SFT

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Domácí úkol č. 5

1, Harmonický oscilátor $\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2 = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2 x^2$

platí : $\hat{H}|n\rangle = \hbar\omega \cdot (n+\frac{1}{2})|n\rangle$

$\hat{N}$ – operátor počtu částic $\hat{N}|n\rangle = n|n\rangle$

$$\hat{N}_{n,m} = \delta_{nm} = \begin{pmatrix} 1 & 0 & 0 & \cdot & \cdot \\ 0 & 2 & 0 & \cdot & \cdot \\ 0 & 0 & 3 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & & & \cdot \end{pmatrix}$$

$$\hat{H} = \hbar\omega\left(\hat{N}+\frac{1}{2}\right) \longrightarrow \hat{H} = \begin{pmatrix} \frac{\hbar\omega}{2} & 0 & 0 & \cdot & \cdot \\ 0 & \frac{3\hbar\omega}{2} & 0 & \cdot & \cdot \\ 0 & 0 & \frac{5\hbar\omega}{2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \cdot \end{pmatrix} =$$

$$\hat{H} = \hbar\omega \cdot \begin{pmatrix} \frac{1}{2} & 0 & 0 & \cdot \\ 0 & \frac{3}{2} & 0 & \cdot \\ 0 & 0 & \frac{5}{2} & \cdot \\ \cdot & \cdot & & \cdot \end{pmatrix}$$

žebříčkové operátory

kreační : $\hat{a}^{+}|n\rangle = \sqrt{n+1}\,|n+1\rangle$

anihilační : $\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$

$$\langle n|\hat{a}|m\rangle = \langle n|\sqrt{m}\,|m-1\rangle = \sqrt{m}\,\underbrace{\langle n|m-1\rangle}_{n=m-1 \text{ pro nenul. prvky}} = \sqrt{m}\cdot\delta_{n,m-1}$$

$$\langle n|\hat{a}^{+}|m\rangle = \langle n|\sqrt{m+1}\,|m+1\rangle = \sqrt{m+1}\,\langle n|m+1\rangle = \sqrt{m+1}\,\delta_{n,m+1}$$

$$\hat{a} = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ 0 & 0 & 0 & \sqrt{3} & \cdots \\ \vdots & & & & \ddots \end{pmatrix} \qquad \hat{a}^{+} = \begin{pmatrix} 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \cdots \\ \vdots & & & \ddots \end{pmatrix}$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\,(\hat{a}+\hat{a}^{+}) = \sqrt{\frac{\hbar}{2m\omega}} \cdot \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & & & \ddots \end{pmatrix}$$

$$\hat{p} = -i\sqrt{\frac{\hbar m\omega}{2}}\,(\hat{a}-\hat{a}^{+}) = +i\sqrt{\frac{\hbar m\omega}{2}} \begin{pmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & -\sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & \vdots & \vdots & & & \ddots \end{pmatrix}$$

$$\hat{x}^2 = \hat{x}\cdot\hat{x} = \frac{\hbar}{2m\omega} \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix} \cdot \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & 0 & \cdots \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 & \cdots \\ 0 & 0 & \sqrt{3} & 0 & \sqrt{4} & \cdots \\ 0 & 0 & 0 & \sqrt{4} & 0 & \cdots \\ \vdots & & & & & \ddots \end{pmatrix} = \frac{\hbar}{2m\omega} \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & \cdots \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdots \\ \sqrt{2} & 0 & 5 & 0 & 2\sqrt{3} & \cdots \\ 0 & \sqrt{6} & 0 & 7 & 0 & \cdots \\ 0 & 0 & 2\sqrt{3} & 0 & \cdots \end{pmatrix}$$

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1) $$\hat{X}^4 = \hat{X}^2 \cdot \hat{X}^2 = \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & \cdot \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdot \\ \sqrt{2} & 0 & 5 & 0 & 2\sqrt{3} & \cdot \\ 0 & \sqrt{6} & 0 & 7 & 0 & \cdot \\ 0 & 0 & 2\sqrt{3} & 0 & 4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \sqrt{2} & 0 & 0 & \cdot \\ 0 & 3 & 0 & \sqrt{6} & 0 & \cdot \\ \sqrt{2} & 0 & 5 & 0 & 2\sqrt{3} & \cdot \\ 0 & \sqrt{6} & 0 & 7 & 0 & \cdot \\ 0 & 0 & 2\sqrt{3} & 0 & 4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix} =$$

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$$= \begin{pmatrix} 3 & 0 & 6\sqrt{2} & 0 & 2\sqrt{6} & \cdot \\ 0 & 15 & 0 & 10\sqrt{6} & 0 & \cdot \\ 6\sqrt{2} & 0 & 39 & 0 & 18\sqrt{3} & \cdot \\ 0 & 10\sqrt{6} & 0 & 55 & 0 & \cdot \\ 2\sqrt{6} & 0 & 10\sqrt{3} & 0 & 28 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}$$

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2) $c_V = \left(\frac{\partial E}{\partial T}\right)_V$ $\qquad E = F + TS = F - T\left(\frac{\partial F}{\partial T}\right)_V$

$$Z = \frac{1}{N!}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}N} \cdot Q_N = \left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right]^N \cdot Q_N$$

$N! \approx \left(\frac{N}{e}\right)^N$

$$F = -kT \ln(Z) = -NkT \cdot \ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right] - kT \ln Q_N$$

$$Q_N = \int d^{3N} \cdot r \cdot e^{-\frac{U(r)}{kT}} = Q_N(T)$$

$$\left(\frac{\partial F}{\partial T}\right)_V = -Nk \cdot \ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right] - NkT \cdot \frac{\cancel{\frac{e}{N}\left(\frac{mk}{2\pi\hbar^2}\right)^{\frac{3}{2}}}}{\cancel{\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}}} \cdot \frac{3}{2} T^{\frac{1}{2}} - k \cdot \ln Q_N -$$

$$- kT \cdot \frac{1}{Q_N} \cdot \frac{\partial Q_N}{\partial T} = -Nk \ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right] - \frac{3}{2} Nk\cancel{T} - k \cdot \ln Q_N - \frac{kT}{Q_N}\frac{\partial Q_N}{\partial T}$$

$$E = \cancel{-NkT \cdot \ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right]} \cancel{- kT \ln Q_N} + \cancel{NkT \cdot \ln\left[\frac{e}{N}\left(\frac{mkT}{2\pi\hbar^2}\right)^{\frac{3}{2}}\right]} + \frac{3}{2} NkT +$$

$$\cancel{+ kT \cdot \ln Q_N} + \frac{kT^2}{Q_N} \cdot \frac{\partial Q_N}{\partial T} \quad = E = \frac{3}{2} NkT + \frac{kT^2}{Q_N} \cdot \frac{\partial Q_N}{\partial T}$$

pro id. plyn $E = \frac{3}{2} NkT$

$c_V = \frac{3}{2} Nk$

$$c_V = \frac{3}{2} Nk + k \cdot \left(\frac{2T}{Q_N} \cdot \frac{\partial Q_N}{\partial T} + T^2\left[\frac{\partial}{\partial T}\left(\frac{1}{Q_N}\right) \cdot \frac{\partial Q_N}{\partial T} + \frac{1}{Q_N}\frac{\partial^2 Q_N}{\partial T^2}\right]\right)$$