Intro...

Hopf Fibration: Geometry of pure states

- . . . part of my Diploma Thesis
- First ideas: Ekert, Knight (1995)
- Big step: Bernevig, Chen (2003) Levay (2003)
- Kiss

Background 1.: Complex projective space

- Complex projective space $\mathbb{C}P^{n-1}$ is a set of equivalence classes in $\mathbb{C}^n$ defined by the equivalence relation

$$|\psi\rangle \sim \lambda|\psi\rangle \quad \text{for} \quad \lambda \in \mathbb{C} \setminus \{0\}$$

- State of a n-level quantum system is represented by point in $\mathbb{C}P^{n-1}$. State of a qubit is represented by point in $\mathbb{C}P^1$
- Complex projective space is not a linear space, i.e. sum of two points is not defined.

Background 2.: Spheres and Sets of unit vectors

- S^n denotes a sphere in $\mathbb{R}^{n+1}$, i.e. set of points that satisfy the condition $\sum x_j^2 = 1$.
- The set of unit vectors in $\mathbb{C}^n \equiv \mathbb{R}^{2n}$ is a Sphere S^{2n-1}.
- S^{2n-1} is not a linear space.

Background 3.: NDAs

- Normed division algebra is
  - vector space over $\mathbb{R}$
  - …endowed with multiplication
  - …no divisors of zero: $ab = 0$ means that $a = 0$ or $b = 0$
  - …multiplication is norm-preserving: $\|ab\| = \|a\|\|b\|$
- Theorem: there are only four NDAs. The Real ($\mathbb{R}$), Complex ($\mathbb{C}$), Quaternions ($\mathbb{H}$), Octonions ($\mathbb{O}$)
- Corresponding dimensions of the vector spaces are $1, 2, 4, 8$.

What and Why?

- What:
  - Geometrical relation between the Hilbert space, the set of unit vectors and the projective space.
  - . . . just for some specific (small) dimensions of the Hilbert space.
- Why:
  - . . . it's interesting, it's fun. (How to understand our thinking?)
  - Operator concept vs. projective space
  - The concept of Geometrical phase (Berry's phase) works naturally in set of unit vectors. . .
  - Hopf fibration is entanglement sensitive. (to be explained later)

The Bloch sphere

- Take a normalized state of a qubit

$$|\psi\rangle = x_0|0\rangle + x_1|1\rangle, \qquad |x_0|^2 + |x_1|^2 = 1$$

- ...and make a mapping $\pi : (x_0, x_1) \longrightarrow (\xi_0, \xi_1, \xi_2)$

$$\langle\psi|\sigma_z|\psi\rangle = |x_0|^2 - |x_1|^2 \equiv \xi_0$$
$$\langle\psi|\sigma_x|\psi\rangle = \bar{x}_0 x_1 + \bar{x}_1 x_0 = \mathfrak{Re}(2x_1\bar{x}_0) \equiv \xi_1$$
$$\langle\psi|\sigma_y|\psi\rangle = \mathrm{i}\bar{x}_0 x_1 - \mathrm{i}\bar{x}_1 x_0 = \mathfrak{Im}(2x_1\bar{x}_0) \equiv \xi_2$$
$$\rho_\psi = \frac{1}{2}\begin{pmatrix} 1+\xi_0 & \xi_1 - \mathrm{i}\xi_2 \\ \xi_1 + \mathrm{i}\xi_2 & 1-\xi_0 \end{pmatrix}$$

- ...these are coordinates on a sphere S^2:

$${\xi_0}^2 + {\xi_1}^2 + {\xi_2}^2 = 1$$

Hopf Fibration

- The concept of Bloch sphere uses mapping from the set of unit vectors to S^2.
- Points $|\psi\rangle$ and $\mathrm{e}^{\mathrm{i}\phi}|\psi\rangle$ are mapped to the same point on the target S^2. So essentially the S^2 represents $\mathbb{C}P^1$.
- The whole idea can be written as $S^3 \xrightarrow{S^1} S^2$. All this is caled Fibred Manifold.
- S^3 is the Total space, S^1 the Fibre and S^2 the Base space.
- Locally, S^3 looks like a cartesian product of S^2 and S^1.

Hopf Fibration: visualization

We can visualize S^3 via stereographic projection.

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

$$X = \frac{x_1}{1 - x_4}, \qquad Y = \frac{x_2}{1 - x_4}, \qquad Z = \frac{x_3}{1 - x_4}$$

- Properties of the visualization map:
  - The south pole is mapped to the origin
  - The equator is a unit sphere $X^2 + Y^2 + Z^2 = 1$
  - The north pole is in infinity...
  - The meridians are mapped to straight lines form origin.
- The visualization map preserves all important geometrical properties of S^3.

Hopf Fibration: pictures no.1

We want to see how are the S^1 fibres embeded in the S^3:

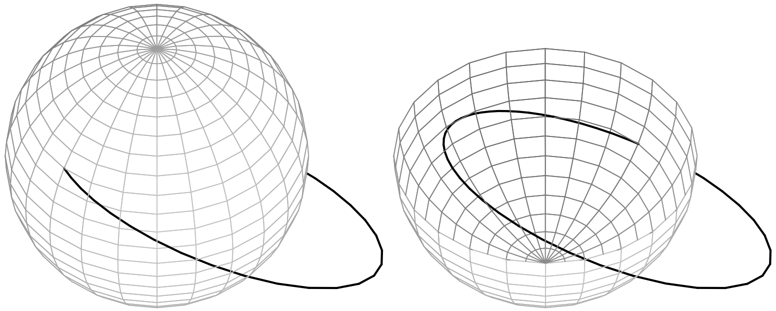

Figure 1: Fibre-circle for parameters $\theta = \frac{\pi}{3}, \phi = 0$.

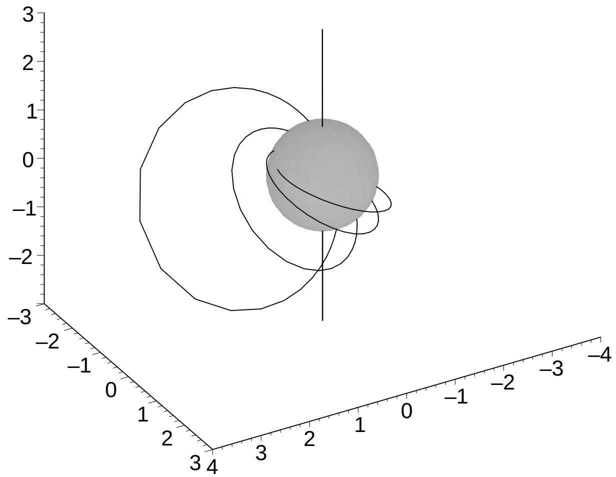

Figure 2: Dependence of radius and inclination of the fibre-circle on θ. Fibres with parameters $(\theta, \phi) = (\frac{\pi}{4}, 0), (\frac{\pi}{3}, \frac{\pi}{6}), (\frac{\pi}{2}, \frac{\pi}{4}), (\frac{2\pi}{3}, \frac{\pi}{3}), (\pi, 0)$ The parameter ϕ was changed just to make the picture more comprehensible.

Hopf Fibration: pictures no.2

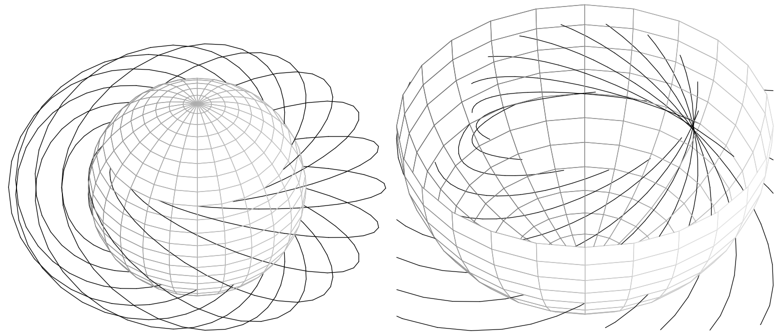

Figure 3: Dependence of position of a fibre-circle on ϕ. Difference in ϕ for two neighbouring fibre-circles is $\frac{\pi}{4}$, $\theta = \frac{\pi}{2}$ for all pictured fibre-circles.

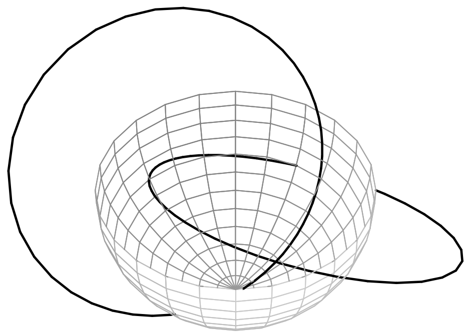

Figure 4: Linking of the two fibre-circles.

Hopf Fibration: general concept

- There are three Hopf Fibrations:

$$S^3 \xrightarrow{S^1} S^2 \qquad S^7 \xrightarrow{S^3} S^4 \qquad S^{15} \xrightarrow{S^7} S^8$$

- Corresponding Total spaces are sets of unit vectors in $\mathbb{C}^2$, $\mathbb{H}^2 \equiv \mathbb{C}^4$ and $\mathbb{O}^2 \equiv \mathbb{C}^8$ respectively. Those are pure states of one, two and three qubits, respectively.

- The prescribtion for fibration mapping has the general form

$$\pi : (x_0, x_1) \longrightarrow (\xi_0, \xi_i) \in S^n$$

$$\xi_0 = |x_0|^2 - |x_1|^2, \qquad \xi_i = \mathrm{i}^{th}\mathfrak{part}\,(2x_1\bar{x}_0) \tag{1}$$

Here $\mathrm{i}^{th}\mathfrak{part}$ is a generalisation of $\mathfrak{Re}$ and $\mathfrak{Im}$ for higher division algebras. So the expression for ξ_i means simply:

$$2x_1\bar{x}_0 = \xi_1 + \mathrm{i}\xi_2 \qquad \text{for} \quad x_0, x_1 \in \mathbb{C}$$

$$2x_1\bar{x}_0 = \xi_1 + \vec{i}\xi_2 + \vec{j}\xi_3 + \vec{k}\xi_4 \qquad \text{for} \quad x_0, x_1 \in \mathbb{H}$$

$$2x_1\bar{x}_0 = \xi_1 + \mathrm{i}_1\xi_2 + \cdots + \mathrm{i}_7\xi_8 \qquad \text{for} \quad x_0, x_1 \in \mathbb{O}$$

- The hopf fibrations are unique concepts!

Second HF and two-qubit entanglement

- The two qubit state reads:
$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$$
$$z \equiv \bar{a}c + \bar{b}d \qquad \zeta \equiv \bar{a}b + \bar{c}d \qquad w \equiv ad - bc.$$
The parameter w is called concurence and is measure of entanglement.

- introduce the quaternionic notation:
$$\begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} a + b\vec{j} \\ c + d\vec{j} \end{pmatrix}$$
The Hopf fibration $\pi : S^7 \rightarrow S^4$ reads:
$$(x_0, x_1) \rightarrow (\xi_0, \xi_i) \equiv (|x_0|^2 - |x_1|^2, 2x_1\overline{x_0})$$
$$\xi_0 = \pm\sqrt{1 - 4|z|^2 - 4|w|^2}, \qquad \xi_1 + \mathrm{i}\xi_2 = 2z, \qquad \xi_3 + \mathrm{i}\xi_4 = 2w.$$

- This means that the Hopf fibration is **entanglement sensitive**. The separable states are mapped to the points of S^4 with vanishing coordinates ξ_3, ξ_4, i.e. they are mapped on $S^2 \subset S^4$ described by the constraint $\xi_0^2 + \xi_1^2 + \xi_2^2 = 1$. The maximally entangled states have $w = \frac{1}{2}$, then $\xi_0 = \xi_1 = \xi_2 = 0$.

- Coordinates ξ on are invariant with respect to multiplication of the quaternionic spinor by a unit quaternion q from the right. This mathematical operation corresponds to the local transformation of the second qubit.

Third HF and three-qubit entanglement

- A pure 3-qubit state reads:

$$|\psi\rangle = a_0|000\rangle + a_1|001\rangle + a_2|010\rangle + a_3|011\rangle + b_0|100\rangle + b_1|101\rangle + b_2|110\rangle + b_3|111\rangle$$
$$\sum |a_i|^2 = 1.$$

- Unlike the two-qubit entanglement, here we have three ways how one qubit could be separated from the other two. If the first qubit is separated, then

$$|\psi\rangle = (c_0|0\rangle + c_1|1\rangle) \otimes (d_0|00\rangle + d_1|01\rangle + d_2|10\rangle + d_3|11\rangle)$$

- We represent the 3-qubit state using the Cayley-Dickinson construction:

$$x_0 = ((a_0^\star, a_1), (a_2, a_3^\star)) \qquad x_1 = ((b_0^\star, b_1), (b_2, b_3^\star))$$

- Then the Hopf fibration gives:

$$2x_1\overline{x_0} = ((C_0, C_1), (C_2, C_3))$$
$$C_0 = a_0 b_0^\star + a_1 b_1^\star + a_2 b_2^\star + a_3 b_3^\star$$
$$C_1 = (a_0 b_1 - a_1 b_0) + (a_3 b_2 - a_2 b_3)^\star$$
$$C_2 = (a_0 b_2 - a_2 b_0) + (a_1 b_3 - a_3 b_1)^\star$$
$$C_3 = (a_2 b_1 - a_1 b_2) + (a_0 b_3 - a_3 b_0)^\star$$

.

- separable states are mapped onto

$$2x_1\overline{x_0} = ((C_0, 0), (0, 0)),$$

i.e. "purely complex" octonions. By showing this we conclude that the third Hopf fibration is also entanglement sensitive.

The whole beautiful picture

The End

There is much more of interesting differential geometry

- metric properties (entanglement as a distance from a separable state)
- connection form (parallel transport, Berry's geometric phase)
- ... but we can do this only for small dimensions
- ... and only for pure states