5. domácí úkol

1) Maticová reprezentace operátorů harm. oscilátoru v bázi $|n\rangle$

a) Hamiltoniánu

$$\hat{H}_{ij} = \langle n_i | \hat{H} | n_j \rangle = \langle n_i | \hbar\omega (n+\tfrac{1}{2}) | n_j \rangle =$$

$$= \hbar\omega \begin{pmatrix} \frac{1}{2} & 0 & 0 & \dots \\ 0 & \frac{3}{2} & 0 & \dots \\ 0 & 0 & \frac{5}{2} & \dots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

$$\hat{A}_{ij} = \langle n_i | \hat{A} | n_j \rangle$$

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + \frac{i}{m\omega} \hat{p} \right)$$

$$\hat{a}^+ = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - \frac{i}{m\omega} \hat{p} \right)$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (\hat{a}^+ + \hat{a})$$

$$\hat{p} = i \sqrt{\frac{\hbar m\omega}{2}} (\hat{a}^+ - \hat{a})$$

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^+|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$$\hat{N} = \hat{a}^+\hat{a}, \quad \hat{N}|n\rangle = n|n\rangle$$

$$\hat{H} = \hbar\omega (\hat{N} + \tfrac{1}{2})$$

b) počtu částic

$$\hat{N}_{ij} = \langle n_i | \hat{N} | n_j \rangle = \langle n_i | n | n_j \rangle =$$

$$= \begin{pmatrix} 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 2 & \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

c) $\hat{a}^+, \hat{a}$

$$\hat{a}^+_{ij} = \langle n_i | \hat{a}^+ | n_j \rangle = \langle n_i | \sqrt{n+1}\, | n_j + 1 \rangle = \sqrt{n+1} \langle n_i | n_j + 1 \rangle =$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & \sqrt{2} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \end{pmatrix}$$

$$\hat{a}_{ij} = \langle n_i | \sqrt{n} | n_j - 1 \rangle = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots \\ & & & & \ddots \end{pmatrix}$$

d) $\hat{x}, \hat{p}$

$$\hat{x}_{ij} = \langle n_i|\hat{x}|n_j\rangle = \langle n_i|\sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^+ + \hat{a})|n_j\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\langle n_i|\hat{a}^+|n_j\rangle + \langle n_i|\hat{a}|n_j\rangle\right) = \sqrt{\frac{\hbar}{2m\omega}} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 & \dots \\ 1 & 0 & \sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ \vdots & \vdots & & & \ddots \end{pmatrix}$$

$$\hat{p}_{ij} = \langle n_i|\hat{p}|n_j\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\langle n_i|\hat{a}^+|n_j\rangle - \langle n_i|\hat{a}|n_j\rangle\right) =$$

$$= i\sqrt{\frac{\hbar m\omega}{2}} \begin{pmatrix} 0 & -1 & 0 & 0 & \dots \\ 1 & 0 & -\sqrt{2} & 0 & \dots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & \dots \\ 0 & 0 & \sqrt{3} & 0 & \dots \\ & & & \ddots & \end{pmatrix}$$

e) $\hat{x}^4$

$$\sum_k |n_k\rangle\langle n_k| = \hat{1}$$

$$\hat{x}_{ij} = \langle n_i|\hat{x}^4|n_j\rangle = \langle n_i|\hat{x}^2 \sum_k |n_k\rangle\langle n_k|\hat{x}^2|n_j\rangle =$$

$$= \frac{\hbar^2}{4m^2\omega^2}\langle n_i|(\overset{\hat{a}^+\hat{a} - \hat{a}\hat{a}^+}{\cancel{\dots}})\sum_k |n_k\rangle\langle n_k|(\overset{\hat{a}^+\hat{a} - \hat{a}\hat{a}^+}{\cancel{\dots}})|n_j\rangle =$$

$$= \frac{\hbar^2}{4m^2\omega^2}\langle n_i|(\quad)\sum_k |n_k\rangle\langle n_k| \underbrace{\left(\overset{n|n_j\rangle + (n+1)|n_j\rangle}{\cancel{\dots}}\right)}_{(n\delta_{kj} + (n+1)\delta_{kj})}$$

= Vubec nevim . . .

$$= \frac{\hbar^2}{4m^2\omega^2}\langle n_i|\hat{a}^+\hat{a} - \hat{a}\hat{a}^+|\sum |n_k\rangle\left(n\delta_{kj} + (n+1)\delta_{kj}\right) = \frac{\hbar^2}{4m^2\omega^2}\langle n_i|\hat{a}^+\hat{a} - \hat{a}\hat{a}^+|$$

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1) e) $\hat{x}^4$

$$x^4_{ij} = \langle n_i|\hat x^4|n_j\rangle = \langle n_i|\hat x^2 \sum_k |n_k\rangle\langle n_k|\hat x^2|n_j\rangle =$$

$$= \langle n_i|\hat x^2 \sum_k |n_k\rangle\langle n_k|\hat a^\dagger\hat a^\dagger + \hat a^\dagger \hat a + \hat a\hat a^\dagger + \hat a\hat a|n_j\rangle =$$

$$= \langle n_i|\hat x^2\sum_k|n_k\rangle\langle n_k|\Big(\sqrt{n+1}\sqrt{n+2}|n_j+2\rangle + \sqrt n\cdot\sqrt n|n_j\rangle + \sqrt{n+1}\cdot\sqrt{n+1}\cdot|n_j\rangle + \sqrt n\sqrt{n-1}|n_j-2\rangle\Big)$$

$$= \langle n_i|\hat x^2\sum_k|n_k\rangle\Big(\sqrt{n+1}\sqrt{n+2}\,\delta_{kj+2} + n\,\delta_{kj} + (n+1)\,\delta_{kj} + \sqrt n\sqrt{n-1}\,\delta_{kj-2}\Big)$$

$$= \langle n_i|\hat x^2\cdot\Big(\sqrt{n+1}\sqrt{n+2}|n_j+2\rangle + \underbrace{n|n_j\rangle + (n+1)|n_j\rangle}_{(2n+1)|n_j\rangle} + \sqrt n\sqrt{n-1}|n_j-2\rangle\Big)$$

$$= \langle n_i|\hat a^\dagger\hat a^\dagger+\hat a^\dagger\hat a+\hat a\hat a^\dagger+\hat a\hat a\Big(\sqrt{n+1}\sqrt{n+2}|n_j+2\rangle + (2n+1)|n_j\rangle+\sqrt n\sqrt{n-1}|n_j-2\rangle\Big)$$

$$= \langle n_i|\Big(\sqrt{n+1}\sqrt{n+2}\cancel{\sqrt{n+3}}\sqrt{n+4}|n_j+4\rangle + \sqrt{n+1}\sqrt{n+2}\cdot\cancel{\sqrt{n+2}}\sqrt{n+2}|n_j+2\rangle +$$

$$+\sqrt{n+1}\sqrt{n+2}\cdot\sqrt{n+1}\cancel{\sqrt{n+1}}|n_j+2\rangle + \sqrt{n+1}\cancel{\sqrt{n+2}}\cdot\sqrt{n+2}\sqrt{n+1}|n_j\rangle +$$

$$+(2n+1)\cdot\sqrt{n+1}\cancel{\sqrt{n+2}}|n_j+2\rangle + (2n+1)\cdot\cancel{\sqrt n}\sqrt n|n_j\rangle + (2n+1)\cancel{\sqrt{n+1}}\sqrt{n+1}|n_j\rangle$$

$$+(2n+1)\sqrt n\cancel{\sqrt{n-1}}|n_j-2\rangle + \sqrt n\sqrt{n-1}\cancel{\sqrt{n-1}}\sqrt n|n_j\rangle + \sqrt n\sqrt{n-1}\cancel{\sqrt{n-2}}\sqrt{n-2}|n_j-2\rangle$$

$$+\sqrt n\sqrt{n-1}\sqrt{n-1}\cancel{\sqrt{n-1}}|n_j-2\rangle + \sqrt n\sqrt{n-1}\cancel{\sqrt{n-2}}\sqrt{n-3}|n_j-4\rangle\Big) =$$

$$= \delta_{ij+4}\cdot\sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4} + \delta_{ij-4}\sqrt n\sqrt{n-1}\sqrt{n-2}\sqrt{n-3} + \delta_{ij+2}\Big(\sqrt{n+1}\sqrt{n+2}(n+2)+$$

$$+\sqrt{n+1}\sqrt{n+2}(n+1) + (2n+1)\sqrt{n+1}\sqrt{n+2}\Big) + \delta_{ij-2}\Big(\sqrt n\sqrt{n-1}\cdot(n-2)+$$

$$+\sqrt n\sqrt{n-1}(n-1) + (2n+1)\sqrt n\sqrt{n-1}\Big) + \delta_{ij}\Big((n+1)(n+2) + n(n-1) + (2n+1)\cdot(n+n+1)\Big) =$$

$$= \delta_{ij+4} \cdot \sqrt{n+1}\sqrt{n+2}\sqrt{n+3}\sqrt{n+4} + \delta_{ij-4}\sqrt{n}\sqrt{n-1}\sqrt{n-2}\sqrt{n-3} + \delta_{ij+2}\cdot\Big(\sqrt{n+1}\sqrt{n+2}\cdot$$

$$\cdot(n+2+n+1+2n+1)\Big) + \delta_{ij-2}\Big(\sqrt{n}\sqrt{n-1}\cdot(n-1+n-2+2n+1)\Big) +$$

$$+ \delta_{ij}\cdot\Big((n+1)(n+2) + n(n-1) + (2n+1)^2\Big) = \;.....$$

2) Boltzmannův neideální plyn

$$\rightarrow Z \simeq \left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right]^N \cdot Q_N$$

Spočtete přibližně c_V, meziatomový $U(r)$

$$F = -kT\ln\left\{\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right]^N Q_N\right\}$$

$$E = F + TS$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V$$

$$S = -\frac{\partial}{\partial T}\left(NkT\ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right] - kT\ln Q_N\right)$$

$$S = Nk\ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right] + NkT\,\frac{N}{e}\left(\frac{mkT}{2\pi\hbar}\right)^{-3/2}\cdot\frac{e}{N}\left(\frac{mk}{2\pi\hbar^2}\right)^{3/2}\cdot T^{1/2}\cdot\left(\frac{3}{2}\right) +$$

$$S = Nk\ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{3/2}\right] + \frac{3}{2}Nk + kT\frac{1}{Q_N}\frac{\partial Q_N}{\partial T} + k\ln Q_N$$

$$+ k\ln Q_N + kT\frac{1}{Q_N}\frac{\partial Q_N}{\partial T}$$

$$E = F + TS = \cancel{-NkT\ln\left\{(\;)^{3/2}\right\}} - \cancel{kT\ln Q_N} + \cancel{NkT\ln(\;)^{3/2}} + \frac{3}{2}NkT +$$

$$+ \cancel{kT\ln Q_N} + kT^2\frac{1}{Q_N}\frac{\partial Q_N}{\partial T} = \frac{3}{2}NkT + kT^2\frac{1}{Q_N}\frac{\partial Q_N}{\partial T}$$

$$c_V = \left(\frac{\partial E}{\partial T}\right)_V = \frac{3}{2}Nk + 2kT\frac{1}{Q_N}\frac{\partial Q_N}{\partial T} + kT^2\left(\frac{-1}{Q_N^2}\frac{\partial Q_N}{\partial T}\cdot\frac{\partial Q_N}{\partial T} + \frac{1}{Q_N}\frac{\partial^2 Q_N}{\partial T^2}\right)$$

$$c_V = \frac{3}{2}Nk + kT\frac{2}{Q_N}\frac{\partial Q_N}{\partial T} + kT^2\frac{\partial^2 Q_N}{\partial T^2}\left(\frac{1}{Q_N} - \frac{1}{Q_N^2}\right)$$

5. domácí úkol

3) Tlak na vesmírné lodi – válec H, R, Ω, ideální plyn

– určete poměr tlaku $p_p(r=R)$ a $p_0(r=0)$

a) pomocí hydrostatické rovnováhy

$$\operatorname{grad} p = -\rho \vec{a}$$

$$pV = Nk_BT$$

$$\frac{m}{V} = \rho$$

$$p = \frac{N}{V}kT = \frac{N\mu}{V\mu} = \frac{\rho}{\mu}kT$$

$$\frac{\partial p}{\partial r} = -\rho r \Omega^2$$

$$p = \frac{\rho}{m}kT \qquad \rho = \frac{pm}{kT}$$

$$\frac{\partial p}{\partial r} = -\frac{mp}{kT}\Omega^2 r$$

$$\frac{\partial p}{p} = -\frac{m}{kT}\Omega^2 r\,dr \Rightarrow \ln p = -\frac{m}{kT}\frac{\Omega^2}{2}r^2 + C$$

$$\Rightarrow p = De^{-\frac{m\Omega^2}{2kT}r^2} \qquad p(0) = p_0$$

$$p = p_0 e^{-\frac{m\Omega^2}{2kT}r^2} \Rightarrow \underline{\frac{p_p}{p_0} = e^{-\frac{m\Omega^2}{2kT}R^2}}$$

b) pomocí Boltzmannova rozdělení

$$Z = \frac{1}{h^{6N}}\int_{\mathbb{R}^3}\int_{▭} e^{\frac{1}{kT}\left[\frac{p^2}{2m} + \frac{m\Omega^2r^2}{2kT}\right]} d^3r\, d^3p \quad , \quad T = \text{konst.}$$

$$F = -kT\ln Z \qquad p = -\left(\frac{\partial F}{\partial V}\right)_T$$

$$Z = \frac{1}{h^{6N}}\int e^{\frac{p^2}{kT2m}} d^3p \int e^{\frac{m\omega^2r^2}{2kT}} d^3r$$

$$Z = \frac{1}{h^{6N}}(2\pi mkT)^{3/2} \cdot \int_{▭} e^{\frac{m\Omega^2r^2}{2kT}} d^3r = \frac{1}{h^{6N}}(2\pi mkT)^{3/2} \cdot \int_0^H\int_0^R\int_0^{2\pi} re^{\frac{m\Omega^2r^2}{2kT}}\, dr\, d\varphi\, dz$$

$$Z = \frac{1}{h^{6N}}(2\pi mkT)^{3/2} \cdot 2\pi H \int_0^R re^{\frac{m\Omega^2r^2}{2kT}} dr = \frac{1}{h^{6N}}(2\pi mkT)^{3/2}\, 2\pi \cancel{H}\, \frac{e^{\frac{m\Omega^2R^2}{2kT}} - 1}{m\Omega^2R^2} \cdot kT$$

$$p = kT\left(\frac{\partial \ln z}{\partial V}\right)_T = kT\,\frac{1}{z}\left(\frac{\partial z}{\partial V}\right)_T$$

$V = \pi R^2 H$

$dV = \pi\, 2 r H\, dr +$
$+ \pi \cancel{r^2}\, dH$

$dV = \pi R\, dR\, dH$

$$p = kT\,\frac{1}{z}\,\frac{1}{2\pi R H}\,\frac{\partial z}{\partial R}$$

$$p = \frac{m\Omega^2 R^2}{e^{\frac{m\Omega^2R^2}{2}} - 1} \cdot \frac{\partial}{\partial R}\left(\frac{e^{m\Omega^2R^2} - 1}{m\Omega^2R^2}\right) \cdot kT\,\frac{1}{2\pi R H}$$

$p =$ nevím.......!

1) e) $\hat{x}^4$

$$\hat{x}^2 = \frac{\hbar}{2m\omega}\left(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+\right)$$

$$\langle n|\hat{x}^4|n\rangle = \langle n|\hat{x}^2 \sum_m |m\rangle\langle m|\hat{x}^2|n\rangle$$

$$= \delta_{ij}\left((n+1)(n+2) + n(n-1) + (2n+1)^2\right) =$$

$$= \delta_{ij}\left(n^2 + 3n + 2 + n^2 - n + 4n^2 + 4n + 1\right) =$$

$$= \delta_{ij}\left(6n^2 + 6n + 3\right)$$

4) vůbec nechápu co se myslí tou chybou