B

1) Vyřešte

$$y' \sin x = y \ln y, \quad y\left(\frac{\pi}{2}\right) = e$$

$$\frac{dy}{y \ln y} = \frac{dx}{\sin x} \qquad / y \neq 0,\ y \neq 1$$

nevyhovuje poč. podm.

nevyhovuje ani zadání

$$\int \frac{1}{y \ln y}\, dy \left| \begin{array}{l} t = \ln y \\ dt = \frac{1}{y} dy \end{array} \right| = \int \frac{1}{t}\, dt = \ln|t| = \ln|\ln y|$$

$$\int \frac{1}{\sin x}\, dx = \ln\left|\operatorname{tg}\frac{x}{2}\right|$$

$$\ln|\ln y| = \ln\left|\operatorname{tg}\frac{x}{2}\right| + c$$

$$\ln\left|\frac{\ln y}{\operatorname{tg}\frac{x}{2}}\right| = C, \quad C \in \mathbb{R}$$

$$\left|\frac{\ln y}{\operatorname{tg}\frac{x}{2}}\right| = k, \quad k > 0$$

$$\frac{\ln y}{\operatorname{tg}\frac{x}{2}} = L, \quad L \in \mathbb{R} \setminus \{0\}$$

def. obor:

$(0, \pi)$

$$\ln y = L \operatorname{tg}\frac{x}{2}$$

$$y = e^{L \operatorname{tg}\frac{x}{2}}$$

poč. podm.

$$e = e^{L \cdot \operatorname{tg}\frac{\pi}{4}} \Rightarrow L = 1$$

$$\underline{\underline{y = e^{\operatorname{tg}\frac{x}{2}}}}$$

2)

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$$\left(\frac{1}{y^2+1}-\frac{y}{x^2}\right)dx+\left(\frac{1}{x}-\frac{2xy}{(y^2+1)^2}\right)dy=0$$

$$\downarrow \frac{\partial}{\partial y} \qquad \downarrow \frac{\partial}{\partial x}$$

$$-\frac{2y}{(y^2+1)^2}-\frac{1}{x^2} = -\frac{1}{x^2}-\frac{2y}{(y^2+1)^2}$$ je exaktní ✓

$$F(x,y)=\int\left(\frac{1}{y^2+1}-\frac{y}{x^2}\right)dx+C(y)=\frac{x}{y^2+1}+\frac{y}{x}+C(y)$$

$$\downarrow \frac{\partial}{\partial y}$$

$$-\frac{x2y}{(y^2+1)^2}+\frac{1}{x}+C'(y)=\frac{1}{x}-\frac{2xy}{(y^2+1)^2}$$

$$C'(y)=0$$

$$C(y)=K$$

$$\underline{\frac{x}{y^2+1}+\frac{y}{x}=K,\ K\in\mathbb{R}}$$

3) Vyřešte

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$$xy' + 2y + x^5 y^3 e^x = 0 \quad / \cdot \frac{1}{y^3},\ y \neq 0$$

vyhovuje

$$\frac{xy'}{y^3} + \frac{2}{y^2} + x^5 e^x = 0$$

$$\frac{y'}{y^3} + \frac{2}{xy^2} = -x^4 e^x$$

$$z = \frac{1}{y^2}, \quad z' = -2 \cdot \frac{1}{y^3} \cdot y'$$

$$z' = -2\left(-\frac{2}{x} z - x^4 e^x\right)$$

$$z' - \frac{4}{x} z = 2x^4 e^x \quad \Big/ \ e^{-\int \frac{4}{x} dx} = e^{-4\ln|x|} = \frac{1}{x^4}$$

$$\frac{z'}{x^4} - \frac{4}{x^5} z = 2e^x$$

$$\left(\frac{z}{x^4}\right)' = 2e^x$$

$$\frac{z}{x^4} = 2e^x + C$$

$$z = (2e^x + C)x^4, \quad C \in \mathbb{R}$$

$$\underline{\underline{\frac{1}{y^2} = (2e^x + C)x^4}} \quad \& \quad \underline{\underline{y \equiv 0}}$$

B

4) Vyriešte

$$y'' + y = 4xe^x$$

$$\lambda^2 + 1 = 0$$

$$\lambda_{1,2} = \pm i$$

$$y_H = C_1 \sin x + C_2 \cos x$$

$$y_p = (Ax+B)e^x$$

$$y_p' = Ae^x + (Ax+B)e^x = (Ax+A+B)e^x$$

$$y_p'' = (Ax+2A+B)e^x$$

$$(Ax+2A+B)e^x + (Ax+B)e^x = 4xe^x$$

$$A + A = 4 \qquad 2A + B + B = 0$$

$$A = 2 \qquad 4 + 2B = 0$$

$$B = -2$$

$$\underline{\underline{y = C_1 \sin x + C_2 \cos x + (2x-2)e^x}}$$

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5) Nechť $X=\mathbb{R}^2$ a pro $x=[x_1,x_2]$, $y=[y_1,y_2]$ uvažme funkci

$$\rho(x,y)=\sqrt{(2x_1-y_1)^2+(2x_2-y_2)^2}.$$

Tvoří dvojice (X,ρ) metrický prostor?

Řešení: Ne. Uvažme bod $\tilde{x}=[1,0]$. Pak

$$\rho(\tilde{x},\tilde{x})=\sqrt{(2-1)^2+(0-0)^2}=1,$$

tedy $\rho(\tilde{x},\tilde{x})\neq 0$.