Example 6.12: Prove the relation

$$s = \int_{\theta_1}^{\theta_2} \sqrt{f^2 + \dot{g}^2 \left(f^2 \sin^2\theta + \dot{f}^2\right)}\, \mathrm{d}\theta, \tag{1}$$

where s is the length of a smooth curve, expressed in the spherical coordinates as

$$r = f(\phi),\ \ \phi = g(\theta), \qquad \text{and where} \qquad \dot{f} = \mathrm{d}f/\mathrm{d}\phi,\ \ \dot{g} = \mathrm{d}g/\mathrm{d}\theta. \tag{2}$$

Solution: Following the definition of line (path) integral in its basic Cartesian form,

$$s = \int_{s_1}^{s_2} \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2}, \tag{3}$$

where

$$x(r,\theta,\phi) = r\sin\theta\cos\phi, \qquad y(r,\theta,\phi) = r\sin\theta\sin\phi, \qquad z(r,\theta) = r\cos\theta, \tag{4}$$

we express the total differentials of the functions x, y, and z as

$$\mathrm{d}x = \frac{\partial x}{\partial r}\,\mathrm{d}r + \frac{\partial x}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial x}{\partial \phi}\,\mathrm{d}\phi = \sin\theta\cos\phi\,\mathrm{d}r + r\cos\theta\cos\phi\,\mathrm{d}\theta - r\sin\theta\sin\phi\,\mathrm{d}\phi, \tag{5}$$

$$\mathrm{d}y = \frac{\partial y}{\partial r}\,\mathrm{d}r + \frac{\partial y}{\partial \theta}\,\mathrm{d}\theta + \frac{\partial y}{\partial \phi}\,\mathrm{d}\phi = \sin\theta\sin\phi\,\mathrm{d}r + r\cos\theta\sin\phi\,\mathrm{d}\theta + r\sin\theta\cos\phi\,\mathrm{d}\phi, \tag{6}$$

$$\mathrm{d}z = \frac{\partial z}{\partial r}\,\mathrm{d}r + \frac{\partial z}{\partial \theta}\,\mathrm{d}\theta = \cos\theta\,\mathrm{d}r - r\sin\theta\,\mathrm{d}\theta. \tag{7}$$

Expanding the squares of the trinomials or the binomial on the right-hand sides and eliminating the trigonometric unit identities (you can try it yourselves), we obtain the expression

$$s = \int_{s_1}^{s_2} \sqrt{\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2}. \tag{8}$$

Expanding this by $\mathrm{d}\theta$, we explicitly write

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2 + r^2\sin^2\theta\,\left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta, \tag{9}$$

where, however, we may expand even the first term within the square root, using the chain rule for derivatives, as

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)^2 + r^2 + r^2\sin^2\theta\,\left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta. \tag{10}$$

Using Eq. (2), we rewrite the latter as

$$s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\dot{f}\dot{g}\right)^2 + f^2 + f^2\sin^2\theta\,\left(\dot{g}\right)^2}\,\mathrm{d}\theta, \tag{11}$$

that is

$$s = \int_{\theta_1}^{\theta_2} \sqrt{f^2 + \dot{g}^2 \left(f^2 \sin^2\theta + \dot{f}^2\right)}\, \mathrm{d}\theta. \tag{12}$$