1) Vyřešte

A

$$(x^2-1)y' + 2xy^2 = 0, \quad y(0) = 1$$

$$y' = -\frac{2xy^2}{x^2-1} \quad \Big| \; y \neq 0$$ je řešení, ale nevyhovuje

$$\frac{dy}{y^2} = -\frac{2x}{x^2-1}$$

def. obor

$(-1,1)$

$$-\frac{1}{y} = -\ln|x^2-1| + C$$

V poč. podm.

$$-\frac{1}{1} = -\underbrace{\ln|0-1|}_{0} + C$$

$$C = -1$$

$$\Downarrow$$

$$+\frac{1}{y} = +\ln|x^2-1| + 1$$

$$y(\ln|x^2-1|+1) = 1$$

$$y(\ln(1-x^2)+1) = 1$$

2)

A

$$\left(\operatorname{arctg} y + \frac{1}{x^2+1}\right) dx + \frac{x}{1+y^2}\, dy = 0$$

$$\downarrow \frac{\partial}{\partial y} \qquad \downarrow \frac{\partial}{\partial x}$$

$$\frac{1}{1+y^2} = \frac{1}{1+y^2}$$ je exaktní

$$F(x,y) = \int \frac{x}{1+y^2}\, dy + C(x) = x \cdot \operatorname{arctg} y + C(x)$$

$$\downarrow \frac{\partial}{\partial x}$$

$$\operatorname{arctg} y + C'(x) = \operatorname{arctg} y + \frac{1}{x^2+1}$$

$$C'(x) = \frac{1}{x^2+1}$$

$$C(x) = \operatorname{arctg} x$$

$$x \cdot \operatorname{arctg} y + \operatorname{arctg} x = k, \quad k \in \mathbb{R}$$

3) (A)

Metodou variace konstant vyřešte

$$y' + \frac{y}{\cos^2 x} = \frac{\sin x}{\cos^3 x}$$

$$y' + \frac{y}{\cos^2 x} = 0 \quad \Big| \; e^{\int \frac{1}{\cos^2 x} dx} = e^{tg x}$$

$$y' e^{tg x} + \frac{y e^{tg x}}{\cos^2 x} = 0$$

$$\left(y e^{tg x}\right)' = 0$$

$$y e^{tg x} = C$$

$$y = C e^{-tg x}$$

$$y = C(x) e^{-tg x} \Rightarrow \quad y' = C'(x) e^{-tg x} + \underbrace{C(x)\, e^{-tg x} \cdot \left(-\frac{1}{\cos^2 x}\right)} = -\underbrace{\frac{C(x) e^{-tg x}}{\cos^2 x}} + \frac{\sin x}{\cos^3 x}$$

$$C'(x) = \frac{\sin x}{\cos^3 x} e^{+tg x} = \frac{1}{\cos^2 x} \cdot tg x \cdot e^{+tg x}$$

$$\int \frac{1}{\cos^2 x} \cdot tg x \cdot e^{+tg x} dx \; \left| \begin{array}{l} t = tg x \\ dt = \frac{1}{\cos^2 x} dx \end{array} \right| =$$

$$= \int t e^{+t} dt \; \left| \begin{array}{ll} u = t & u' = 1 \\ v = + e^{+t} & v' = e^{+t} \end{array} \right| =$$

$$= + t e^{+t} - \int e^{+t} dt = + t e^{+t} - e^{+t} =$$

$$= + e^{+tg x} (tg x - 1)$$

$$\underline{\underline{y = C e^{-tg x} + tg x - 1}}$$

4)

A

$$y'' + y' - 2y = 3xe^x$$

$$\lambda^2 + \lambda - 2 = 0$$

$$\lambda_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} = \frac{-1 \pm 3}{2} \begin{cases} -2 \\ 1 \end{cases}$$

$$y_H = C_1 e^{-2x} + C_2 e^x$$

$$y_p = x(Ax+B)e^x$$

$$y_p' = (Ax+B)e^x + xAe^x + x(Ax+B)e^x = (Ax^2 + Bx + 2Ax + B)$$

$$y_p'' = e^x(Ax^2 + 4Ax + Bx + 2B + 2A)$$

$$e^x(Ax^2 + 4Ax + Bx + 2B + 2A) + e^x(Ax^2 + Bx + 2Ax + B) - 2e^x(Ax^2 + Bx) = 3xe^x$$

$A + A - 2A = 0$ ✓

$4A + B + B + 2A - 2B = 3$

$6A = 3$

$A = 1/2$

$2B + 2A + B = 0$

$3B = -1$

$B = -1/3$

$$\underline{y = C_1 e^{-2x} + C_2 e^x + \left(\frac{x^2}{2} - \frac{x}{3}\right)e^x}$$

(A)

5) Nechť $X=\mathbb{R}^2$ a pro $x=[x_1,x_2]$, $y=[y_1,y_2]$ uvažme funkci

$$\varrho(x,y)=\min\{|x_1-y_1|,|x_2-y_2|\}.$$

Tvoří dvojice (X,ϱ) metrický prostor?

Řešení: Ne. Uvažme body $\tilde{x}=[0,0]$, $\tilde{y}=[0,1]$. Pak

$$\varrho(\tilde{x},\tilde{y})=\min\{|0-0|,|0-1|\}=\min\{0,1\}=0,$$

očem $\tilde{x}\neq\tilde{y}$, tedy funkce ϱ není metrika.