12. domácí úkol

$$\hat{H} = E_n |n\rangle\langle n|$$

$$\hat{H}|n\rangle = E_n |n\rangle$$

$$E_n = \hbar\omega\left(n + \tfrac{1}{2}\right)$$

1) Matice hustoty harmonického oscilátoru

– v reprezentaci:

a) energiové

– kanonické rozdělení: $$\hat{\rho} = \frac{e^{-\beta\hat{H}}}{\underbrace{\mathrm{Tr}\{e^{-\beta\hat{H}}\}}_{Z}} = \sum_n \frac{e^{-\beta E_n}}{Z} |n\rangle\langle n| =$$

$$Z = \sum_n e^{-\beta E_n} = \sum_n^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = \frac{e^{-\beta\hbar\omega\frac{1}{2}}}{e^{\beta\hbar\omega} - 1} = \frac{1}{e^{\frac{\beta\hbar\omega}{2}} - e^{-\frac{\beta\hbar\omega}{2}}} = \frac{1}{2\sinh\frac{\beta\hbar\omega}{2}}$$

$$= \left(e^{\beta\hbar\omega} - 1\right) \sum_n e^{-\beta\hbar\omega(n+\frac{1}{2})} \cdot e^{\frac{\beta\hbar\omega}{2}} |n\rangle\langle n|$$

$$= \left(e^{\beta\hbar\omega} - 1\right) \sum_n e^{-(2n+1)} |n\rangle\langle n|$$

$$\hat{\rho} = \left(e^{\beta\hbar\omega} - 1\right) \begin{pmatrix} e^{-1} & & & \\ & e^{-3} & & \\ & & e^{-5} & \\ & & & \ddots \end{pmatrix}$$

$$\hat{\rho} = 2\sinh\frac{\beta\hbar\omega}{2} \sum e^{-\beta\hbar\omega(n+\frac{1}{2})} |n\rangle\langle n|$$

b) souřadnicové

$$x = \sqrt{\frac{m\omega}{\hbar}}\, q$$

$$\psi_n(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{H_n(x)}{\sqrt{2^n n!}} e^{-\frac{x^2}{2}}$$

integrální reprezentace

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} = \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (-2iu)^n e^{-u^2 + 2iux}\, du$$

$$\rho(x, x') = \langle x|\hat{\rho}|x'\rangle =$$

$$= \sum w_i\, \rho_i(x) \cdot \rho_i^*(x')$$

$$= \sum_n \frac{e^{-\beta E_n}}{Z} \psi_n(x)\, \psi_n^*(x')$$

další strana

$$\langle x' | \rho | x \rangle = \sum_{m,n=0}^{\infty} \psi_m(x') \underbrace{\rho_{mn}}_{2\sinh\frac{\beta\hbar\omega}{2} \cdot e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)}} \psi_n(x) = \sum_{n=0}^{\infty} 2\sinh\frac{\beta\hbar\omega}{2} \cdot e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} \cdot$$

$$\cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{1}{2^n n!} \cdot e^{-\frac{x^2+x'^2}{2}} H_n(x) \cdot H_n(x')$$

vyjádříme v int. tvaru

$$= 2\sinh\frac{\beta\hbar\omega}{2} \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)}$$

$$\cdot \frac{1}{2^n \cdot n!} e^{-\frac{x^2-x'^2}{2}} \cdot \frac{e^{x^2}}{\pi^{1/2}} \int_{-\infty}^{\infty} (-2iu)^n e^{-u^2+2iux} \, du \cdot \frac{e^{x'^2}}{\pi^{1/2}} \int_{-\infty}^{\infty} (-2iu')^n e^{-u'^2+2iu'x'} \, du'$$

$$= 2\sinh\frac{\beta\hbar\omega}{2} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{1}{\pi} e^{+\frac{x^2+x'^2}{2}} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)} \frac{1}{2^n n!} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$\cdot \underbrace{(-2iu)^n \cdot (-2iu')^n}_{(-4uu')^n} \cdot e^{-u^2-u'^2+2iu'x'+2iux} \, du \, du'$$

$$= 2\sinh\frac{\beta\hbar\omega}{2} \cdot \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{1}{\pi} e^{\frac{x^2+x'^2}{2}} \cdot e^{-\frac{\beta\hbar\omega}{2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} \frac{(-\cancel{4}^{\,2}uu')^n}{\cancel{2^n} n!} e^{-u^2-u'^2+2i(ux+u'x')} \, du \, du'$$

$$= \text{———} \; || \; \text{————} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sum_{n=0}^{\infty} e^{-\beta\hbar\omega n} \frac{(-2uu')^n}{n!} e^{\cdots} \, du \, du'$$

dál už to asi nedám ....