MASARYKOVA UNIVERZITA
Přírodovědecká fakulta
Ústav matematiky a statistiky

Subriemannovské geometrie

Diplomová práce

Pavla Musilová

Vedoucí práce:
Prof. RNDr. Jan Slovák, DrSc.

Brno 2019

RIEMANNOVSKÁ STRUKTURA

$(M, \tilde{g})$

$\downarrow$

$\downarrow$

NEHOLONOMNÍ STRUKTURA

$(M, \mathcal{D}, \mathcal{D}^{\perp}, g)$

$\downarrow$

$\downarrow$

SUBRIEMANNOVSKÁ STRUKTURA

$(M, \mathcal{D}, g)$

Subriemannovská struktura $(M, \mathcal{D}, g)$

Délka horizontální křivky:

$$L(\gamma) = \int_{t_1}^{t_2} ||\dot{\gamma}|| \mathrm{d}t = \int_{t_1}^{t_2} \sqrt{\langle \dot{\gamma}(t) | \dot{\gamma}(t) \rangle} \mathrm{d}t.$$

$\longrightarrow$ **vzdálenost** $\longrightarrow$ **metrický prostor**

Chow-Rashewski theorem

kometrika: hladký řez $\overline{G}$ bandlu

$$S^2(TM) \subset TM \otimes TM \to M$$

$$\beta : T^*M \to TM, \quad \nu(\beta_q(\mu)) = \overline{G}(\nu, \mu)_q$$

hamiltonián:

$$H : T^*M \ni [q, \nu] \longrightarrow H(q, \nu) = \frac{1}{2} \overline{G}(q)(\nu, \nu) \in \mathbf{R}$$

Věta o normálních geodetikách: *Nechť* $\zeta(t) = (\gamma(t), p(t))$ *je řešení Hamiltonových diferenciálních rovnic na* T^*M *pro subriemannovský hamiltonián* H*:*

$$\dot{x}^i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial x^i},$$

a nechť $\gamma(t)$ *je projekce tohoto řešení na* M*. Pak každý dostatečně malý oblouk křivky* γ *je (jediná) subriemannovská geodetika spojující jeho koncové body.*

Subriemannovská konexe:

$\nabla : \Gamma(T^*M) \times \Gamma(T^*M) \ni [\eta, \nu] \to \nabla_\eta \nu \in \Gamma(T^*M)$

- ∇ je $\mathbf{R}$-lineární v obou argumentech
- ∇ je $C^\infty M$-lineární v prvním argumentu
- $\nabla_\eta(f\nu) = f\nabla_\eta \nu + (\beta \circ \eta)(f)\nu$

normální:

- $\nabla_\eta \nu + \nabla_\nu \eta = \partial_{\beta(\eta)}\nu + \partial_{\beta(\nu)}\eta - \mathrm{d}(\nu(\beta(\eta)))$

Rovnice geodetik normální konexe, tj. autoparalelních křivek:

$$\dot{x}^i = \overline{g}^{ij}(x(t))p_j(t)$$

$$\dot{p}_j(t) = -\Gamma_j^{ik}(x(t))p_i(t)p_k(t)$$

$$\Gamma_k^{ij} + \Gamma_k^{ji} = \frac{\partial \overline{g}^{ij}}{\partial x^k}$$

Moje rovnice:

$$\mathcal{D} = \langle X_1, \ldots, X_k \rangle$$

$$X_i = \sum_{a=1}^{n} X_i^a \frac{\partial}{\partial x^a}$$

$$g(X_i, X_j) = \delta_{ij}$$

$$\dot{x}^a = \left(\sum_{i=1}^{k} X_i^a X_i^b \right) p_b$$

$$\dot{p}_c = -\sum_{i=1}^{k} \left(\frac{\partial X_i^a}{\partial x^c} X_i^b \right) p_a p_b.$$

Neholonomní struktura $(M, \mathcal{D}, \mathcal{D}^{\perp}, g)$

Neholonomní (Koszulova) konexe:

$$\nabla : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \longrightarrow \Gamma(\mathcal{D})$$

- $\nabla_{fX} Y = f \nabla_X Y$
- $\nabla_X (fY) = X(f) Y + f \nabla_X Y$

metrická a bez torze:

- $\nabla g \equiv 0$
- $\nabla_X Y - \nabla_Y X - [X, Y]_{\mathcal{D}} = 0$

Rovnice neholonomních geodetik, tj. autoparalelních křivek:

$$\ddot{\gamma}^c(t) + \Gamma^c_{ab}(\gamma(t)) \dot{\gamma}^a(t) \dot{\gamma}^b(t) = 0$$

$$\Gamma^c_{ab} = \frac{1}{2}(c^c_{ab} - c^b_{ac} - c^a_{bc})$$

Kdy je neholonomní geodetika zároveň subriemannovskou?

- *Je-li distribuce $\mathcal{D}$ geodeticky invariantní v TM, pak každá neholonomní geodetika struktury $(M, \mathcal{D}, \mathcal{D}^{\perp}, g)$ je normální subriemannovskou geodetikou subriemannovské struktury $(M, \mathcal{D}, g)$.*
- *Distribuce $\mathcal{D}$ je geodeticky invariantní v TM právě tehdy, když množina neholonomních geodetik struktury $(M, \mathcal{D}, \mathcal{D}^{\perp}, g)$ splývá s množinou Riemannových geodetik struktury $(M, \tilde{g})$ (kde $g = \tilde{g}_{\mathcal{D}}$) tečných k $\mathcal{D}$.*

$\underline{\textbf{Riemannova varieta } (M, g)}$

Variační mechanika neholonomních systémů

$$\overline{E}_l \circ J^2\gamma = 0 \quad \bar{L} = L \circ \iota$$

$$\overline{E}_l = \varepsilon_l(\bar{L}) - \bar{L}_a \varepsilon_l(g^a)$$

$$\bar{L}_a = \frac{\partial L}{\partial \dot{x}^{m-k+a}} \circ \iota$$

$$\varepsilon_l = \frac{\partial_c}{\partial x^l} - \frac{\mathrm{d}_c}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}^l}$$

$$\frac{\mathrm{d}_c}{\mathrm{d}t} = \frac{\partial}{\partial t} + \sum_{l=1}^{m-k} \dot{x}^l \frac{\partial}{\partial x^l} +$$

$$+ \sum_{a=1}^{k} g^a \frac{\partial}{\partial x^{m-k+a}} + \sum_{l=1}^{m-k} \ddot{x}^l \frac{\partial}{\partial \dot{x}^l}$$

$$\frac{\partial_c}{\partial x^l} = \frac{\partial}{\partial x^l} + \sum_{a=1}^{k} \frac{\partial g^a}{\partial \dot{x}^l} \frac{\partial}{\partial x^{m-k+a}}.$$

Alternativa — Četajevovy rovnice

$$\frac{\partial L}{\partial x^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^i} + \mu^a \frac{\partial f^a}{\partial x^i} = 0$$

Příklad 1.

$$X_1 = \frac{\partial}{\partial x} - y\frac{\partial}{\partial z}$$
$$X_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$$

$$\mathcal{D} = \langle X_1, X_2 \rangle$$
$$g(X_i, X_j) = \delta_{ij}$$

Subriemannovské rovnice:

$$\begin{aligned}
\dot{x} &= \alpha - y\gamma, \\
\dot{y} &= \beta + x\gamma, \\
\dot{z} &= (x^2 + y^2)\gamma + (\beta x - \alpha y), \\
\dot{\alpha} &= -x\gamma^2 - \beta\gamma, \\
\dot{\beta} &= -y\gamma^2 + \alpha\gamma, \\
\dot{\gamma} &= 0.
\end{aligned}$$

Subriemannovské geodetiky:

$$\begin{aligned} x &= \frac{A}{K}\cos(Kt+\varphi)+C, \\ y &= \frac{A}{K}\sin(Kt+\varphi)+D, \\ z &= \frac{A^2}{K}t-\frac{DA}{K}\cos(Kt+\varphi)+ \\ &\quad +\frac{CA}{K}\sin(Kt+\varphi)+E. \end{aligned}$$

$$\begin{aligned} x &= At+C, \\ y &= Bt+D, \\ z &= (BC-AD)t+E. \end{aligned}$$

Neholonomní struktura s obecnou volbu doplňku:

$$\mathcal{D}^\perp = \langle X_3 \rangle$$

$$X_3 = f\frac{\partial}{\partial x} + g\frac{\partial}{\partial y} + h\frac{\partial}{\partial z}$$

$$h + fy - gx \neq 0$$

Neholonomní rovnice:

$$\begin{aligned}
\ddot{x} &= \frac{2\dot{y}}{h + fy - gx}(g\dot{y} + f\dot{x}), \\
\ddot{y} &= -\frac{2\dot{x}}{h + fy - gx}(g\dot{y} + f\dot{x}), \\
\dot{z} &= -\dot{x}y + \dot{y}x.
\end{aligned}$$

Diskuse:

Doplněk	f	g	h	$\ddot{x} =$	$\ddot{y} =$
1.	0	0	1	0	0
2.	y	$-x$	1	$\frac{-2\dot{y}\dot{z}}{1+x^2+y^2}$	$\frac{2\dot{x}\dot{z}}{1+x^2+y^2}$
3.	x	y	1	$2\dot{y}(y\dot{y}+x\dot{x})$	$-2\dot{x}(y\dot{y}+x\dot{x})$
4.	x	y	-1	$-2\dot{y}(y\dot{y}+x\dot{x})$	$2\dot{x}(y\dot{y}+x\dot{x})$
5.	1	0	$1-y$	$2\dot{y}\dot{x}$	$-2\dot{x}\dot{x}$
6.	1	0	$1+x$	$2\dot{y}\dot{y}$	$-2\dot{x}\dot{y}$
7.	1	1	$1-y+x$	$2\dot{y}(\dot{y}+\dot{x})$	$-2\dot{x}(\dot{y}+\dot{x})$
8.	-1	1	$1+y+x$	$2\dot{y}(\dot{y}-\dot{x})$	$-2\dot{x}(\dot{y}-\dot{x})$

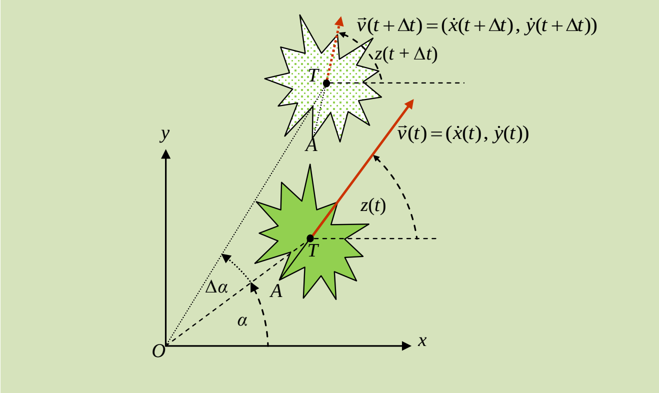
Třetí rovnice je vždy stejná, je to rovnice vazby:

$$\dot{z} = \dot{y}x - \dot{x}y.$$

Geodetiky Riemannovy struktury s neholonomní vazbou (první a druhý doplněk)

První doplněk — distribuce je geodeticky invariantní

Fyzikální úlohy - brusle



$$\frac{\dot{y}}{\dot{x}} = \tan z \quad \Longrightarrow \quad \dot{y}\cos z - \dot{x}\sin z = 0$$

Distribuce $\mathcal{D}$:

$$X_1 = \cos z \frac{\partial}{\partial x} + \sin z \frac{\partial}{\partial y}$$

$$X_2 = \frac{\partial}{\partial z} \quad g(X_i, X_j) = \delta_{ij}$$

Doplněk $\mathcal{D}^{\perp}$:

$$X_3 = [X_1, X_2] = \sin z \frac{\partial}{\partial x} - \cos z \frac{\partial}{\partial y}$$

Neholonomní geodetiky:

$$\begin{aligned} x(t) &= \frac{C}{A}\sin\left(At + B\right) + D \\ y(t) &= -\frac{C}{A}\cos\left(At + B\right) + E \\ z(t) &= At + B \end{aligned}$$

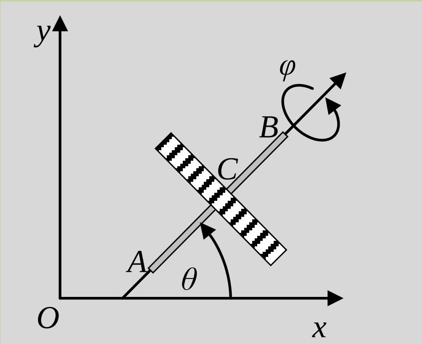
$$\begin{aligned} x(t) &= (C\cos B)t + D \\ y(t) &= (C\sin B)t + E \\ z(t) &= B \end{aligned}$$

Rovnice pro subriemannovsté geodetiky:

$$\begin{aligned}
&\dot{\alpha} = 0 \implies \alpha = R,\\
&\dot{\beta} = 0 \implies \beta = S,\\
&\dot{\gamma} = \frac{1}{2}[R^2 - S^2]\sin 2z - RS\cos 2z,\\
&\dot{x} = \cos z[R\cos z + S\sin z],\\
&\dot{y} = \sin z[R\cos z + S\sin z],\\
&\dot{z} = \gamma.
\end{aligned}$$

Fyzikální úlohy - planimetr

$$\dot{\varphi} = \dot{x}\sin\theta - \dot{y}\cos\theta$$



Rovnice geodetik Riemannovy struktury s neholonomní vazbou

$$\begin{aligned}
&\ddot{x}(1+\sin^2\theta) - \ddot{y}\sin\theta\cos\theta + \dot{x}\dot{\theta}\sin\theta\cos\theta + \dot{y}\dot{\theta}\sin^2\theta = 0,\\
&-\ddot{x}\sin\theta\cos\theta + \ddot{y}(1+\cos^2\theta) - \dot{x}\dot{\theta}\cos^2\theta - \dot{y}\dot{\theta}\sin\theta\cos\theta = 0,\\
&\ddot{\theta} = 0.
\end{aligned}$$

Rovnice geodetik neholonomní konexe:

$$\begin{aligned}\ddot{\gamma}^1 &= -\frac{\sqrt{2}}{2}\dot{\gamma}^2\dot{\gamma}^3,\\ \ddot{\gamma}^2 &= \frac{\sqrt{2}}{2}\dot{\gamma}^1\dot{\gamma}^3,\\ \ddot{\gamma}^3 &= 0.\end{aligned}$$

Neholonomní geodetiky:

$$\begin{aligned}x &= \frac{1-\frac{\sqrt{2}}{2}}{2+\sqrt{2}}\sin\left[\left(1+\frac{\sqrt{2}}{2}\right)Ct+D+\omega\right]+\frac{\frac{\sqrt{2}}{2}+1}{2-\sqrt{2}}\sin\left[\left(1-\frac{\sqrt{2}}{2}\right)Ct+D+\omega\right],\\ y &= -\frac{\frac{\sqrt{2}}{2}+1}{2+\sqrt{2}}\cos\left[\left(1+\frac{\sqrt{2}}{2}\right)Ct+D+\omega\right]-\frac{\frac{\sqrt{2}}{2}-1}{2-\sqrt{2}}\cos\left[\left(1-\frac{\sqrt{2}}{2}\right)Ct+D+\omega\right],\\ \theta &= Ct+D,\\ \varphi &= -\frac{1}{C}\cos\left(\frac{\sqrt{2}}{2}Ct+\omega\right).\end{aligned}$$

$$\begin{aligned}x &= Kt+L,\\ y &= Mt+N,\\ \theta &= P,\\ \varphi &= (K\sin P-M\cos P)t+R.\end{aligned}$$

Subriemannovské rovnice:

$$
\begin{aligned}
\dot{x} &= \alpha \cos^2\theta + \beta \cos\theta \sin\theta + \alpha \frac{\sin^2\theta}{2} - \beta \frac{\sin\theta\cos\theta}{2} + \delta \frac{\sin\theta}{2}, \\
\dot{y} &= \beta \sin^2\theta + \alpha \cos\theta \sin\theta + \beta \frac{\cos^2\theta}{2} - \alpha \frac{\sin\theta\cos\theta}{2} - \delta \frac{\cos\theta}{2}, \\
\dot{\theta} &= \gamma, \\
\dot{\varphi} &= \alpha \frac{\sin\theta}{2} - \beta \frac{\cos\theta}{2} + \frac{1}{2}\delta, \\
\dot{\alpha} &= 0, \\
\dot{\beta} &= 0, \\
\dot{\gamma} &= \frac{1}{4}[\alpha^2 - \beta^2] \sin 2\theta - \frac{1}{2}\alpha\beta \cos 2\theta - \alpha\delta \frac{\cos\theta}{2} - \beta\delta \frac{\sin\theta}{2}, \\
\dot{\delta} &= 0.
\end{aligned}
$$

Závěr

Literatura

[1] Agrachev, A., Barilari, D., Boscain, U. (2016). *Introduction to Riemannian and sub-Riemannian geometry*, Sissa.

[2] Agrachev, A., Sachkov, Y. (2004). *Control Theory from the Geometric Viewpoint*. Springer-Verlag.

[3] Barilari, D., Boscain, U., Siagalotti, M. (2016). *Geometry, Analysis an Dynamics on sub-Riemannian Manifolds*, Vol. I, European Mathematical Society.

[4] Barilari, D., Boscain, U., Siagalotti, M. (2016). *Geometry, Analysis an Dynamics on sub-Riemannian Manifolds*, Vol. II., European Mathematical Society.

[5] Barrett, D.I. (2016), *Contributions to the Study of Nonholonomic Riemannian Manifolds*. Ph.D thesis, Rhodes University.

[6] Calin, O., Mangione, V. (2003). *Variational Calculus on Sub-Riemannian Manifolds*. Balkan Journal of Geometry and Its Applications, Vol.8, No.1, 2003, p. 21-32. Geometry Balkan Press.

[7] Cantrijn, F., Langerock, B. (2003). *Generalised Connections over a Vector Bundle Map*. Differential Geometry and its Applications 18 (2003), p. 295-317. Elsevier.

[8] Cortés, J., de Léon, D., de Diego, M, Marínez, S. (2003) *Geometric Description of Vakonomic and Nonholonomic Dynamics. Comparsion of Solutions.* SIAM J. Control Optim. 41 (5)(2003) p. 1389-1412.

[9] Czudková, L., Musilová, J. (2013). *Nonholonomic mechanics: A practical application of the geometrical theory on fibred manifolds to a planimeter motion.* International Journal of Non-linear Mechanics. 50 (2013), p. 19-24.

[10] Čech, M. (2018), *Teorém Emmy Noetherové a zákony zachování v neholonomní mechanice.* Disertační práce MU Brno.

[11] Doležal, M. (2017), *Reprezentační techniky v geometrii a analýze.* Diplomová práce MU Brno.

[12] Hrdina, J., Návrat, A., Vašík, P. (2015). *Control of 3-Link Robotic Snake Based on Conformal Geometric Algebra*, Advances in Applied Clifford Algebras, Springer Basel.

[13] Krupková, O. (1997). *Mechanical Systems with Non-holonomic Constraints.* Journal of Math. Physics, Vol. 38, 1997, p. 5098-5126.

[14] Krupková, O. (1997). *The Geometry of Ordinary Variational Equations.* Lecture Notes in Mathematics. Vol. 1678, 1997 Springer Verlag Berlin.

[15] Krupková, O. (2009). *The Nonholonomic Variational Principle.* J. Phys. A: Math. Theor. Vol. 42, 2009, p. 185-201.

[16] Krupková, O. (2010). *Geometric Mechanics on Nonholonomic Submanifolds.* Communications in Mathematics, Vol. 18, 2010, 1, p. 51-78.

[17] Lang, S. (1999). *Fundamentals of Differential Geometry.* Springer-Verlag.

[18] Langerock, B. (2003). *A Connection Theoretic Approach to Sub-Riemannian Geometry.* Journal of Geometry ani Physics, Vol. 43, 2003, p. 203-230, Elsevier.

[19] Montgomery, R. (2002). *A Tour of Subriemannian Geometries, Their Geodesics and Applications.* American Mathematical Society.

[20] Neǐmark, J. I., Fufaev, N. A. (1972). *Dynamics of Nonholonomic Systems.* Translations of Mathematical Monographs, Vol. 33, 1972 AMS Providence Rhode Island.

[21] Strichartz, R. (1986). *Subriemannian Geometry*. Journal of Differential Geometry, Vol. 24, No.2, 1986, p. 221-263, International Press of Boston.

[22] Vašík, P. (2017). *Mathematical Models of Robotic Mechanisms Control*, Habilitační práce, VUT Brno.

[23] Wang, Z. Ma, S., Li, B., Wang, Y. (2011). *A Unified Dynamic Model for Locomotion and Manipulation of a Snake-like Robot Based on Differential Geometry.* Science China 2011, Vol. 54 No. 2, p. 318-333, China press and Springer-Verlag.