# 7. domácí úkol

## 1) Kdyby byl Vesmír obrovská dutina

- sférická dutina s poloměrem $10^{28}$ cm

### a) $T = 3\,K$, určete celkový počet fotonů

- celková energie z Plancka $E = V\int_0^\infty \rho(\nu)\,d\nu = V\int_0^\infty \frac{8\pi\nu^2}{c^3}\frac{h\nu}{e^{\frac{h\nu}{kT}}-1}\,d\nu$

$$N = \frac{E_{celk}}{E_{fotonu}} = V\cdot\int_0^\infty \frac{8\pi\nu^2}{c^3}\frac{1}{e^{\frac{h\nu}{kT}}-1}\,d\nu = \left|\begin{array}{l}\frac{h\nu}{kT} = t\\ d\nu = \frac{kT}{h}\,dt\end{array}\right| =$$

$$= V\cdot\frac{8\pi}{c^3}\cdot\left(\frac{kT}{h}\right)^3 \underbrace{\int_0^\infty t^2\frac{1}{e^t-1}\,dt}_{\Gamma(3)\cdot\zeta(3)\,\doteq\,2{,}404} \doteq V\cdot 5{,}48\cdot 10^8 = \frac{4\pi}{3}\cdot(10^{26})^3\cdot 5{,}48\cdot 10^8$$

$\doteq 2{,}29\cdot 10^{87}$ fotonů

### b) $T = 0\,K$, $N_{e^-} = 10^{80}$, určete Fermiho hybnost

$$\rho(p) = \frac{4\pi p^2}{(2\pi\hbar)^3}Vg = \frac{p^2V}{\pi^2\hbar^3}$$

$$N = \int_0^\infty N(p)\rho(p)\,dp = \int_0^\infty \theta(p_F - p)\frac{p^2V}{\pi^2\hbar^3}\,dp = \int_0^{p_F}\frac{p^2V}{\pi^2\hbar^2}\,dp = \frac{1}{3}\frac{p_F^3V}{\pi^2\hbar^3}$$

$$p_F = (3\pi^2)^{1/3}\left(\frac{N}{V}\right)^{1/3}\hbar = (3\pi^2)^{1/3}\left(\frac{10^{80}}{\frac{4}{3}\pi(10^{26})^3}\right)^{1/3}\hbar = 9{,}39\cdot 10^{-34}\ \text{kg m/s}$$

2) n-rozměrný vesmír

a) hustota energie záření AČT

$$S_{n-1} = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$$

$$\varepsilon = \frac{E}{V} = \frac{1}{V}\int_0^\infty E\rho(E)\,dE \qquad \rho(E) = \frac{gV}{(2\pi)^n}\left(k(E)\right)^{n-1} S_{n-1}\left|\frac{dk}{dE}\right| dE$$

$$\rho(\nu)\,d\nu = \rho(E)\,dE \Rightarrow \rho(\nu) = \rho(E)\frac{dE}{d\nu}$$

$$\varepsilon = \mathrm{const.}\int_0^\infty h\nu \cdot \rho(\nu)\,d\nu \qquad \rho(\nu) \approx \mathrm{const.}\ \nu^{n-1}\left(e^{\frac{h\nu}{kT}}-1\right)^{-1}$$

$$\varepsilon = \mathrm{const.}\int_0^\infty \frac{\nu\cdot\nu^{n-1}}{e^{\frac{h\nu}{kT}}-1}\,d\nu = \left|\begin{array}{l}\frac{h\nu}{kT} = t \\ d\nu = \frac{kT}{h}dt\end{array}\right| = \mathrm{const.}\left(\frac{kT}{h}\right)^{n+1}\underbrace{\int_0^\infty \frac{t^n}{e^t-1}\,dt}_{\text{číslo...}}$$

$$= \mathrm{const.}\ T^{n+1}$$

b) poměr $c_p$ a $c_V$

$$c_p = \left(\frac{\partial E}{\partial T}\right)_p = T\left(\frac{\partial S}{\partial T}\right)_p$$

$$c_V = \left(\frac{\partial E}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V$$

$$F = -kT\ln Z$$

$$Z_1 = \frac{1}{(2\pi\hbar)^n}\int e^{-\frac{E}{kT}}\,d^np\,d^nq = \frac{1}{(2\pi\hbar)^n}\int e^{-\frac{(p_x^2+\dots+p_n^2)}{2mkT}}\,d^np\,d^nq =$$

$$= \frac{V_n}{(2\pi\hbar)^n}\underbrace{\int e^{-\frac{(p_x^2+\dots+p_n^2)}{2mkT}}\,d^np}_{\left(\int e^{-\frac{p_x^2}{2mkT}}dp_x\right)^n = (2\pi mkT)^{\frac{1}{2}\cdot n}} = \frac{V_n}{(2\pi\hbar)^n}\cdot(2\pi mkT)^{\frac{n}{2}}$$

$$\Rightarrow Z_N = \frac{1}{N!}\left[\frac{V_n}{(2\pi\hbar)^n}(2\pi mkT)^{\frac{n}{2}}\right]^N$$

$$F = -kT\left[\underbrace{\ln\frac{1}{N!}}_{\to -N\ln N+N} + N\ln\left(\frac{V_n}{(2\pi\hbar)^n}(2\pi mkT)^{\frac{n}{2}}\right)\right] = -NkT\left[\ln\left(\frac{V_n(2\pi mkT)^{\frac{n}{2}}}{N(2\pi\hbar)^n}\right)+1\right]$$

2b) $$F = -NkT\left[\ln\left(\frac{V_n(2\pi mkT)^{\frac{n}{2}}}{N(2\pi\hbar)^n}\right)+1\right]$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = -\frac{F}{T} + NkT\,\frac{\cancel{N(2\pi\hbar)^n}}{\cancel{V_n(2\pi mkT)^{\frac{n}{2}}}}\cdot\frac{n}{2}\frac{\cancel{(2\pi mk)^{\frac{n}{2}}}\cancel{V_n}}{\cancel{N(2\pi\hbar)^n}}T^{\frac{n}{2}-1}$$

$$S = -\frac{F}{T} + NkT\frac{n}{2}\cdot\frac{1}{T} = -\frac{F}{T} + Nk\frac{n}{2}$$

$$\left(\frac{\partial S}{\partial T}\right)_V = +\frac{F}{T^2} - \frac{1}{T}\underbrace{\left(\frac{\partial F}{\partial T}\right)_V}_{-S} = \cancel{\frac{F}{T^2}} - \frac{1}{T}\left(\cancel{+\frac{F}{T}} - Nk\frac{n}{2}\right) = +\frac{Nkn}{2T}$$

$$\left(\frac{\partial S}{\partial T}\right)_p = \left(\frac{\partial S}{\partial T}\right)_V + \left(\frac{\partial S}{\partial V}\right)_T\left(\frac{\partial V}{\partial T}\right)_p \qquad \left(\frac{\partial V}{\partial T}\right)_p \underset{\text{ideální plyn}}{=} \frac{Nk}{p}$$

$$\left(\frac{\partial S}{\partial V}\right)_T = -\frac{1}{T}\left(\frac{\partial F}{\partial V}\right)_T = +\frac{1}{\cancel{T}}\cdot(+Nk\cancel{T})\cdot\left(\frac{\cancel{N(2\pi\hbar)^3}}{V_n\cancel{(2\pi mkT)^{\frac{n}{2}}}}\cdot\frac{\cancel{(2\pi mkT)^{\frac{n}{2}}}}{\cancel{N(2\pi\hbar)^3}}\right) = \frac{Nk}{V_n}$$

$$\Rightarrow \left(\frac{\partial S}{\partial T}\right)_p = \frac{Nkn}{2T} + \frac{Nk}{V_n}\cdot\frac{Nk}{p} = \frac{Nkn}{2T} + \frac{N^2k^2}{pV_n}$$

$$pV = NkT \qquad \frac{NkT}{pV} = 1$$

$$\frac{c_p}{c_V} = \frac{T\left(\frac{\partial S}{\partial T}\right)_p}{T\left(\frac{\partial S}{\partial T}\right)_V} = \frac{\frac{Nkn}{2T}+\frac{N^2k^2}{pV_n}}{\frac{Nkn}{2T}} = 1 + \frac{2T\cancel{Nk}}{\cancel{pV_n}\cdot n} = \underline{\underline{1+\frac{2}{n}}}$$