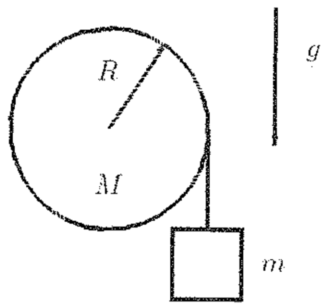
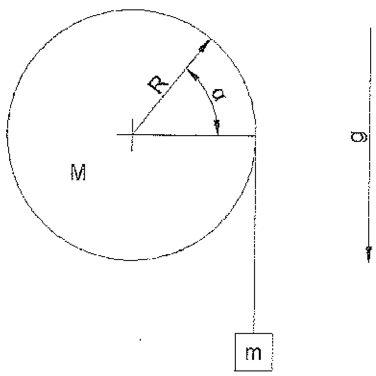
Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

Příklady z teoretické mechaniky pro domácí počítání

1. *Počet bodů: 1*

   Těleso o hmotnosti m je spojené s lanem, které se bez tření navinuje na kladku o hmotnosti M, poloměru R a momentu setrvačnosti $I = MR^2/2$. Gravitační síla působí vertikálně směrem dolů. Určete zrychlení tělesa o hmotnosti m.





$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}MR^2\dot{\alpha}^2$$
$$v = \omega R$$
$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}M\dot{y}^2$$

$$V = mgy$$

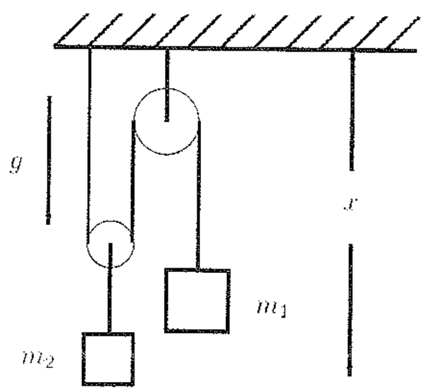
$$L = \frac{1}{2}m\dot{y}^2 + \frac{1}{4}MR^2\dot{\alpha}^2 - mgy$$

$$\frac{\partial L}{\partial \dot{y}} = m\dot{y} + \frac{1}{2}M\dot{y}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = m\ddot{y} + \frac{1}{2}M\ddot{y}$$
$$\frac{\partial L}{\partial y} = -mg$$
$$m\ddot{y} + \frac{1}{2}M\ddot{y} = -mg$$

$$\boldsymbol{\ddot{y} = -\frac{mg}{m + \frac{1}{2}M}}$$

2. *Počet bodů: 1*

   Určete zrychlení tělesa o hmotnosti m_1 pomocí Lagrangeových rovnic.



$$x_1 = x$$
$$\dot{x}_1 = \dot{x}$$
$$\dot{x}_1^2 = \dot{x}^2$$

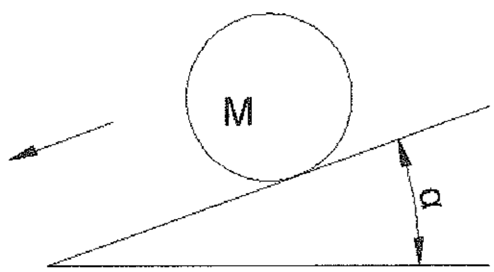
$$x_2 = -\frac{x}{2}$$
$$\dot{x}_2 = -\frac{\dot{x}}{2}$$
$$\dot{x}_2^2 = \frac{\dot{x}^2}{4}$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{8}m_2\dot{x}^2$$
$$V = m_1 g x_1 + m_2 x_2 g = m_1 g x - \frac{1}{2}m_2 g x$$
$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{8}m_2\dot{x}^2 - m_1 g x + \frac{1}{2}m_2 g x$$

$$\frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + \frac{1}{4}m_2\dot{x}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m_1\ddot{x} + \frac{1}{4}m_2\ddot{x}$$
$$\frac{\partial L}{\partial x} = -m_1 g + \frac{1}{2}m_2 g$$

$$\ddot{x} = \boldsymbol{\frac{2m_2 g - 4m_1 g}{4m_1 + m_2}}$$

3. *Počet bodů: 1*
Válec o hmotnosti M, poloměru R a momentu setrvačnosti $I = MR^2/2$ se valí bez klouzání dolu po nakloněné rovině. Určete zrychlení válce.



$$T = \frac{1}{2}I\omega^2 = \frac{1}{4}MR^2\omega^2 = \frac{1}{4}MR^2\dot{\varphi}^2$$

$$V = Mgy$$

kde
$$y = l\sin\alpha = R\varphi\sin\alpha$$

$$V = MgR\varphi\sin\alpha$$

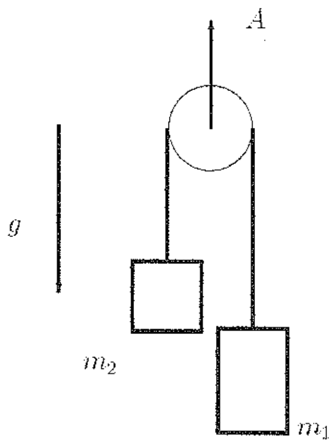
$$L = \frac{1}{4}MR^2\dot{\varphi}^2 - MgR\varphi\sin\alpha$$

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{2}MR^2\dot{\varphi}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}MR^2\ddot{\varphi}$$
$$\frac{\partial L}{\partial y} = -MgR\sin\alpha$$

$$\ddot{\varphi} = -\boldsymbol{\frac{2\sin\alpha}{R}}\ g$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

4. *Počet bodů: 1*
Určete zrychlení tělesa o hmotnosti m_1 pomocí Lagrangeových rovnic. Kladka se pohybuje se zrychlením A nahoru. Zrychlením rozumíme zrychlení vzhledem k Zemi.



$$x_1 = x$$
$$\dot{x}_1 = \dot{x}$$
$$\dot{x}_1^2 = \dot{x}^2$$

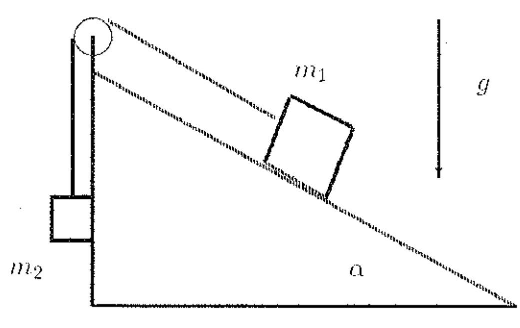
$$x_2 = -x$$
$$\dot{x}_2 = -\dot{x}$$
$$\dot{x}_2^2 = \dot{x}^2$$

$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2$$
$$V = m_1(A+g)x - m_2(A+g)x$$
$$L = \frac{1}{2}m_1\dot{x}^2 + \frac{1}{2}m_2\dot{x}^2 - m_1(A+g)x + m_2(A+g)x$$

$$\frac{\partial L}{\partial \dot{x}} = m_1\dot{x} + m_2\dot{x}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = m_1\ddot{x} + m_2\ddot{x}$$
$$\frac{\partial L}{\partial x} = (A+g)(m_2 - m_1)$$

$$\ddot{x} = (A+g)\frac{\boldsymbol{m_2 - m_1}}{\boldsymbol{m_2 + m_1}}$$

5. *Počet bodů: 2*
Určete zrychlení tělesa o hmotnosti m_1. Nakloněná rovina je v klidu.



l...délka lana
r...vzdálenost od kladky k m_1

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

$$x_1 = r\cos\alpha \qquad y_1 = -r\sin\alpha$$
$$\dot{x}_1 = \dot{r}\cos\alpha \qquad \dot{y}_1 = -\dot{r}\sin\alpha$$
$$\dot{x}_1^2 = \dot{r}^2\cos^2\alpha \qquad y_1^2 = \dot{r}^2\sin^2\alpha$$

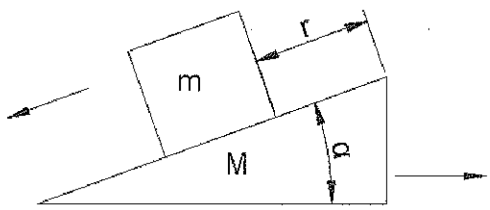
$$x_2 = 0 \qquad y_2 = l - r$$
$$\dot{x}_2 = 0 \qquad \dot{y}_2 = -\dot{r}$$
$$\dot{x}_2^2 = 0 \qquad y_2^2 = \dot{r}^2$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{r}^2$$
$$V = -m_1 g r\sin\alpha + m_2 g(l - r)$$
$$L = \frac{1}{2}(m_1 + m_2)\dot{r}^2 + m_1 g r\sin\alpha - m_2 g(l - r)$$

$$\frac{\partial L}{\partial \dot{r}} = \dot{r}(m_1 + m_2)$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \ddot{r}(m_1 + m_2)$$
$$\frac{\partial L}{\partial r} = m_1 g\sin\alpha + m_2 g$$

$$\boldsymbol{\ddot{r} = \frac{m_1 g\sin\alpha + m_2 g}{m_2 + m_1}}$$

6. *Počet bodů: 2*
Blok o hmotnosti m klouže bez tření po tělese ve tvaru nakloněné roviny o hmotnosti M, které se může pohybovat v horizontální rovině také bez tření. Určete pohybové rovnice bloku a nakloněné roviny.



$$x_M = x \qquad y_M = 0$$
$$\dot{x}_M = \dot{x} \qquad \dot{y}_M = 0$$
$$\dot{x}_M^2 = \dot{x}^2 \qquad \dot{y}_M^2 = 0$$

$$x_m = x - r\cos\alpha \qquad y_m = -r\sin\alpha$$
$$\dot{x}_m = \dot{x} - \dot{r}\cos\alpha \qquad \dot{y}_m = -\dot{r}\sin\alpha$$
$$\dot{x}_m^2 = (\dot{x} - \dot{r}\cos\alpha)^2 \qquad \dot{y}_m^2 = \dot{r}^2\sin^2\alpha$$

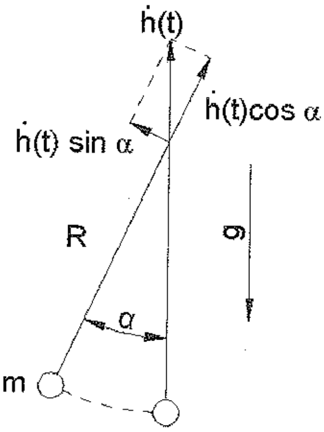
$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} - \dot{r}\cos\alpha)^2 + \frac{1}{2}m\dot{r}^2\sin^2\alpha$$
$$V = -mgr\sin\alpha$$
$$L = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x} - \dot{r}\cos\alpha)^2 + \frac{1}{2}m\dot{r}^2\sin^2\alpha + mgr\sin\alpha$$

$$\frac{\partial L}{\partial \dot{r}} = -m\dot{x}\cos\alpha + m\dot{r}\cos^2\alpha + m\dot{r}\sin^2\alpha = -m\dot{x}\cos\alpha + m\dot{r}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = -m\ddot{x}\cos\alpha + m\ddot{r}$$
$$\frac{\partial L}{\partial r} = mg\sin\alpha$$
$$\boldsymbol{mg\sin\alpha = -m\ddot{x}\cos\alpha + \ddot{r}}$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} - m\dot{r}\cos\alpha + m\dot{x}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = M\ddot{x} - m\ddot{r}\cos\alpha + m\ddot{x}$$
$$\frac{\partial L}{\partial x} = 0$$
$$\mathbf{0 = M\ddot{x} - m\ddot{r}\cos\alpha + m\ddot{x}}$$

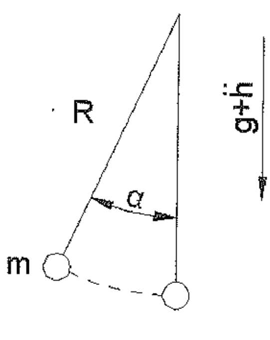
7. *Počet bodů: 2*

Bod závěsu matematického kyvadla koná vertikální pohyb v gravitačním poli podle zákona $y = h(t)$, kde $h(t)$ je daná funkce. Zapište Lagrangeovy pohybové rovnice. Ukažte, že pohybující se kyvadlo se chová jako matematické kyvadlo na pevném závěsu v gravitačním poli $g + \ddot{h}$, kde tečky značí příslušnou derivaci podle času.



$$T = \frac{1}{2}m\left(\left(R\dot{\alpha} + \dot{h}\sin\alpha\right)^2 + \dot{h}^2\cos^2\alpha\right)$$
$$V = -mgR\cos\alpha + mgh$$
$$L = \frac{1}{2}m\left(\left(R\dot{\alpha} + \dot{h}\sin\alpha\right)^2 + \dot{h}^2\cos^2\alpha\right) + mgR\cos\alpha - mgh$$
$$L = \frac{1}{2}mR^2\dot{\alpha}^2 + mR\dot{\alpha}\dot{h}\sin\alpha + \frac{1}{2}m\dot{h}^2 + mgR\cos\alpha - mgh$$

$$\frac{\partial L}{\partial \dot{\alpha}} = mR^2\dot{\alpha} + mR\dot{h}\sin\alpha$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = mR^2\ddot{\alpha} + mR\ddot{h}\sin\alpha + mR\dot{h}\dot{\alpha}\cos\alpha$$
$$\frac{\partial L}{\partial \alpha} = mR\dot{h}\dot{\alpha}\cos\alpha - mgR\sin\alpha$$
$$mR^2\ddot{\alpha} + mR\ddot{h}\sin\alpha = -\,mgR\sin\alpha$$
$$\ddot{\alpha} = -\frac{\mathbf{(g + \ddot{h})\sin\alpha}}{\mathbf{R}}$$

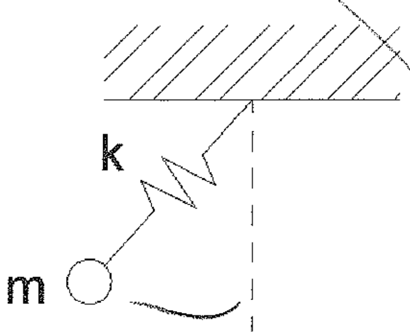


$$T = \frac{1}{2}mR^2\dot{\alpha}^2$$
$$V = -m(g + \ddot{h})R\cos\alpha$$
$$L = \frac{1}{2}mR^2\dot{\alpha}^2 + m(g + \ddot{h})R\cos\alpha$$

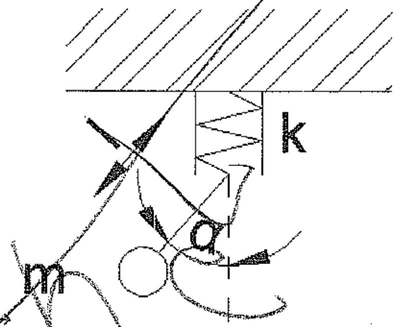
$$\frac{\partial L}{\partial \dot{\alpha}} = mR^2\dot{\alpha}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = mR^2\ddot{\alpha}$$
$$\frac{\partial L}{\partial \alpha} = -mR(g + \ddot{h})\sin\alpha$$
$$\ddot{\alpha} = -\frac{\mathbf{(g + \ddot{h})\sin\alpha}}{\mathbf{R}}$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

9. *Počet bodů: 2*
Kyvadlo je tvořeno tělesem o hmotnosti m a pružinou o konstantě pružnosti k. Délka nenapjaté pružiny je l_0. Systém je umístěn ve vertikálním gravitačním poli. Zapište Lagrangeovy pohybové rovnice.



← lze překreslit na →



Špatně

$$x = l\sin\alpha \qquad y = l\cos\alpha$$
$$\dot{x} = l\cos\alpha \qquad \dot{y} = -l\sin\alpha$$
$$\dot{x}^2 = l^2\cos^2\alpha \qquad \dot{y}^2 = l^2\sin^2\alpha$$

$$T = \frac{1}{2}ml^2(\cos^2\alpha + \sin^2\alpha) = \frac{1}{2}ml^2$$
$$V = -kl\cos\alpha + mgl\sin\alpha$$
$$L = \frac{1}{2}ml^2 + kl\cos\alpha - mgl\sin\alpha$$
$$\frac{\partial L}{\partial\dot{\alpha}} = 0$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}} = 0$$
$$\frac{\partial L}{\partial\alpha} = -kl\sin\alpha - mgl\cos\alpha$$

$$\boldsymbol{kl\sin\alpha + mgl\cos\alpha = 0}$$

10. *Počet bodů: 2*
Korálek o hmotnosti m klouže bez tření podél drátu, který má tvar paraboly $y = Ax^2$. Gravitační pole je vertikální. Zapište Lagrangeovy pohybové rovnice.

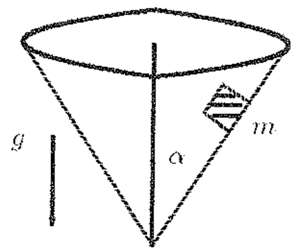
$$x = x \qquad y = Ax^2$$
$$\dot{x} = \dot{x} \qquad \dot{y} = 2Ax\dot{x}$$
$$\dot{x}^2 = \dot{x}^2 \qquad \dot{y}^2 = 4A^2x^2\dot{x}^2$$

$$T = \frac{1}{2}m(\dot{x}^2 + 4A^2x^2\dot{x}^2) = \frac{1}{2}m\dot{x}^2 + 2mA^2x^2\dot{x}^2$$
$$V = mgAx^2$$
$$L = \frac{1}{2}m\dot{x}^2 + 2mA^2x^2\dot{x}^2 - mgAx^2$$

$$\frac{\partial L}{\partial\dot{x}} = m\dot{x} + 4mA^2x^2\dot{x}$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = m\ddot{x} + 8mA^2x\dot{x}^2 + 4mA^2x^2\ddot{x}$$
$$\frac{\partial L}{\partial x} = 4mA^2x\dot{x}^2 - 2mgAx$$
$$m\ddot{x} + 8mA^2x\dot{x}^2 + 4mA^2x^2\ddot{x} = 4mA^2x\dot{x}^2 - 2mgAx$$
$$\boldsymbol{\ddot{x} + 4A^2x^2\ddot{x} + 4A^2x\dot{x}^2 + 2gAx = 0}$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

11. *Počet bodů: 2*
Částice o hmotnosti m klouže po vnitřní ploše kuželu. Osa kuželu je orientovaná vertikálně s vrcholem dolů. Zapište Lagrangeovy pohybové rovnice a vyjádřete moment hybnosti částice ve vertikálním směru (osa z).



2 stupně volnosti r, φ
r...vzdálenost tělesa od osy kužele
φ...úhel mezi tělesem a osou x

$$x = r\cos\varphi\sin\alpha$$
$$\dot{x} = \dot{r}\cos\varphi\sin\alpha - r\dot{\varphi}^2\sin\varphi\sin\alpha$$
$$\dot{x}^2 = (\dot{r}\cos\varphi\sin\alpha - r\dot{\varphi}^2\sin\varphi\sin\alpha)^2$$

$$y = r\sin\varphi\sin\alpha$$
$$\dot{y} = \dot{r}\sin\varphi\sin\alpha + r\dot{\varphi}^2\cos\varphi\sin\alpha$$
$$\dot{y}^2 = (\dot{r}\sin\varphi\sin\alpha + r\dot{\varphi}^2\cos\varphi\sin\alpha)^2$$

$$z = r\cos\alpha$$
$$\dot{z} = \dot{r}\cos\alpha$$
$$\dot{z}^2 = \dot{r}^2\cos^2\alpha$$

$$T = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2\sin^2\alpha$$
$$V = mgr\cos\alpha$$
$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\varphi}^2\sin^2\alpha - mgr\cos\alpha$$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}$$
$$\frac{\partial L}{\partial r} = mr\dot{\varphi}^2\sin^2\alpha - mg\cos\alpha$$
$$\boldsymbol{m\ddot{r} = mr\dot{\varphi}^2\sin^2\alpha - mg\cos\alpha}$$

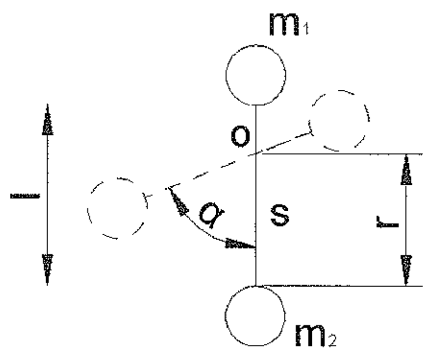
$$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}\sin^2\alpha$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 2mr\dot{r}\dot{\varphi}\sin^2\alpha + mr^2\ddot{\varphi}\sin^2\alpha$$
$$\frac{\partial L}{\partial \varphi} = 0$$
$$\boldsymbol{2r\dot{r}\dot{\varphi}\sin^2\alpha + r^2\ddot{\varphi}\sin^2\alpha = 0}$$

$$\underbrace{\frac{d}{dt}(mr^2\dot{\varphi}\sin^2\alpha) = 0}_{mr^2\dot{\varphi}\sin^2\alpha = l = const}$$
$$mr^2\dot{\varphi}\sin^2\alpha = mz^2\dot{\varphi}\frac{\sin^2}{\cos^2}\alpha = mz^2\dot{\varphi}\,tg^2\,\alpha = l$$
$$\boldsymbol{mz^2\dot{\varphi}\,tg^2\,\alpha = l}$$

l...moment hybnosti

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

12. *Počet bodů: 2*
Dvě částice o hmotnostech m_1, m_2 jsou spojeny tuhou tyčí zanedbatelné hmotnosti a délky l. Této činky má být použito jako kyvadla. V jaké vzdálenosti r od středu tyče musí procházet osa otáčení, má-li být doba kmitu tohoto kyvadla minimální?



$$T = \frac{1}{2} m_1 (l-r)^2 \dot{\alpha}^2 + \frac{1}{2} m_2 r^2 \dot{\alpha}^2$$
$$V = m_1 g (l-r) \cos\alpha - m_2 g r \cos\alpha$$
$$L = \frac{1}{2} m_1 (l-r)^2 \dot{\alpha}^2 + \frac{1}{2} m_2 r^2 \dot{\alpha}^2 - m_1 g (l-r) \cos\alpha + m_2 g r \cos\alpha$$

$$\frac{\partial L}{\partial \dot{\alpha}} = m_1 (l-r)^2 \dot{\alpha} + m_2 r^2 \dot{\alpha}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\alpha}} = m_1 (l-r)^2 \ddot{\alpha} + m_2 r^2 \ddot{\alpha}$$
$$\frac{\partial L}{\partial \alpha} = m_1 g (l-r) \sin\alpha - m_2 g r \sin\alpha$$

$$\ddot{\alpha} + \frac{m_2 g r \sin\alpha - m_1 g (l-r) \sin\alpha}{m_1 (l-r)^2 + m_2 r^2} = 0$$

Pro malé výchylky platí $\sin\alpha \doteq \alpha$

$$\ddot{\alpha} + \frac{\big(m_2 r - m_1 (l-r)\big) g}{m_1 (l-r)^2 + m_2 r^2} \alpha = 0$$

Jde o diferenciální rovnici 2. řádu kde

$$\omega^2 = \frac{\big(m_2 r - m_1 (l-r)\big) g}{m_1 (l-r)^2 + m_2 r^2}$$

Kde

$$\omega = \frac{2\pi}{T} \rightarrow aby\ T\ bylo\ minimální\ \rightarrow\ \omega\ musí\ být\ maximální\ \rightarrow\ \omega^2\ musí\ být\ maximální$$

$$f(r) = \frac{\big(m_2 r - m_1 (l-r)\big) g}{m_1 (l-r)^2 + m_2 r^2} = \frac{(m_1 + m_2) r g - m_1 l g}{(m_1 + m_2) r^2 - 2 m_1 l r + m_1 l^2}$$

$$\frac{df(r)}{dr} = \frac{(m_1 + m_2) g [(m_1 + m_2) r^2 - 2 m_1 l r + m_1 l^2] - [(m_1 + m_2) r g - m_1 l g][2(m_1 + m_2) r - 2 m_1 l]}{[(m_1 + m_2) r^2 - 2 m_1 l r + m_1 l^2]^2} = 0$$

$$(m_1 + m_2) g [(m_1 + m_2) r^2 - 2 m_1 l r + m_1 l^2] = [(m_1 + m_2) r g - m_1 l g][2(m_1 + m_2) r - 2 m_1 l]$$

Po úpravě:

$$-g (m_1 + m_2)^2 r^2 + 2 g l m_1 (m_1 + m_2) r + g l^2 m_1 (m_2 - m_1) = 0$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

$$D = 4g^2l^2m_1m_2(m_1+m_2)^2$$
$$\sqrt{D} = 4gl(m_1+m_2)\sqrt{m_1m_2}$$

$$r_{12} = \frac{-2glm_1(m_1+m_2) \pm 2gl(m_1+m_2)\sqrt{m_1m_2}}{-2g(m_1+m_2)^2}$$

$$r_{12} = \frac{lm_1 \mp l\sqrt{m_1m_2}}{m_1+m_2}\begin{cases} r_1 = \dfrac{lm_1 - l\sqrt{m_1m_2}}{m_1+m_2} \ldots NEVYHOVUJE \\ \boldsymbol{r_2 = \dfrac{lm_1 + l\sqrt{m_1m_2}}{m_1+m_2} \ldots VYHOVUJE} \end{cases}$$

Nejmenší periodu kmitů má kyvadlo ve vzdálenosti $\boldsymbol{r} = \frac{\boldsymbol{lm_1+l\sqrt{m_1m_2}}}{\boldsymbol{m_1+m_2}}$ od spodního závaží, tedy $\left|\boldsymbol{r} - \frac{\boldsymbol{l}}{2}\right|$ od středu.

13. *Počet bodů: 3*
Lagrangeova funkce nabité částice v elektromagnetickém poli popsaném skalárním potenciálem ϕ a vektorovým potenciálem $\vec{A}$ je

$$L = \frac{1}{2}mv^2 + q\vec{A}\vec{v} - \varphi.$$

(a) Nalezněte Lagrangeovy pohybové rovnice a ukažte, že pro zrychlení částice platí

$$m\frac{\mathrm{d}^2\vec{r}}{\mathrm{d}t^2} = q(\vec{E} + \vec{v}\times\vec{B}),$$

kde $\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t}$ a $\vec{B} = \nabla\times A$.
(b) Nalezněte Hamiltonovu funkci a Hamiltonovy pohybové rovnice. Je kanonická hybnost $\vec{p}$ rovna mechanické hybnosti $m\vec{v}$?

a)

V zadání je chyba má tam být $-q\varphi$

$$L = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) + q(A_xv_x + A_yv_y + A_zv_z) - q\varphi$$

$$\frac{\partial L}{\partial\dot{x}} = mv_x + qA_x$$
$$\frac{\partial L}{\partial\dot{y}} = mv_y + qA_y$$
$$\frac{\partial L}{\partial\dot{z}} = mv_z + qA_z$$

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{x}} = m\dot{v_x} + q\dot{A_x}$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{y}} = m\dot{v_y} + q\dot{A_y}$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{z}} = m\dot{v_z} + q\dot{A_z}$$

$$\frac{\partial L}{\partial x} = q\left(\frac{\partial A_x}{\partial x}v_x + \frac{\partial A_y}{\partial x}v_y + \frac{\partial A_z}{\partial x}v_z\right) - q\frac{\partial\varphi}{\partial x}$$
$$\frac{\partial L}{\partial y} = q\left(\frac{\partial A_x}{\partial y}v_x + \frac{\partial A_y}{\partial y}v_y + \frac{\partial A_z}{\partial y}v_z\right) - q\frac{\partial\varphi}{\partial y}$$
$$\frac{\partial L}{\partial z} = q\left(\frac{\partial A_x}{\partial z}v_x + \frac{\partial A_y}{\partial z}v_y + \frac{\partial A_z}{\partial z}v_z\right) - q\frac{\partial\varphi}{\partial z}$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

$$m\dot{v}_x + q\dot{A}_x = q\left(\frac{\partial A_x}{\partial x}v_x + \frac{\partial A_y}{\partial x}v_y + \frac{\partial A_z}{\partial x}v_z\right) - q\frac{\partial \varphi}{\partial x}$$
$$m\dot{v}_y + q\dot{A}_y = q\left(\frac{\partial A_x}{\partial y}v_x + \frac{\partial A_y}{\partial y}v_y + \frac{\partial A_z}{\partial y}v_z\right) - q\frac{\partial \varphi}{\partial y}$$
$$m\dot{v}_z + q\dot{A}_z = q\left(\frac{\partial A_x}{\partial z}v_x + \frac{\partial A_y}{\partial z}v_y + \frac{\partial A_z}{\partial z}v_z\right) - q\frac{\partial \varphi}{\partial z}$$

$$\vec{E} = -\left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z}\right) - q\left(\frac{\partial A_x}{\partial t}, \frac{\partial A_y}{\partial t}, \frac{\partial A_z}{\partial t}\right)$$
$$E_x = -\frac{\partial \varphi}{\partial x} - \frac{\partial A_x}{\partial t}$$
$$E_y = -\frac{\partial \varphi}{\partial y} - \frac{\partial A_y}{\partial t}$$
$$E_z = -\frac{\partial \varphi}{\partial z} - \frac{\partial A_z}{\partial t}$$

$$\vec{B} = \nabla \times \vec{A}$$
$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\vec{\imath} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\vec{k}$$
$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$
$$B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}$$
$$B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}$$

$$\vec{v} \times \vec{B} = \vec{v} \times \nabla \times \vec{A}$$
$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ v_x & v_y & v_z \\ \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} & \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} & \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \end{vmatrix} =$$
$$\left(\frac{\partial A_y}{\partial x}v_y - \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_z}{\partial x}v_z - \frac{\partial A_x}{\partial z}v_z\right)\vec{\imath} +$$
$$\left(\frac{\partial A_x}{\partial y}v_x - \frac{\partial A_y}{\partial x}v_x + \frac{\partial A_z}{\partial y}v_z - \frac{\partial A_y}{\partial z}v_z\right)\vec{\jmath} +$$
$$\left(\frac{\partial A_x}{\partial z}v_x - \frac{\partial A_z}{\partial x}v_x + \frac{\partial A_y}{\partial z}v_y - \frac{\partial A_z}{\partial y}v_y\right)\vec{k}$$

$$q\dot{A}_x = q\frac{dA_x}{dt} = q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z\right)$$
$$q\dot{A}_y = q\frac{dA_y}{dt} = q\left(\frac{\partial A_y}{\partial t} + \frac{\partial A_y}{\partial x}v_x + \frac{\partial A_y}{\partial y}v_y + \frac{\partial A_y}{\partial z}v_z\right)$$
$$q\dot{A}_z = q\frac{dA_z}{dt} = q\left(\frac{\partial A_z}{\partial t} + \frac{\partial A_z}{\partial x}v_x + \frac{\partial A_z}{\partial y}v_y + \frac{\partial A_z}{\partial z}v_z\right)$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

$$m\dot{v}_x + q\left(\frac{\partial A_x}{\partial t} + \frac{\partial A_x}{\partial x}v_x + \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_x}{\partial z}v_z\right) = q\left(\frac{\partial A_x}{\partial x}v_x + \frac{\partial A_y}{\partial x}v_y + \frac{\partial A_z}{\partial x}v_z\right) - q\frac{\partial \varphi}{\partial x}$$
$$m\dot{v}_y + q\left(\frac{\partial A_y}{\partial t} + \frac{\partial A_y}{\partial x}v_x + \frac{\partial A_y}{\partial y}v_y + \frac{\partial A_y}{\partial z}v_z\right) = q\left(\frac{\partial A_x}{\partial y}v_x + \frac{\partial A_y}{\partial y}v_y + \frac{\partial A_z}{\partial y}v_z\right) - q\frac{\partial \varphi}{\partial y}$$
$$m\dot{v}_z + q\left(\frac{\partial A_z}{\partial t} + \frac{\partial A_z}{\partial x}v_x + \frac{\partial A_z}{\partial y}v_y + \frac{\partial A_z}{\partial z}v_z\right) = q\left(\frac{\partial A_x}{\partial z}v_x + \frac{\partial A_y}{\partial z}v_y + \frac{\partial A_z}{\partial z}v_z\right) - q\frac{\partial \varphi}{\partial z}$$

$$m\dot{v}_x = q\left(\frac{\partial A_y}{\partial x}v_y - \frac{\partial A_x}{\partial y}v_y + \frac{\partial A_z}{\partial x}v_z - \frac{\partial A_x}{\partial z}v_z\right) - q\frac{\partial \varphi}{\partial x} - q\frac{\partial A_x}{\partial t}$$
$$m\dot{v}_y = q\left(\frac{\partial A_x}{\partial y}v_x - \frac{\partial A_y}{\partial x}v_x + \frac{\partial A_z}{\partial y}v_z - \frac{\partial A_y}{\partial z}v_z\right) - q\frac{\partial \varphi}{\partial y} - q\frac{\partial A_y}{\partial t}$$
$$m\dot{v}_z = q\left(\frac{\partial A_x}{\partial z}v_x - \frac{\partial A_z}{\partial x}v_x + \frac{\partial A_y}{\partial z}v_y - \frac{\partial A_z}{\partial y}v_y\right) - q\frac{\partial \varphi}{\partial z} - q\frac{\partial A_z}{\partial t}$$

$$m\dot{v}_x = q\left(E_x + \left(\vec{v}\times\nabla\times\vec{A}\right)_x\right)$$
$$m\dot{v}_y = q\left(E_y + \left(\vec{v}\times\nabla\times\vec{A}\right)_y\right)$$
$$m\dot{v}_z = q\left(E_x + \left(\vec{v}\times\nabla\times\vec{A}\right)_z\right)$$

$$m\,\frac{d^2\vec{r}}{dt^2} = q(\vec{E} + \vec{v}\times\vec{B})$$

b)

$$H = \sum p_k \dot{q}_k - L$$
$$H = mv_x^2 + qA_xv_x + mv_y^2 + qA_yv_y + mv_z^2 + qA_zv_z - \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) - q\left(A_xv_x + A_yv_y + A_zv_z\right) + q\varphi$$
$$H = \frac{1}{2}m\left(v_x^2 + v_y^2 + v_z^2\right) + q\varphi$$
$$H = \frac{1}{2}\frac{p_x^2 + p_y^2 + p_z^2 - 2p_xqA_x - 2p_yqA_y - 2p_zqA_z + q^2A_x^2 + q^2A_y^2 + q^2A_z^2}{m} + q\varphi$$

$$\dot{x} = \frac{p_x}{m} - \frac{qA_x}{m}$$
$$\dot{y} = \frac{p_y}{m} - \frac{qA_y}{m}$$
$$\dot{z} = \frac{p_z}{m} - \frac{qA_z}{m}$$

$$-\dot{p}_x = \frac{-p_xq\frac{\partial A_x}{\partial x} + q^2A_x\frac{\partial A_x}{\partial x}}{m} + q\frac{\partial \varphi}{\partial x}$$
$$-\dot{p}_y = \frac{-p_yq\frac{\partial A_y}{\partial y} + q^2A_y\frac{\partial A_y}{\partial y}}{m} + q\frac{\partial \varphi}{\partial y}$$
$$-\dot{p}_z = \frac{-p_zq\frac{\partial A_z}{\partial z} + q^2A_x\frac{\partial A_z}{\partial z}}{m} + q\frac{\partial \varphi}{\partial z}$$

Dle $\frac{\partial L}{\partial \vec{r}}$ je patrné, že kanonická hybnost $\vec{p}$ není rovna mechanické hybnosti $m\vec{v}$

14. *Počet bodů: 2*
Řetěz délky l je položen rovně na stole tak, že jeho úsek délky a visí volně přes hranu stolu. V okamžiku $t = 0$ řetěz uvolníme, takže začne bez tření klouzat ze stolu. Vyšetřete pohyb řetězu. Za jakou dobu sklouzne řetěz celý ze stolu, je-li $l = 2$ m a $a = 0,2$ m?

l...délka lana
ρ...délková hustota

$$T = \frac{1}{2}l\,\rho\,\dot{y}^2$$
$$dV = -dy\,\rho\,g\,y$$
$$V = -\rho\,g\int y\,dy = -\rho\,g\frac{y^2}{2}$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

$$L = \frac{1}{2} l \rho \dot{y}^2 + \frac{1}{2} \rho g y^2$$

$$\frac{\partial L}{\partial \dot{y}} = l \rho \dot{y}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} = l \rho \ddot{y}$$

$$\frac{\partial L}{\partial y} = \rho g y$$

$$\ddot{y} - \frac{g}{l} y = 0$$

$$\lambda^2 - \frac{g}{l} = 0$$

$$\lambda_{12} = \pm\sqrt{\frac{g}{l}}$$

$$y = C_1 e^{\sqrt{\frac{g}{l}}t} + C_2 e^{-\sqrt{\frac{g}{l}}t}$$

$$\dot{y} = C_1\sqrt{\frac{g}{l}} e^{\sqrt{\frac{g}{l}}t} - C_2\sqrt{\frac{g}{l}} e^{-\sqrt{\frac{g}{l}}t}$$

Počáteční podmínky:

$$t = 0 \rightarrow y = a$$

$$t = 0 \rightarrow \dot{y} = 0$$

$$a = C_1 + C_2$$

$$0 = C_1\sqrt{\frac{g}{l}} - C_2\sqrt{\frac{g}{l}}$$

$$C_1 = C_2 = \frac{a}{2}$$

$$y = \frac{a}{2} e^{\sqrt{\frac{g}{l}}t} + \frac{a}{2} e^{-\sqrt{\frac{g}{l}}t}$$

Dle zadání:

$$l = 2m$$

$$a = 0{,}2m$$

$$y = 0$$

$$2 = 0{,}1 e^{\sqrt{\frac{g}{2}}t} + 0{,}1 e^{-\sqrt{\frac{g}{2}}t}$$

$$e^{\sqrt{\frac{g}{2}}t + \ln 20} = e^{2\sqrt{\frac{g}{2}}t}$$

$$\sqrt{\frac{g}{2}}t + \ln 20 = 2\sqrt{\frac{g}{2}}t$$

$$\ln 20 = \sqrt{\frac{g}{2}}t$$

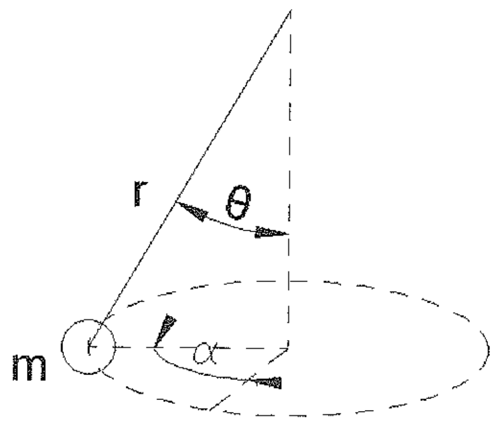
$$t = \sqrt{\frac{2}{g}}\ln 20$$

$$t = 1{,}35s$$

V čase $t = 1,35s$ spadne celý řetěz ze stolu.

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

15. *Počet bodů: 2*
Najděte pohybové rovnice sférického kyvadla (jde o matematické kyvadlo, které nemusí kývat v rovině). Za zobecněné souřadnice zvolte úhly θ, φ známé ze sférických souřadnic. Jaká veličina se zachovává během pohybu?



$$x = r\sin\theta\cos\alpha$$
$$y = r\sin\theta\sin\alpha$$
$$z = r\cos\theta$$

$$\dot{x} = r(\cos\theta\ \dot{\theta}\cos\alpha - \sin\theta\sin\alpha\ \dot{\alpha})$$
$$\dot{y} = r(\cos\theta\ \dot{\theta}\sin\alpha + \sin\theta\cos\alpha\ \dot{\alpha})$$
$$\dot{z} = -r\sin\theta\ \dot{\theta}$$

$$\dot{x}^2 = r^2\cos^2\theta\ \dot{\theta}^2\cos^2\alpha - 2r^2\cos\theta\ \dot{\theta}\cos\alpha\sin\theta\sin\alpha\ \dot{\alpha} + r^2\sin^2\theta\sin^2\alpha\ \dot{\alpha}^2$$
$$\dot{y}^2 = r^2\cos^2\theta\ \dot{\theta}^2\sin^2\alpha + 2r^2\cos\theta\ \dot{\theta}\sin\alpha\sin\theta\cos\alpha\ \dot{\alpha} + r^2\sin^2\theta\cos^2\alpha\ \dot{\alpha}^2$$
$$\dot{z}^2 = r^2\sin^2\theta\ \dot{\theta}^2$$

$$T = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\alpha}^2$$
$$V = mgr\cos\theta$$
$$L = \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}mr^2\sin^2\theta\,\dot{\alpha}^2 - mgr\cos\theta$$

$$\frac{\partial L}{\partial\dot{\theta}} = mr^2\dot{\theta}$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\theta}} = mr^2\ddot{\theta}$$
$$\frac{\partial L}{\partial\theta} = mr^2\sin\theta\cos\theta\,\dot{\alpha}^2 + mgr\sin\theta$$
$$\boldsymbol{mr^2\ddot{\theta} = mr^2\sin\theta\cos\theta\,\dot{\alpha}^2 + mgr\sin\theta}$$

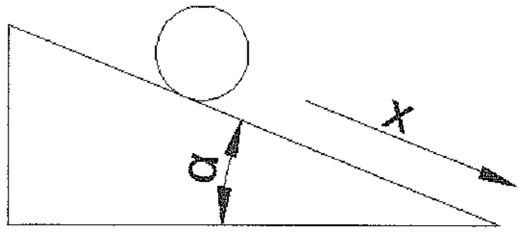
$$\frac{\partial L}{\partial\dot{\alpha}} = mr^2\sin^2\theta\,\dot{\alpha}$$
$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\alpha}} = mr^2\sin^2\theta\,\ddot{\alpha}$$
$$\frac{\partial L}{\partial\alpha} = 0$$
$$\boldsymbol{mr^2\sin^2\theta\,\ddot{\alpha} = 0}$$

$$\boldsymbol{\frac{d}{dt}(mr^2\sin^2\theta\,\dot{\alpha}) = 0 \ \rightarrow mr^2\sin^2\theta\,\ddot{\alpha} = l = const}$$

Moment hybnosti vzhledem k ose se zachovává

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

16. *Počet bodů: 1*
Válec o poloměru r_1 a hustotě ρ_1 a koule o poloměru r_2 a hustotě ρ_2 se valí po téže nakloněné rovině. Které z obou těles dosáhne většího zrychlení? Jak závisí odpověď na hodnotách $\rho_{1,2}$ a $r_{1,2}$?



$$\omega = \frac{v}{r}$$

Koule

$$J_k = \frac{2}{5} m_2 r_2^2$$
$$T = \frac{1}{2} J_k \omega^2 + \frac{1}{2} m v^2 = \frac{1}{5} m_2 \dot{x}_2^2 + \frac{1}{2} m_2 \dot{x}_2^2 = \frac{7}{10} m_2 \dot{x}_2^2$$
$$V = -m_2 g x \sin\alpha$$
$$L = \frac{7}{10} m_2 \dot{x}_2^2 + m_2 g x_2 \sin\alpha$$

$$\frac{\partial L}{\partial \dot{x}_2} = \frac{7}{5} m_2 \dot{x}_2$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_2} = \frac{7}{5} m_2 \ddot{x}_2$$
$$\frac{\partial L}{\partial x_2} = m_2 g \sin\alpha$$

$$\frac{7}{5} m_2 \ddot{x}_2 = m_2 g \sin\alpha$$
$$\boldsymbol{\ddot{x}_2 = \frac{5}{7} g \sin\alpha}$$

Válec

$$J_v = \frac{1}{2} m_1 r_1^2$$
$$T = \frac{1}{2} J_v \omega^2 + \frac{1}{2} m v^2 = \frac{1}{4} m_1 \dot{x}_1^2 + \frac{1}{2} m_1 \dot{x}_1^2 = \frac{3}{4} m_1 \dot{x}_1^2$$
$$V = -m_1 g x \sin\alpha$$
$$L = \frac{3}{4} m_1 \dot{x}_1^2 + m_1 g x_1 \sin\alpha$$

$$\frac{\partial L}{\partial \dot{x}_1} = \frac{3}{2} m_1 \dot{x}_1$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_1} = \frac{3}{2} m_1 \ddot{x}_1$$
$$\frac{\partial L}{\partial x_1} = m_1 g \sin\alpha$$
$$\frac{3}{2} m_1 \ddot{x}_1 = m_1 g \sin\alpha$$
$$\boldsymbol{\ddot{x}_1 = \frac{2}{3} g \sin\alpha}$$

Zrychlení koule je vždy větší než zrychlení válce bez závislosti na ρ a r

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

24. *Počet bodů: 1*
Vypočtěte Poissonovy závorky složek L_x, L_y vektoru momentu hybnosti a dále Poissonovu závorku $\{L^2, L_z\}$, kde $L^2 = L_x^2 + L_y^2 + L_z^2$.

$$L = r \times p$$

$$\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x & y & z \\ p_x & p_y & p_z \end{vmatrix} = (yp_z - zp_y)\vec{\imath} + (zp_x - xp_z)\vec{\jmath} + (xp_y - yp_x)\vec{k}$$

$$L_x = yp_z - zp_y$$
$$L_y = zp_x - xp_z$$
$$L_z = xp_y - yp_x$$

$$L^2 = y^2p_z^2 - 2yp_zzp_y + z^2p_y^2 + z^2p_x^2 - 2zp_xxp_z + x^2p_z^2 + x^2p_y^2 - 2xp_yyp_x + y^2p_x^2$$

$$\{L_x, L_y\} = \left[\frac{\partial L_x}{\partial x}\frac{\partial L_y}{\partial p_x} - \frac{\partial L_x}{\partial p_x}\frac{\partial L_y}{\partial x}\right] + \left[\frac{\partial L_x}{\partial y}\frac{\partial L_y}{\partial p_y} - \frac{\partial L_x}{\partial p_y}\frac{\partial L_y}{\partial y}\right] + \left[\frac{\partial L_x}{\partial z}\frac{\partial L_y}{\partial p_z} - \frac{\partial L_x}{\partial p_z}\frac{\partial L_y}{\partial z}\right]$$

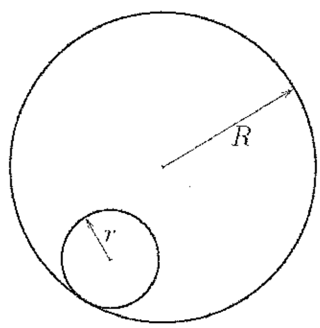
$$\{L_x, L_y\} = p_yx - yp_x$$

$$\{L^2, L_z\} = \left[\frac{\partial L^2}{\partial x}\frac{\partial L_z}{\partial p_x} - \frac{\partial L^2}{\partial p_x}\frac{\partial L_z}{\partial x}\right] + \left[\frac{\partial L^2}{\partial y}\frac{\partial L_z}{\partial p_y} - \frac{\partial L^2}{\partial p_y}\frac{\partial L_z}{\partial y}\right] + \left[\frac{\partial L^2}{\partial z}\frac{\partial L_z}{\partial p_z} - \frac{\partial L^2}{\partial p_z}\frac{\partial L_z}{\partial z}\right]$$

$$\{L^2, L_z\} =$$
$$(2xp_z^2 + 2xp_y^2 - 2zp_xp_z - 2p_yyp_x)(-y) - (2z^2p_x + 2y^2p_x - 2zxp_z - 2xp_yy)p_y +$$
$$(2yp_z^2 + 2yp_x^2 - 2p_zzp_y - 2xp_yp_x)x - (2z^2p_y + 2x^2p_y - 2yp_zz - 2xyp_x)(-p_x)$$

$$\{L^2, L_z\} = 0$$

25. *Počet bodů: 2*
Homogenní válec o poloměru r se valí uvnitř válcové dutiny o poloměru R. Najděte jeho Lagrangeovu funkci a určete frekvenci malých kmitů válce.



$$J = \frac{3}{2}mr^2$$
$$v = R\dot{\varphi} + (R - 2r)\dot{\varphi}$$

$$\omega = \frac{v}{2r}$$
$$\omega = \frac{(R - r)\dot{\varphi}}{r}$$

$$T = \frac{1}{2}J\omega^2$$
$$T = \frac{3}{4}m(R - r)^2\dot{\varphi}^2$$
$$V = -mg(R - r)\cos\varphi$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

$$L = \frac{3}{4} m(R-r)^2 \dot{\varphi}^2 + mg(R-r)\cos\varphi$$

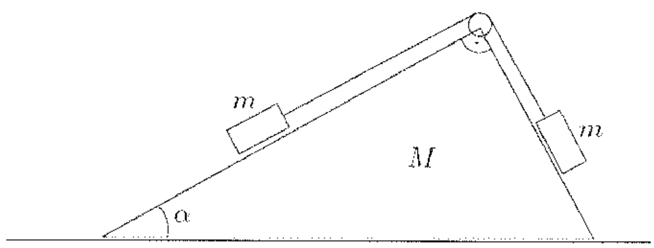
$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{3}{2} m(R-r)^2 \dot{\varphi}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = \frac{3}{2} m(R-r)^2 \ddot{\varphi}$$
$$\frac{\partial L}{\partial \varphi} = -mg(R-r)\sin\varphi$$

$$\frac{3}{2} m(R-r)^2 \ddot{\varphi} + mg(R-r)\sin\varphi = 0$$
$$\ddot{\varphi} + \frac{2}{3}\frac{g}{(R-r)}\sin\varphi = 0$$

Pro malé výchylky platí $\sin\alpha \doteq \alpha$
$$\ddot{\varphi} + \frac{2}{3}\frac{g}{(R-r)}\varphi = 0$$

$$\omega = \sqrt{\frac{2}{3}\frac{g}{(R-r)}} = 2\pi f$$
$$\boldsymbol{f = \frac{1}{2\pi}\sqrt{\frac{2}{3}\frac{g}{(R-r)}}}$$

26. *Počet bodů: 2*
Určete zrychlení klínu na obrázku. Všechna tření zanedbáváme.



r...délka lana od prvního tělesa ke kladce
l...délka lana
Počet stupňů volnosti $2 - r, x$

$$x_1 = x \qquad y_1 = 0$$
$$\dot{x}_1 = \dot{x} \qquad \dot{y}_1 = 0$$
$$\dot{x}_1^2 = \dot{x}^2 \qquad \dot{y}_1^2 = 0$$

$$x_2 = x - r\cos\alpha \qquad y_2 = -r\sin\alpha$$
$$\dot{x}_2 = \dot{x} - \dot{r}\cos\alpha \qquad \dot{y}_2 = -\dot{r}\sin\alpha$$
$$\dot{x}_2^2 = \dot{x}^2 - 2\dot{x}\dot{r}\cos\alpha + \dot{r}^2\cos^2\alpha \qquad \dot{y}_2^2 = \dot{r}^2\sin^2\alpha$$

$$x_3 = x + (l-r)\sin\alpha \qquad y_3 = (l-r)\cos\alpha$$
$$\dot{x}_3 = \dot{x} - \dot{r}\sin\alpha \qquad \dot{y}_3 = -\dot{r}\cos\alpha$$
$$\dot{x}_3^2 = \dot{x}^2 - 2\dot{x}\dot{r}\sin\alpha + \dot{r}^2\sin^2\alpha \qquad \dot{y}_3^2 = \dot{r}^2\cos^2\alpha$$

Michal PAVERA 77475
VUT – Ústav fyzikálního inženýrství

$$T = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\dot{x}^2 - m\dot{x}\dot{r}\cos\alpha + \frac{1}{2}m\dot{r}^2\cos^2\alpha + \frac{1}{2}m\dot{r}^2\sin^2\alpha + \frac{1}{2}m\dot{x}^2 - m\dot{x}\dot{r}\sin\alpha + \frac{1}{2}m\dot{r}^2\sin^2\alpha + \frac{1}{2}m\dot{r}^2\cos^2\alpha$$

$$T = \frac{1}{2}M\dot{x}^2 + m\dot{x}^2 - m\dot{x}\dot{r}(\cos\alpha + \sin\alpha) + m\dot{r}^2$$

$$V = -mgr\sin\alpha + mg(l - r)\cos\alpha$$

$$L = \frac{1}{2}M\dot{x}^2 + m\dot{x}^2 - m\dot{x}\dot{r}(\cos\alpha + \sin\alpha) + m\dot{r}^2 + mgr(\sin\alpha + \cos\alpha) - mgl\cos\alpha$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + 2m\dot{x} - m\dot{r}(\cos\alpha + \sin\alpha)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = M\ddot{x} + 2m\ddot{x} - m\ddot{r}(\cos\alpha + \sin\alpha)$$

$$\frac{\partial L}{\partial x} = 0$$

$$\boldsymbol{M\ddot{x} + 2m\ddot{x} - m\ddot{r}(\cos\alpha + \sin\alpha) = 0}$$

$$\frac{\partial L}{\partial \dot{r}} = -m\dot{x}(\cos\alpha + \sin\alpha) + 2m\dot{r}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = -m\ddot{x}(\cos\alpha + \sin\alpha) + 2m\ddot{r}$$

$$\frac{\partial L}{\partial r} = mg(\sin\alpha + \cos\alpha)$$

$$\boldsymbol{-m\ddot{x}(\cos\alpha + \sin\alpha) + 2m\ddot{r} = mg(\sin\alpha + \cos\alpha)}$$

$$\boldsymbol{\ddot{r} = \frac{1}{2}(\sin\alpha + \cos\alpha)(g + \ddot{x})}$$

$$\boldsymbol{\ddot{x} = \frac{mg(\cos\alpha + \sin\alpha)^2}{2M + 4m - m(\cos\alpha + \sin\alpha)^2}}$$