1.) x_n KONV. $\Rightarrow$ JE CAUCHY

D.: $\varepsilon > 0$ LIB., MÁME $x_n \xrightarrow{\varrho} x$ $\Rightarrow$ K $\frac{\varepsilon}{2}$ $\exists n_0 \in \mathbb{N}$: $\forall n > n_0$ PLATÍ $\varrho(x_n, x) < \frac{\varepsilon}{2}$.

NECHŤ JSOU $m, n \in \mathbb{N}$, $m, n > n_0$, JINAK LIBOVOLNÉ!

PAK $\varrho(x_m, x_n) \leq \varrho(x_m, x) + \varrho(x_n, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \varrho(x_m, x_n) < \varepsilon$ ($\exists n_0$ TAK, ŽE OD NĚJ DÁL TO PLATÍ), TJ. POSL. JE CAUCHY.

2.) $\ln\sqrt{x^2+y^2} - \operatorname{arctg}\frac{y}{x} = 0$, $D(F) = \{[x,y] \in \mathbb{R}^2 : x^2+y^2 \neq 0, x \neq 0\} = \{[x,y] \in \mathbb{R}^2 : x \neq 0\}$

$F(x,y) := \ln\sqrt{x^2+y^2} - \operatorname{arctg}\frac{y}{x}$

$$F_y = \frac{1}{\sqrt{x^2+y^2}} \cdot \frac{2y}{2\sqrt{x^2+y^2}} - \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{y}{x^2+y^2} - \frac{x^{\cancel{2}}}{(x^2+y^2)\cdot\cancel{x}} =$$

$$= \frac{y-x}{x^2+y^2} = 0 \Leftrightarrow x = y$$

tj. IMPL. ZAD. FCE $y = f(x) \Leftrightarrow y \neq x \wedge x \neq 0$

$$F_x = \frac{x}{x^2+y^2} + \frac{x^2}{x^2+y^2} \cdot \frac{y}{x^2} = \frac{x+y}{x^2+y^2}$$

$$\Rightarrow f'(x) = -\frac{F_x}{F_y} = -\frac{\frac{x+y}{x^2+y^2}}{\frac{y-x}{x^2+y^2}} = \frac{x+y}{x-y} = \frac{x+f(x)}{x-f(x)}$$

3.) $\dfrac{\partial f}{\partial v}(x,y) = f'(x,y,v) = \lim\limits_{h \to 0} \dfrac{f(x+ah,\, y+bh) - f(x,y)}{h}$

$f(x,y) = x^2 y^3 \ln x \Rightarrow \left.\begin{aligned} f_x &= y^3 \cdot (2x \cdot \ln x + x) \\ f_y &= 3y^2 x^2 \ln x \end{aligned}\right\}$ SP. V $[e,-2]$ $\Rightarrow$ f JE DIFEREN.

$\left.\begin{aligned} f_x(e,-2) &= -8 \cdot (2e+e) = -24e \\ f_y(e,-2) &= 3 \cdot 4 \cdot e^2 = 12e^2 \end{aligned}\right\} \Rightarrow \nabla f(e,-2) = \begin{pmatrix} -24e \\ 12e^2 \end{pmatrix}$

$\dfrac{\partial f}{\partial v}(e,-2) = \left\langle \begin{pmatrix} -24e \\ 12e^2 \end{pmatrix}, \begin{pmatrix} 3e \\ 5 \end{pmatrix} \right\rangle = -72e^2 + 60e^2 = \underline{\underline{-12e^2}}$

5.) Věta 43 (Stokesova věta)

Nechť plochu S, která je omezená uzavřenou prostorovou křivkou C tvořící kraj S, lze rozložit na konečný počet částí, které jsou grafy funkcí proměnných x, y, totéž pro x, z a y, z. Nechť funkce $P, Q, R\colon S \to \mathbb{R}$ jsou na S spojité spolu se svými parciálními derivacemi prvního řádu. Dále nechť křivka C je orientována souhlasně s S. Pak platí

$$\oint_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$
$$= \iint_S (R_y - Q_z)\,dy\,dz + (P_z - R_x)\,dx\,dz + (Q_x - P_y)\,dx\,dy.$$

8.) $3x^2y\,dx + (x^3+1)\,dy \qquad (3x^2y)'_y = 3x^2 = (x^3+1)'_x$

$\int 3x^2y\,dx = x^3y + c(y) \quad / \frac{\partial}{\partial y} \Rightarrow \quad x^3 + c'(y) = x^3+1$

$c'(y) = 1$

$c(y) = y + c$

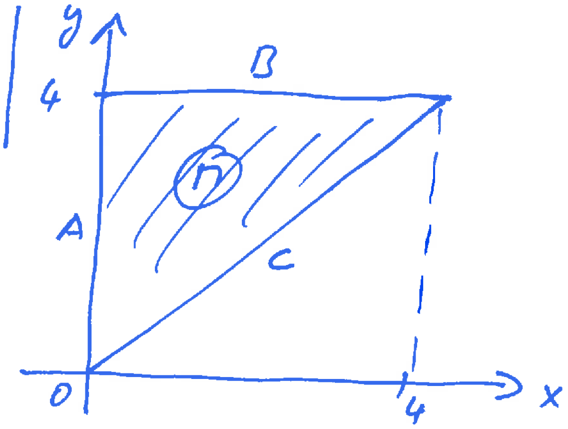
$\Rightarrow$ KMEN. FCE $\varphi(x,y) = x^3y + y + c$

NEZÁV. NA INT. CESTĚ

a), b), c): $\int_C 3x^2y\,dx + (x^3+1)\,dy = [x^3y + y]_{[0,0]}^{[1,1]} = 2 - 0 = \underline{\underline{2}}$

4) $f(x,y) = x^2 - 3y^2 - x + 18y + 4$, $0 \le x \le y \le 4 \ldots M$

→ MNA KOMPAKTNÍ, f SPOJ. ⇒ EXTR. EXIST.



→ STAC. BODY:

$f_x = 2x - 1 = 0$

$f_y = -6y + 18 = 0$

$\Rightarrow [\tfrac{1}{2}, 3] \in M$

→ HRANICE

(A) $x = 0 \Rightarrow g(y) = -3y^2 + 18y + 4$, $g'(y) = -6y + 18 = 0 \Leftrightarrow y = 3$

$\Rightarrow$ BOD $[0,3] \in M$

(B) $y = 4 \Rightarrow g(x) = x^2 - x + 28$, $g'(x) = 2x - 1 = 0 \Leftrightarrow x = \tfrac{1}{2}$

$\Rightarrow$ BOD $[\tfrac{1}{2}, 4] \in M$

(C) $y = x \Rightarrow g(x) = -2x^2 + 17x + 4$, $g'(x) = -4x + 17 = 0 \Leftrightarrow x = \tfrac{17}{4}$

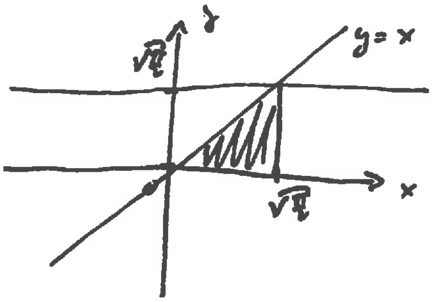
$\Rightarrow$ BOD $[\tfrac{17}{4}, \tfrac{17}{4}] \notin M$

→ „DĚLICÍ BODY" NA HRANICI ⇒ $[0,0], [0,4], [4,4]$

BOD	$[\tfrac{1}{2},3]$	$[0,3]$	$[\tfrac{1}{2},4]$	$[0,0]$	$[0,4]$	$[4,4]$
$f(x,y)$	$\tfrac{123}{4}$	31	$\tfrac{111}{4}$	4	28	40

$$\boxed{\begin{array}{l}\text{GLOB. MAX. V } [4,4] \text{ S HODN. } 40 \\ \text{GLOB. MIN. V } [0,0] \text{ S HODN. } 4\end{array}}$$

5) $$\int_0^{\sqrt{\frac{\pi}{2}}}\int_y^{\sqrt{\frac{\pi}{2}}} y^2 \sin x^2 \, dx \, dy = \int_0^{\sqrt{\frac{\pi}{2}}}\int_0^{x} y^2 \sin x^2 \, dy \, dx =$$

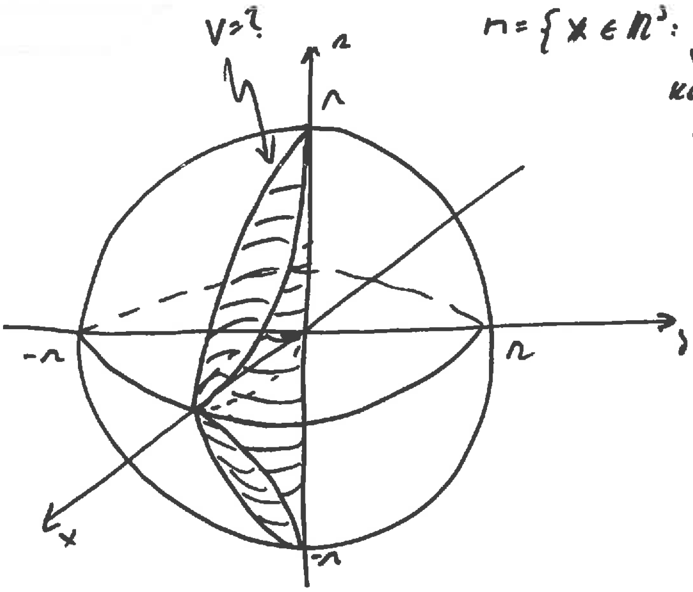


$$= \int_0^{\sqrt{\frac{\pi}{2}}} \sin x^2 \cdot \left[\frac{y^3}{3}\right]_0^x dx = \int_0^{\sqrt{\frac{\pi}{2}}} \frac{x^3}{3}\cdot \sin x^2 \, dx =$$

$$= \left| \begin{matrix} t = x^2 \\ dt = 2x\,dx \end{matrix} \right| \begin{matrix} x=\sqrt{\frac{\pi}{2}} \Rightarrow t = \frac{\pi}{2} \\ x = 0 \Rightarrow t = 0 \end{matrix} \right| = \frac{1}{3}\cdot\frac{1}{2}\cdot\int_0^{\frac{\pi}{2}} t\cdot \sin t \, dt =$$

$$= \frac{1}{6}\cdot\int_0^{\frac{\pi}{2}} t\cdot\sin t \, dt = \left| \begin{matrix} u = t & u' = 1 \\ v' = \sin t & v = -\cos t \end{matrix} \right| = \frac{1}{6}\cdot\left[-t\cdot\cos t\right]_0^{\frac{\pi}{2}} + \frac{1}{6}\cdot\int_0^{\frac{\pi}{2}} \cos t \, dt =$$

$$= \frac{1}{6}\cdot\left[-t\cdot\cos t + \sin t\right]_0^{\frac{\pi}{2}} = \frac{1}{6}\cdot 1 = \underline{\underline{\frac{1}{6}}}$$



$$M = \{ x \in \mathbb{R}^3 : \underbrace{x^2+y^2+z^2 \le r^2}_{\text{KOULE, } S=[0,0,0],\ \text{POLOM.} = r},\ \underbrace{x^2+y^2 \le rx}_{} \}$$

KOULE, $S=[0,0,0]$
POLOM. $= r$

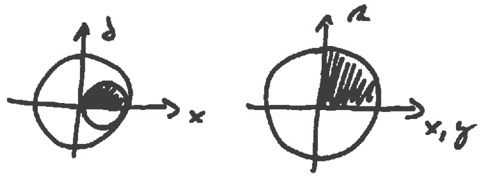
$$x^2 - rx + y^2 = \left(x-\frac{r}{2}\right)^2 + y^2 - \frac{r^2}{4} \le 0$$

$$\left(x-\frac{r}{2}\right)^2 + y^2 \le \frac{r^2}{4}$$

VÁLEC, POLOM. $\frac{r}{2}$, OSA $[x,y] = [\frac{r}{2}, 0]$

CHCEME V ⇒ STAČÍ UVAŽOVAT I. OKTANT A VZÍT $4\times$

tj.



$x = \rho \cdot \cos\varphi$
$y = \rho \cdot \sin\varphi$
$z = z$
$|J| = \rho$

$z \in [0, \sqrt{r^2-\rho^2}]$
$(z^2 \le r^2 - (x^2+y^2) = r^2 - \rho^2)$

$\varphi \in [0, \frac{\pi}{2}]$

$\rho \in [0, r \cdot \cos\varphi]$
$(x^2+y^2 \le rx$
$\rho^2 \le r\rho\cos\varphi)$

$$\frac{1}{4}V = \int_0^{\frac{\pi}{2}} \int_0^{r\cdot\cos\varphi} \int_0^{\sqrt{r^2-\rho^2}} 1 \cdot \rho \, dz\, d\rho\, d\varphi =$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{r\cdot\cos\varphi} \rho \cdot \sqrt{r^2-\rho^2} \, d\rho\, d\varphi =$$

$$= \left| \begin{array}{l} u^2 = r^2-\rho^2 \\ u\,du = -\rho\, d\rho \end{array} \right| \begin{array}{l} \rho = r\cdot\cos\varphi \Rightarrow u^2 = r^2\cdot(1-\cos^2\varphi) \\ \quad u = r\cdot\sin\varphi \\ \rho = 0 \Rightarrow u^2 = r^2,\ u = r \end{array} \Bigg| ,$$

$$\int_0^{\frac{\pi}{2}} \int_r^{r\cdot\sin\varphi} -u\cdot u \, du\, d\varphi = +\int_0^{\frac{\pi}{2}} \left[\frac{u^3}{3}\right]_{r\cdot\sin\varphi}^{r} d\varphi = \frac{1}{3}\cdot\int_0^{\frac{\pi}{2}} r^3 - r^3\cdot\sin^3\varphi \, d\varphi =$$

$$\frac{1}{3}\int_0^{\frac{\pi}{2}} r^3\, d\varphi - r^3\frac{1}{3}\int_0^{\frac{\pi}{2}} \sin\varphi\cdot(1-\cos^2\varphi)\, d\varphi = \left| \begin{array}{l} t = \cos\varphi \\ dt = -\sin\varphi\, d\varphi \end{array} \right| \begin{array}{l} \varphi = 0 \Rightarrow t = 1 \\ \varphi = \frac{\pi}{2} \Rightarrow t = 0 \end{array} \Bigg| =$$

$$\frac{1}{3}\cdot r^3 \cdot \frac{\pi}{2} + \frac{1}{3}\cdot r^3 \cdot \int_1^0 1-t^2\, dt = \frac{\pi r^3}{6} + \frac{r^3}{3}\cdot\left[\frac{t^3}{3} - t\right]_0^1 = \frac{\pi r^3}{6} + \frac{r^3}{3}\cdot\left(-\frac{2}{3}\right)$$

$$\Rightarrow V = 4\cdot\left(\frac{\pi r^3}{6} - \frac{2r^3}{9}\right) = \underline{\underline{\frac{4}{3} r^3 \cdot \left(\frac{\pi}{2} - \frac{2}{3}\right)}}$$