in U$. Prove that a norm satisfying the parallelogram equality comes from an inner product (in other words, show that if $\| \ \|$ is a norm on U satisfying the parallelogram equality, then there is an inner product $\langle \ , \ \rangle$ on U such that $\|u\| = \langle u, u \rangle^{1/2}$ for all $u \in U$).

9. Suppose n is a positive integer. Prove that

$$\Big(\frac{1}{\sqrt{2\pi}}, \frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \dots, \frac{\sin nx}{\sqrt{\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \dots, \frac{\cos nx}{\sqrt{\pi}}\Big)$$

is an orthonormal list of vectors in $C[-\pi, \pi]$, the vector space of continuous real-valued functions on $[-\pi, \pi]$ with inner product

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

This orthonormal list is often used for modeling periodic phenomena such as tides.

10. On $\mathcal{P}_2(\mathbf{R})$, consider the inner product given by

$$\langle p, q \rangle = \int_0^1 p(x) q(x)\, dx.$$

Apply the Gram-Schmidt procedure to the basis $(1, x, x^2)$ to produce an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$.

11. What happens if the Gram-Schmidt procedure is applied to a list of vectors that is not linearly independent?

12. Suppose V is a real inner-product space and $(v_1, \dots, v_m)$ is a linearly independent list of vectors in V. Prove that there exist exactly 2^m orthonormal lists $(e_1, \dots, e_m)$ of vectors in V such that

$$\operatorname{span}(v_1, \dots, v_j) = \operatorname{span}(e_1, \dots, e_j)$$

for all $j \in \{1, \dots, m\}$.

13. Suppose $(e_1, \dots, e_m)$ is an orthonormal list of vectors in V. Let $v \in V$. Prove that

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_m \rangle|^2$$

if and only if $v \in \operatorname{span}(e_1, \dots, e_m)$.

14. Find an orthonormal basis of $\mathcal{P}_2(\mathbf{R})$ (with inner product as in Exercise 10) such that the differentiation operator (the operator that takes p to p') on $\mathcal{P}_2(\mathbf{R})$ has an upper-triangular matrix with respect to this basis.

15. Suppose U is a subspace of V. Prove that
$$\dim U^{\perp} = \dim V - \dim U.$$

16. Suppose U is a subspace of V. Prove that $U^{\perp} = \{0\}$ if and only if $U = V$.

17. Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and every vector in $\operatorname{null} P$ is orthogonal to every vector in $\operatorname{range} P$, then P is an orthogonal projection.

18. Prove that if $P \in \mathcal{L}(V)$ is such that $P^2 = P$ and
$$\|Pv\| \le \|v\|$$
for every $v \in V$, then P is an orthogonal projection.

19. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $P_U T P_U = T P_U$.

20. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U and $U^{\perp}$ are both invariant under T if and only if $P_U T = T P_U$.

21. In $\mathbf{R}^4$, let
$$U = \operatorname{span}\big((1,1,0,0),(1,1,1,2)\big).$$
Find $u \in U$ such that $\|u - (1,2,3,4)\|$ is as small as possible.

22. Find $p \in \mathcal{P}_3(\mathbf{R})$ such that $p(0) = 0$, $p'(0) = 0$, and
$$\int_0^1 |2 + 3x - p(x)|^2\,dx$$
is as small as possible.

23. Find $p \in \mathcal{P}_5(\mathbf{R})$ that makes
$$\int_{-\pi}^{\pi} |\sin x - p(x)|^2\,dx$$
as small as possible. (The polynomial 6.40 is an excellent approximation to the answer to this exercise, but here you are asked to find the exact solution, which involves powers of π. A computer that can perform symbolic integration will be useful.)

24. Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$p(\frac{1}{2}) = \int_0^1 p(x)q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

25. Find a polynomial $q \in \mathcal{P}_2(\mathbf{R})$ such that

$$\int_0^1 p(x)(\cos \pi x)\,dx = \int_0^1 p(x)q(x)\,dx$$

for every $p \in \mathcal{P}_2(\mathbf{R})$.

26. Fix a vector $v \in V$ and define $T \in \mathcal{L}(V, \mathbf{F})$ by $Tu = \langle u, v \rangle$. For $a \in \mathbf{F}$, find a formula for T^*a.

27. Suppose n is a positive integer. Define $T \in \mathcal{L}(\mathbf{F}^n)$ by

$$T(z_1, \dots, z_n) = (0, z_1, \dots, z_{n-1}).$$

Find a formula for $T^*(z_1, \dots, z_n)$.

28. Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbf{F}$. Prove that λ is an eigenvalue of T if and only if $\bar{\lambda}$ is an eigenvalue of T^*.

29. Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V. Prove that U is invariant under T if and only if $U^\perp$ is invariant under T^*.

30. Suppose $T \in \mathcal{L}(V, W)$. Prove that

   (a) T is injective if and only if T^* is surjective;

   (b) T is surjective if and only if T^* is injective.

31. Prove that

$$\dim \operatorname{null} T^* = \dim \operatorname{null} T + \dim W - \dim V$$

and

$$\dim \operatorname{range} T^* = \dim \operatorname{range} T$$

for every $T \in \mathcal{L}(V, W)$.

32. Suppose A is an m-by-n matrix of real numbers. Prove that the dimension of the span of the columns of A (in $\mathbf{R}^m$) equals the dimension of the span of the rows of A (in $\mathbf{R}^n$).

Exercises

1. Make $\mathcal{P}_2(\mathbf{R})$ into an inner-product space by defining

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,dx.$$

Define $T \in \mathcal{L}(\mathcal{P}_2(\mathbf{R}))$ by $T(a_0 + a_1x + a_2x^2) = a_1x$.

(a) Show that T is not self-adjoint.

(b) The matrix of T with respect to the basis $(1, x, x^2)$ is

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix equals its conjugate transpose, even though T is not self-adjoint. Explain why this is not a contradiction.

2. Prove or give a counterexample: the product of any two self-adjoint operators on a finite-dimensional inner-product space is self-adjoint.

3. (a) Show that if V is a real inner-product space, then the set of self-adjoint operators on V is a subspace of $\mathcal{L}(V)$.

(b) Show that if V is a complex inner-product space, then the set of self-adjoint operators on V is not a subspace of $\mathcal{L}(V)$.

4. Suppose $P \in \mathcal{L}(V)$ is such that $P^2 = P$. Prove that P is an orthogonal projection if and only if P is self-adjoint.

5. Show that if $\dim V \geq 2$, then the set of normal operators on V is not a subspace of $\mathcal{L}(V)$.

6. Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\operatorname{range} T = \operatorname{range} T^*.$$

7. Prove that if $T \in \mathcal{L}(V)$ is normal, then

$$\operatorname{null} T^k = \operatorname{null} T \quad \text{and} \quad \operatorname{range} T^k = \operatorname{range} T$$

for every positive integer k.

8. Prove that there does not exist a self-adjoint operator $T \in \mathcal{L}(\mathbf{R}^3)$ such that $T(1,2,3) = (0,0,0)$ and $T(2,5,7) = (2,5,7)$.

9. Prove that a normal operator on a complex inner-product space is self-adjoint if and only if all its eigenvalues are real.

Exercise 9 strengthens the analogy (for normal operators) between self-adjoint operators and real numbers.

10. Suppose V is a complex inner-product space and $T \in \mathcal{L}(V)$ is a normal operator such that $T^9 = T^8$. Prove that T is self-adjoint and $T^2 = T$.

11. Suppose V is a complex inner-product space. Prove that every normal operator on V has a square root. (An operator $S \in \mathcal{L}(V)$ is called a ***square root*** of $T \in \mathcal{L}(V)$ if $S^2 = T$.)

12. Give an example of a real inner-product space V and $T \in \mathcal{L}(V)$ and real numbers α, β with $\alpha^2 < 4\beta$ such that $T^2 + \alpha T + \beta I$ is not invertible.

This exercise shows that the hypothesis that T is self-adjoint is needed in 7.11, even for real vector spaces.

13. Prove or give a counterexample: every self-adjoint operator on V has a cube root. (An operator $S \in \mathcal{L}(V)$ is called a ***cube root*** of $T \in \mathcal{L}(V)$ if $S^3 = T$.)

14. Suppose $T \in \mathcal{L}(V)$ is self-adjoint, $\lambda \in \mathbf{F}$, and $\epsilon > 0$. Prove that if there exists $v \in V$ such that $\|v\| = 1$ and

$$\|Tv - \lambda v\| < \epsilon,$$

then T has an eigenvalue λ' such that $|\lambda - \lambda'| < \epsilon$.

15. Suppose U is a finite-dimensional real vector space and $T \in \mathcal{L}(U)$. Prove that U has a basis consisting of eigenvectors of T if and only if there is an inner product on U that makes T into a self-adjoint operator.

16. Give an example of an operator T on an inner product space such that T has an invariant subspace whose orthogonal complement is not invariant under T.

This exercise shows that 7.18 can fail without the hypothesis that T is normal.

17. Prove that the sum of any two positive operators on V is positive.

18. Prove that if $T \in \mathcal{L}(V)$ is positive, then so is T^k for every positive integer k.

19. Suppose that T is a positive operator on V. Prove that T is invertible if and only if
$$\langle Tv, v\rangle > 0$$
for every $v \in V \setminus \{0\}$.

20. Prove or disprove: the identity operator on $\mathbf{F}^2$ has infinitely many self-adjoint square roots.

21. Prove or give a counterexample: if $S \in \mathcal{L}(V)$ and there exists an orthonormal basis $(e_1, \dots, e_n)$ of V such that $\|Se_j\| = 1$ for each e_j, then S is an isometry.

22. Prove that if $S \in \mathcal{L}(\mathbf{R}^3)$ is an isometry, then there exists a nonzero vector $x \in \mathbf{R}^3$ such that $S^2x = x$.

23. Define $T \in \mathcal{L}(\mathbf{F}^3)$ by
$$T(z_1, z_2, z_3) = (z_3, 2z_1, 3z_2).$$
Find (explicitly) an isometry $S \in \mathcal{L}(\mathbf{F}^3)$ such that $T = S\sqrt{T^*T}$.

Exercise 24 shows that if we write T as the product of an isometry and a positive operator (as in the polar decomposition), then the positive operator must equal $\sqrt{T^*T}$.

24. Suppose $T \in \mathcal{L}(V)$, $S \in \mathcal{L}(V)$ is an isometry, and $R \in \mathcal{L}(V)$ is a positive operator such that $T = SR$. Prove that $R = \sqrt{T^*T}$.

25. Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if there exists a unique isometry $S \in \mathcal{L}(V)$ such that $T = S\sqrt{T^*T}$.

26. Prove that if $T \in \mathcal{L}(V)$ is self-adjoint, then the singular values of T equal the absolute values of the eigenvalues of T (repeated appropriately).

27. Prove or give a counterexample: if $T \in \mathcal{L}(V)$, then the singular values of T^2 equal the squares of the singular values of T.

28. Suppose $T \in \mathcal{L}(V)$. Prove that T is invertible if and only if 0 is not a singular value of T.

29. Suppose $T \in \mathcal{L}(V)$. Prove that $\dim \operatorname{range} T$ equals the number of nonzero singular values of T.

30. Suppose $S \in \mathcal{L}(V)$. Prove that S is an isometry if and only if all the singular values of S equal 1.

31. Suppose $T_1, T_2 \in \mathcal{L}(V)$. Prove that T_1 and T_2 have the same singular values if and only if there exist isometries $S_1, S_2 \in \mathcal{L}(V)$ such that $T_1 = S_1 T_2 S_2$.

32. Suppose $T \in \mathcal{L}(V)$ has singular-value decomposition given by

$$Tv = s_1\langle v, e_1\rangle f_1 + \cdots + s_n\langle v, e_n\rangle f_n$$

for every $v \in V$, where $s_1, \ldots, s_n$ are the singular values of T and $(e_1, \ldots, e_n)$ and $(f_1, \ldots, f_n)$ are orthonormal bases of V.

(a) Prove that

$$T^*v = s_1\langle v, f_1\rangle e_1 + \cdots + s_n\langle v, f_n\rangle e_n$$

for every $v \in V$.

(b) Prove that if T is invertible, then

$$T^{-1}v = \frac{\langle v, f_1\rangle e_1}{s_1} + \cdots + \frac{\langle v, f_n\rangle e_n}{s_n}$$

for every $v \in V$.

33. Suppose $T \in \mathcal{L}(V)$. Let $\hat{s}$ denote the smallest singular value of T, and let s denote the largest singular value of T. Prove that

$$\hat{s}\|v\| \leq \|Tv\| \leq s\|v\|$$

for every $v \in V$.

34. Suppose $T', T'' \in \mathcal{L}(V)$. Let s' denote the largest singular value of T', let s'' denote the largest singular value of T'', and let s denote the largest singular value of $T' + T''$. Prove that $s \leq s' + s''$.

Exercises

1. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by
$$T(w, z) = (z, 0).$$
Find all generalized eigenvectors of T.

2. Define $T \in \mathcal{L}(\mathbf{C}^2)$ by
$$T(w, z) = (-z, w).$$
Find all generalized eigenvectors of T.

3. Suppose $T \in \mathcal{L}(V)$, m is a positive integer, and $v \in V$ is such that $T^{m-1}v \neq 0$ but $T^m v = 0$. Prove that
$$(v, Tv, T^2v, \ldots, T^{m-1}v)$$
is linearly independent.

4. Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Prove that T has no square root. More precisely, prove that there does not exist $S \in \mathcal{L}(\mathbf{C}^3)$ such that $S^2 = T$.

5. Suppose $S, T \in \mathcal{L}(V)$. Prove that if ST is nilpotent, then TS is nilpotent.

6. Suppose $N \in \mathcal{L}(V)$ is nilpotent. Prove (without using 8.26) that 0 is the only eigenvalue of N.

7. Suppose V is an inner-product space. Prove that if $N \in \mathcal{L}(V)$ is self-adjoint and nilpotent, then $N = 0$.

8. Suppose $N \in \mathcal{L}(V)$ is such that $\operatorname{null} N^{\dim V - 1} \neq \operatorname{null} N^{\dim V}$. Prove that N is nilpotent and that
$$\dim \operatorname{null} N^j = j$$
for every integer j with $0 \leq j \leq \dim V$.

9. Suppose $T \in \mathcal{L}(V)$ and m is a nonnegative integer such that
$$\operatorname{range} T^m = \operatorname{range} T^{m+1}.$$
Prove that $\operatorname{range} T^k = \operatorname{range} T^m$ for all $k > m$.

10. Prove or give a counterexample: if $T \in \mathcal{L}(V)$, then

$$V = \operatorname{null} T \oplus \operatorname{range} T.$$

11. Prove that if $T \in \mathcal{L}(V)$, then

$$V = \operatorname{null} T^n \oplus \operatorname{range} T^n,$$

where $n = \dim V$.

12. Suppose V is a complex vector space, $N \in \mathcal{L}(V)$, and 0 is the only eigenvalue of N. Prove that N is nilpotent. Give an example to show that this is not necessarily true on a real vector space.

13. Suppose that V is a complex vector space with $\dim V = n$ and $T \in \mathcal{L}(V)$ is such that

$$\operatorname{null} T^{n-2} \neq \operatorname{null} T^{n-1}.$$

Prove that T has at most two distinct eigenvalues.

14. Give an example of an operator on $\mathbf{C}^4$ whose characteristic polynomial equals $(z-7)^2(z-8)^2$.

15. Suppose V is a complex vector space. Suppose $T \in \mathcal{L}(V)$ is such that 5 and 6 are eigenvalues of T and that T has no other eigenvalues. Prove that

$$(T - 5I)^{n-1}(T - 6I)^{n-1} = 0,$$

where $n = \dim V$.

16. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if every generalized eigenvector of T is an eigenvector of T.

For complex vector spaces, this exercise adds another equivalence to the list given by 5.21.

17. Suppose V is an inner-product space and $N \in \mathcal{L}(V)$ is nilpotent. Prove that there exists an orthonormal basis of V with respect to which N has an upper-triangular matrix.

18. Define $N \in \mathcal{L}(\mathbf{F}^5)$ by

$$N(x_1, x_2, x_3, x_4, x_5) = (2x_2, 3x_3, -x_4, 4x_5, 0).$$

Find a square root of $I + N$.

19. Prove that if V is a complex vector space, then every invertible operator on V has a cube root.

20. Suppose $T \in \mathcal{L}(V)$ is invertible. Prove that there exists a polynomial $p \in \mathcal{P}(\mathbf{F})$ such that $T^{-1} = p(T)$.

21. Give an example of an operator on $\mathbf{C}^3$ whose minimal polynomial equals z^2.

22. Give an example of an operator on $\mathbf{C}^4$ whose minimal polynomial equals $z(z-1)^2$.

For complex vector spaces, this exercise adds another equivalence to the list given by 5.21.

23. Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Prove that V has a basis consisting of eigenvectors of T if and only if the minimal polynomial of T has no repeated roots.

24. Suppose V is an inner-product space. Prove that if $T \in \mathcal{L}(V)$ is normal, then the minimal polynomial of T has no repeated roots.

25. Suppose $T \in \mathcal{L}(V)$ and $v \in V$. Let p be the monic polynomial of smallest degree such that

$$p(T)v = 0.$$

Prove that p divides the minimal polynomial of T.

26. Give an example of an operator on $\mathbf{C}^4$ whose characteristic and minimal polynomials both equal $z(z-1)^2(z-3)$.

27. Give an example of an operator on $\mathbf{C}^4$ whose characteristic polynomial equals $z(z-1)^2(z-3)$ and whose minimal polynomial equals $z(z-1)(z-3)$.

This exercise shows that every monic polynomial is the characteristic polynomial of some operator.

28. Suppose $a_0, \dots, a_{n-1} \in \mathbf{C}$. Find the minimal and characteristic polynomials of the operator on $\mathbf{C}^n$ whose matrix (with respect to the standard basis) is

$$\begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & \ddots & & & -a_2 \\ & & \ddots & & & \vdots \\ & & & & 0 & -a_{n-2} \\ & & & & 1 & -a_{n-1} \end{bmatrix}.$$

This equation, along with 9.23, shows that $\dim V = \dim U$. Because U is a subspace of V, this implies that $V = U$. In other words,

$$V = U_1 + \cdots + U_m + V_1 + \cdots + V_M.$$

This equation, along with 9.23, allows us to use 2.19 to conclude that (a) holds, completing the proof. ■

Exercises

1. Prove that 1 is an eigenvalue of every square matrix with the property that the sum of the entries in each row equals 1.

2. Consider a 2-by-2 matrix of real numbers
$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$
Prove that A has an eigenvalue (in $\mathbf{R}$) if and only if
$$(a-d)^2 + 4bc \geq 0.$$

3. Suppose A is a block diagonal matrix
$$A = \begin{bmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{bmatrix},$$
where each A_j is a square matrix. Prove that the set of eigenvalues of A equals the union of the eigenvalues of $A_1, \dots, A_m$.

Clearly Exercise 4 is a stronger statement than Exercise 3. Even so, you may want to do Exercise 3 first because it is easier than Exercise 4.

4. Suppose A is a block upper-triangular matrix
$$A = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{bmatrix},$$
where each A_j is a square matrix. Prove that the set of eigenvalues of A equals the union of the eigenvalues of $A_1, \dots, A_m$.

5. Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Suppose $\alpha, \beta \in \mathbf{R}$ are such that $T^2 + \alpha T + \beta I = 0$. Prove that T has an eigenvalue if and only if $\alpha^2 \geq 4\beta$.

6. Suppose V is a real inner-product space and $T \in \mathcal{L}(V)$. Prove that there is an orthonormal basis of V with respect to which T has a block upper-triangular matrix
$$\begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{bmatrix},$$
where each A_j is a 1-by-1 matrix or a 2-by-2 matrix with no eigenvalues.

7. Prove that if $T \in \mathcal{L}(V)$ and j is a positive integer such that $j \leq \dim V$, then T has an invariant subspace whose dimension equals $j - 1$ or j.

8. Prove that there does not exist an operator $T \in \mathcal{L}(\mathbf{R}^7)$ such that $T^2 + T + I$ is nilpotent.

9. Give an example of an operator $T \in \mathcal{L}(\mathbf{C}^7)$ such that $T^2 + T + I$ is nilpotent.

10. Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Suppose $\alpha, \beta \in \mathbf{R}$ are such that $\alpha^2 < 4\beta$. Prove that

$$\operatorname{null}(T^2 + \alpha T + \beta I)^k$$

has even dimension for every positive integer k.

11. Suppose V is a real vector space and $T \in \mathcal{L}(V)$. Suppose $\alpha, \beta \in \mathbf{R}$ are such that $\alpha^2 < 4\beta$ and $T^2 + \alpha T + \beta I$ is nilpotent. Prove that $\dim V$ is even and

$$(T^2 + \alpha T + \beta I)^{\dim V/2} = 0.$$

12. Prove that if $T \in \mathcal{L}(\mathbf{R}^3)$ and $5, 7$ are eigenvalues of T, then T has no eigenpairs.

13. Suppose V is a real vector space with $\dim V = n$ and $T \in \mathcal{L}(V)$ is such that

$$\operatorname{null} T^{n-2} \neq \operatorname{null} T^{n-1}.$$

Prove that T has at most two distinct eigenvalues and that T has no eigenpairs.

14. Suppose V is a vector space with dimension 2 and $T \in \mathcal{L}(V)$. Prove that if

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

is the matrix of T with respect to some basis of V, then the characteristic polynomial of T equals $(z - a)(z - d) - bc$.

You do not need to find the eigenvalues of T to do this exercise. As usual unless otherwise specified, here V may be a real or complex vector space.

15. Suppose V is a real inner-product space and $S \in \mathcal{L}(V)$ is an isometry. Prove that if (α, β) is an eigenpair of S, then $\beta = 1$.

Exercises

1. Suppose $T \in \mathcal{L}(V)$ and $(v_1, \dots, v_n)$ is a basis of V. Prove that $\mathcal{M}(T, (v_1, \dots, v_n))$ is invertible if and only if T is invertible.

2. Prove that if A and B are square matrices of the same size and $AB = I$, then $BA = I$.

3. Suppose $T \in \mathcal{L}(V)$ has the same matrix with respect to every basis of V. Prove that T is a scalar multiple of the identity operator.

4. Suppose that $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ are bases of V. Let $T \in \mathcal{L}(V)$ be the operator such that $Tv_k = u_k$ for $k = 1, \dots, n$. Prove that
$$\mathcal{M}(T, (v_1, \dots, v_n)) = \mathcal{M}(I, (u_1, \dots, u_n), (v_1, \dots, v_n)).$$

5. Prove that if B is a square matrix with complex entries, then there exists an invertible square matrix A with complex entries such that $A^{-1}BA$ is an upper-triangular matrix.

6. Give an example of a real vector space V and $T \in \mathcal{L}(V)$ such that $\operatorname{trace}(T^2) < 0$.

7. Suppose V is a real vector space, $T \in \mathcal{L}(V)$, and V has a basis consisting of eigenvectors of T. Prove that $\operatorname{trace}(T^2) \geq 0$.

8. Suppose V is an inner-product space and $v, w \in \mathcal{L}(V)$. Define $T \in \mathcal{L}(V)$ by $Tu = \langle u, v \rangle w$. Find a formula for $\operatorname{trace} T$.

9. Prove that if $P \in \mathcal{L}(V)$ satisfies $P^2 = P$, then $\operatorname{trace} P$ is a nonnegative integer.

10. Prove that if V is an inner-product space and $T \in \mathcal{L}(V)$, then
$$\operatorname{trace} T^* = \overline{\operatorname{trace} T}.$$

11. Suppose V is an inner-product space. Prove that if $T \in \mathcal{L}(V)$ is a positive operator and $\operatorname{trace} T = 0$, then $T = 0$.

12. Suppose $T \in \mathcal{L}(\mathbf{C}^3)$ is the operator whose matrix is

$$\begin{bmatrix} 51 & -12 & -21 \\ 60 & -40 & -28 \\ 57 & -68 & 1 \end{bmatrix}.$$

Someone tells you (accurately) that -48 and 24 are eigenvalues of T. Without using a computer or writing anything down, find the third eigenvalue of T.

13. Prove or give a counterexample: if $T \in \mathcal{L}(V)$ and $c \in \mathbf{F}$, then $\operatorname{trace}(cT) = c \operatorname{trace} T$.

14. Prove or give a counterexample: if $S, T \in \mathcal{L}(V)$, then $\operatorname{trace}(ST) = (\operatorname{trace} S)(\operatorname{trace} T)$.

15. Suppose $T \in \mathcal{L}(V)$. Prove that if $\operatorname{trace}(ST) = 0$ for all $S \in \mathcal{L}(V)$, then $T = 0$.

16. Suppose V is an inner-product space and $T \in \mathcal{L}(V)$. Prove that if $(e_1, \dots, e_n)$ is an orthonormal basis of V, then

$$\operatorname{trace}(T^*T) = \|Te_1\|^2 + \cdots + \|Te_n\|^2.$$

Conclude that the right side of the equation above is independent of which orthonormal basis $(e_1, \dots, e_n)$ is chosen for V.

17. Suppose V is a complex inner-product space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of T, repeated according to multiplicity. Suppose

$$\begin{bmatrix} a_{1,1} & \dots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \dots & a_{n,n} \end{bmatrix}$$

is the matrix of T with respect to some orthonormal basis of V. Prove that

$$|\lambda_1|^2 + \cdots + |\lambda_n|^2 \le \sum_{k=1}^{n} \sum_{j=1}^{n} |a_{j,k}|^2.$$

18. Suppose V is an inner-product space. Prove that

$$\langle S, T \rangle = \operatorname{trace}(ST^*)$$

defines an inner product on $\mathcal{L}(V)$.

Exercise 19 fails on infinite-dimensional inner-product spaces, leading to what are called hyponormal operators, which have a well-developed theory.

19. Suppose V is an inner-product space and $T \in \mathcal{L}(V)$. Prove that if
$$\|T^* v\| \leq \|Tv\|$$
for every $v \in V$, then T is normal.

20. Prove or give a counterexample: if $T \in \mathcal{L}(V)$ and $c \in \mathbf{F}$, then $\det(cT) = c^{\dim V} \det T$.

21. Prove or give a counterexample: if $S, T \in \mathcal{L}(V)$, then $\det(S+T) = \det S + \det T$.

22. Suppose A is a block upper-triangular matrix
$$A = \begin{bmatrix} A_1 & & * \\ & \ddots & \\ 0 & & A_m \end{bmatrix},$$
where each A_j along the diagonal is a square matrix. Prove that
$$\det A = (\det A_1) \dots (\det A_m).$$

23. Suppose A is an n-by-n matrix with real entries. Let $S \in \mathcal{L}(\mathbf{C}^n)$ denote the operator on $\mathbf{C}^n$ whose matrix equals A, and let $T \in \mathcal{L}(\mathbf{R}^n)$ denote the operator on $\mathbf{R}^n$ whose matrix equals A. Prove that $\operatorname{trace} S = \operatorname{trace} T$ and $\det S = \det T$.

24. Suppose V is an inner-product space and $T \in \mathcal{L}(V)$. Prove that
$$\det T^* = \overline{\det T}.$$
Use this to prove that $|\det T| = \det \sqrt{T^* T}$, giving a different proof than was given in 10.37.

25. Let a, b, c be positive numbers. Find the volume of the ellipsoid
$$\left\{(x, y, z) \in \mathbf{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} < 1\right\}$$
by finding a set $\Omega \subset \mathbf{R}^3$ whose volume you know and an operator $T \in \mathcal{L}(\mathbf{R}^3)$ such that $T(\Omega)$ equals the ellipsoid above.