12. domácí úkol

$\hat H = E_n |n\rangle\langle n|$

$\hat H |n\rangle = E_n |n\rangle$

$E_n = \hbar\omega(n+\frac{1}{2})$

1) Matice hustoty harmonického oscilátoru

– v reprezentaci:

a) energiové

$\mathcal{H} = \hbar\omega(\hat n + \frac{1}{2})$

– kanonické rozdělení: $\hat\rho = \dfrac{e^{-\beta H}}{\underbrace{Tr\{\hat 1 e^{-\beta H} \hat 1\}}} = \hat 1 \sum_n \dfrac{e^{-\beta E_n}}{Z} |n\rangle\langle n| \hat 1 = 1$ $\to Tr \hat A = Z$

navíc maticová reprezentace je $\langle n|\hat\rho|n'\rangle Z$

$$= (e^{\beta\hbar\omega} - 1) \sum_n e^{-\beta\hbar\omega(n+\frac{1}{2})} \cdot e^{\frac{\beta\hbar\omega}{2}} |n\rangle\langle n|$$

$$= (e^{\beta\hbar\omega} - 1) \sum_n e^{-(2n+1)} |n\rangle\langle n|$$

není maticový prvek

$$\hat\rho = (e^{\beta\hbar\omega} - 1) \begin{pmatrix} e^{-1} & & & \\ & e^{-3} & & \\ & & e^{-5} & \\ & & & \ddots \end{pmatrix}$$

$$Z = \sum_n e^{-\beta E_n} = \sum_n^{\infty} e^{-\beta\hbar\omega(n+\frac{1}{2})} = \frac{e^{-\beta\hbar\omega\frac{1}{2}}}{e^{\beta\hbar\omega} - 1} = \frac{1}{e^{\frac{\beta\hbar\omega}{2}} - e^{-\frac{\beta\hbar\omega}{2}}} = \frac{1}{2\sinh\frac{\beta\hbar\omega}{2}}$$

$$\hat\rho = 2\sinh\frac{\beta\hbar\omega}{2} \sum e^{-\beta\hbar\omega(n+\frac{1}{2})} |n\rangle\langle n|$$

b) souřadnicové

$x = \sqrt{\frac{m\omega}{\hbar}}\, q$

$$\psi_n(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \frac{H_n(x)}{\sqrt{2^n n!}} e^{-\frac{x^2}{2}}$$

integrální reprezentace

$$H_n(x) = (-1)^n e^{x^2} \left(\frac{d}{dx}\right)^n e^{-x^2} = \frac{e^{x^2}}{\sqrt{\pi}} \int_{-\infty}^{\infty} (-2iu)^n e^{-u^2+2iux} du$$

$\rho(x,x') = \langle x|\hat\rho|x'\rangle =$

$= \sum w_i \rho_i(x) \cdot \rho_i^*(x')$

$= \sum_n \dfrac{e^{-\beta E_n}}{Z} \psi_n(x) \psi_n^*(x')$

další strana

$$\sum_{m,n} \langle x'|m\rangle\langle m|\hat{\rho}|n\rangle\langle n|x\rangle$$

$$\langle x'|\rho|x\rangle = \sum_{m,n=0}^{\infty} \psi_m(x')\,\underbrace{\rho_{mn}}\,\psi_n(x) = \sum_{n=0}^{\infty} 2\sinh\frac{\beta\hbar\omega}{2}\cdot e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)}\cdot$$

$$2\sinh\frac{\beta\hbar\omega}{2}\cdot e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)}$$

vyjádříme v int. tvaru

$$\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\cdot\frac{1}{2^n n!}\cdot e^{-\frac{x^2+x'^2}{2}}H_n(x)\cdot H_n(x') = 2\sinh\frac{\beta\hbar\omega}{2}\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\cdot\sum_{n=0}^{\infty}e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)}$$

$$\cdot\frac{1}{2^n n!}\,e^{-\frac{x^2-x'^2}{2}}\cdot\frac{e^{x^2}}{\pi^{1/2}}\int_{-\infty}^{\infty}(-2iu)^n e^{-u^2+2iux}\,du\cdot\frac{e^{x'^2}}{\pi^{1/2}}\int_{-\infty}^{\infty}(-2iu')^n e^{-u'^2+2iu'x'}\,du'$$

$$= 2\sinh\frac{\beta\hbar\omega}{2}\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\cdot\frac{1}{\pi}e^{+\frac{x^2+x'^2}{2}}\sum_{n=0}^{\infty}e^{-\beta\hbar\omega\left(n+\frac{1}{2}\right)}\frac{1}{2^n n!}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$$

$$\cdot\underbrace{(-2iu)^n\cdot(-2iu')^n}_{(-4uu')^n}\cdot e^{-u^2-u'^2+2iu'x'+2iux}\,du\,du'$$

$$= 2\sinh\frac{\beta\hbar\omega}{2}\cdot\left(\frac{m\omega}{\pi\hbar}\right)^{1/2}\cdot\frac{1}{\pi}e^{\frac{x^2+x'^2}{2}}\cdot e^{-\frac{\beta\hbar\omega}{2}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}e^{-\beta\hbar\omega n}\frac{(-\cancel{4}uu')^n}{\cancel{2^n}n!}e^{-u^2-u'^2+2i(ux+u'x')}\,du\,du'$$

$$= \text{———}\ \ \|\ \ \text{————}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}e^{-\beta\hbar\omega n}\frac{(-2uu')^n}{n!}e^{\cdots}\,du\,du'$$

dál už to asi nedám ....