Zápočtový příklad Fourierova transformace

Konvoluce funkcí f a g:

$$f \star g(y) = \int\limits_{-\infty}^{\infty} f(x)g(y-x)\mathrm{d}x$$

Fourierův obraz funkce f:

$$\widehat{f}(\xi) = \int\limits_{-\infty}^{\infty} f(x)\mathrm{e}^{-\mathrm{i}\xi x}\mathrm{d}x$$

Určete $f \star g$, $\widehat{f}$, $\widehat{g}$ a $\widehat{f \star g}$ a ověřte, že $\widehat{f \star g} = \widehat{f} \cdot \widehat{g}$ pro jeden z následujících příkladů.

1.

$$f(x) = \begin{cases} 1 & x \in [-1,1] \\ 0 & \text{jinak} \end{cases}, \quad g(x) = \begin{cases} 1-x^2 & x \in [-1,1] \\ 0 & \text{jinak} \end{cases}$$

Řešení:

$$\widehat{f}(\xi) = \frac{2}{\xi}\sin\xi, \quad \widehat{g}(\xi) = \frac{4}{\xi^3}\sin\xi - \frac{4}{\xi^2}\cos\xi,$$

$$f * g(y) = \begin{cases} \frac{|y|^3}{3} - y^2 + \frac{4}{3} & y \in [-2,2] \\ 0 & \text{jinak} \end{cases}, \quad \widehat{f * g}(\xi) = \frac{8\sin^2\xi}{\xi^4} - \frac{8\sin\xi\cos\xi}{\xi^3}.$$

2.

$$f(x) = \begin{cases} 1-|x| & x \in [-1,1] \\ 0 & \text{jinak} \end{cases}, \quad g(x) = \begin{cases} 2 & x \in [-1,1] \\ 0 & \text{jinak} \end{cases}$$

Řešení:

$$\widehat{f}(\xi) = \frac{2}{\xi^2}(1-\cos\xi), \quad \widehat{g}(\xi) = \frac{4}{\xi}\sin\xi,$$

$$f*g(y) = \begin{cases} 2y+3-(y+1)|y+1| & y \in [-2,0] \\ -2y+3-(y-1)|y-1| & y \in [0,2] \\ 0 & \text{jinak} \end{cases}, \quad \widehat{f * g}(\xi) = \frac{8\sin\xi}{\xi^3} - \frac{8\sin\xi\cos\xi}{\xi^3}.$$

3.

$$f(x) = \begin{cases} 1 & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 & \text{jinak} \end{cases}, \quad g(x) = \begin{cases} \cos x & x \in [-\frac{\pi}{2}, \frac{\pi}{2}] \\ 0 & \text{jinak} \end{cases}$$

Řešení:

$$\widehat{f}(\xi) = \frac{2}{\xi}\sin\frac{\xi\pi}{2}, \quad \widehat{g}(\xi) = \frac{2\cos\frac{\xi\pi}{2}}{1-\xi^2},$$

$$f * g(y) = \begin{cases} 1+\cos y & y \in [-\pi,\pi] \\ 0 & \text{jinak} \end{cases}, \quad \widehat{f * g}(\xi) = \frac{2\sin(\pi\xi)}{\xi(1-\xi^2)}.$$
