# Absolute optical instruments, classical superintegrability, and separability of the Hamilton-Jacobi equation

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**Abstract**

An absolute optical instrument is a region of space, typically defined by a spatially-varying index of refraction, in which bound ray trajectories are closed. Traditional examples of such devices include Maxwell's fisheye and the Eaton and Luneburg lenses. In this work we employ the close analogy between classical mechanics and geometrical optics to develop a general theory of absolute instruments based on the Hamilton-Jacobi equation. Based on this theory, we derive many general properties of absolute instruments, and design a number of previously unknown examples. We also show how absolute optical instruments are related to superintegrable systems in mechanics and that the optical case is much less restrictive, which leads to an immense design space of absolute optical instruments.

## 1 Introduction

The ability to engineer a refractive index distribution to accomplish a particular optical task without resorting to trial-and-error numerical computation is a long-sought goal in classical optics. With the advent of transformation optics [1, 2] and fabrication advances in recent years which allow the construction of spatially graded refractive index distributions both in two (2D) [3] and three dimensions (3D) [4], there has also been a renewed interest in gradient index optics in general. Here, we present a general and comprehensive theory which allows construction of an entire class of optical devices from first principles–the absolute instruments (AIs) [5, 6, 7, 8, 9]–which cannot be designed through transformation optics and which, until now, only specific examples were known. These lenses are fascinating because, unlike standard convex or concave lenses which have an image plane, an AI can stigmatically image every point in a 3D region of space. Two well-known examples of AIs that were both mentioned in the classic *Principles of Optics* textbook by Max Born and Emil Wolf [5] are a plane mirror (all real points on one side of the mirror are virtually imaged to the half-space beyond the mirror) and Maxwell's fisheye (all real points have real images across a spherical inversion) which is shown in Figure 1 (a). In addition to the two AIs already mentioned, a few other lenses have been discovered within the last two centuries, such as the Eaton lens [10], the Luneburg lens [7] as shown in Figure 1(b), the Miñano lens [8], and the Lissajous lens [11]. Until now, known 3D AIs other than

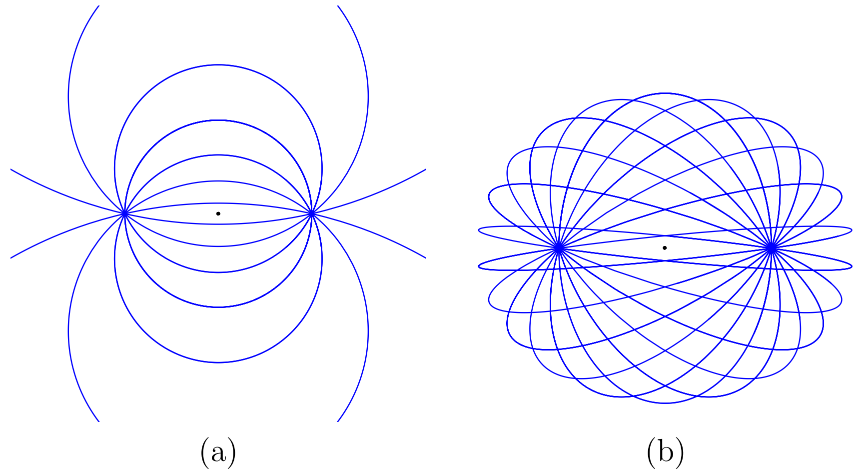


Figure 1: Ray trajectories in the (a) Maxwell fisheye (b) Luneburg lens/harmonic oscillator. The centre of the potential is marked by a black dot.

the Lissajous lens have all had spherical symmetry in the refractive index profile, and this specific case has been well studied [9, 12, 13].

In this work we focus on absolute instruments with bound rays. They have the following known unique properties [5, 8, 9, 14, 15]:

1. Under well-defined limitations, ray trajectories are closed; light follows periodic motion.
2. Every point in space is stigmatically imaged to either itself or to at least one other point. In other words, rays emanating from one point in space will, at some later time in a bound system, converge to a single point.
3. Wave optically, energy passing through a point in space will eventually return to the same point in space in a bound system.
4. The frequency spectrum of the eigenmodes is equidistantly spaced and degenerate, at least approximately [14, 15].

These properties are very restrictive. Our task in this work is to find other general properties that absolute instruments have in common, and use them to design new such devices. The ultimate goal is then to find all possible refractive index distributions $n(\vec{r})$ that result in an absolute optical instrument with bound light trajectories. It is impossible to reach this goal with the methods of transformation optics, coordinate transformations, or other usual methods of gradient index lens design. Instead, we use methods of Hamiltonian mechanics and the Hamilton-Jacobi theory that can be adapted to the optical case very well. This enables us to find very general properties of absolute instruments in both geometrical and wave optics.

The paper is organized as follows. In Sec. 2 we summarize the optical-mechanical analogy, in Sec. 3 we employ Hamilton-Jacobi theory to find many general properties of AIs. In Sec. 4, we relate the optical path length to the classical action and in Sec. 5 we find general properties of the spectra of AIs using the WKB method. Finally, in Sec. 6, we apply our methods to design new absolute instruments, and conclude in Sec. 7.

# 2 Mechanical-optical analogy

We begin with the close relationship between classical mechanics and geometrical optics [16]. Consider a particle of energy $E$ moving in potential $V(\vec{r})$. The geometrical shape of its trajectory is determined by Maupertuis' principle [17]

$$\delta \int_A^B \sqrt{2m[E - V(\vec{r})]}\, \mathrm{d}l = 0\,, \tag{1}$$

where $A, B$ is the initial and final point, respectively, and $\mathrm{d}l$ is the path length element. On the other hand, a light ray trajectory in an optical medium with refractive index distribution $n(\vec{r})$ is determined by the Fermat's principle of stationary time [5]

$$\delta \int_A^B n(\vec{r})\, \mathrm{d}l = 0\,. \tag{2}$$

Now suppose that the refractive index profile $n(\vec{r})$ in the optical problem is related to the potential $V(\vec{r})$ and energy $E$ in the mechanical problem by the relation

$$n = \frac{\sqrt{2m(E - V)}}{\gamma} = \frac{|\vec{p}|}{\gamma} \tag{3}$$

where $\vec{p}$ is the particle's momentum and $\gamma$ is a constant of dimension $\mathrm{kg} \cdot \mathrm{m} \cdot \mathrm{s}^{-1}$. Then Eqs. (1) and (2) become identical, so the two variation problems will have the same solutions; light rays will follow trajectories which are identical in shape to the trajectories of the particle.

This close relation between classical mechanics and geometrical optics allows the transfer of results from one field to the other: for example, if we find a mechanical potential in which all bound trajectories are closed, we have automatically found an absolute optical instrument. However, an important and critical difference exists between the mechanical problem and the optical problem: in classical mechanics, the energy $E$ of a particle traveling through a potential $V$ is a variable related to the particle itself and can have different values; in the case of light rays in this analogy, $E$ is a variable related to the refractive index distribution by Eq. (3), so it is a constant for light rays therein. It turns out that this feature of fixed energy is in fact a great advantage when designing AIs. Indeed, it is enough to find a mechanical system that has closed trajectories for one fixed energy, and we automatically generate an absolute optical instrument. Clearly, it is far less restrictive to require closed trajectories for a single energy than for a range of energies. This is, for example, the case of Maxwell's fisheye [18] shown in Figure 1 (a): its refractive index $n = 2/(1 + r^2)$ corresponds to a mechanical particle with energy $E = 0$ moving in the potential $V = -2/(1 + r^2)^2$, but for a different energy $E \neq 0$, the trajectories would no longer be closed. Moreover, it turns out that in some coordinate systems, separation of the Hamilton-Jacobi equation is possible for a fixed value of $E$ while it is impossible if $E$ can take any value from some range; an example of such a situation is given in Sec. 6.4. This way, the set of AIs is obviously much richer than the set of mechanical potentials that give closed trajectories for a range of energies. Moreover, a situation that is not very interesting from the mechanical point of view (closed trajectories for a single energy) may be very interesting from the optical perspective (leading to an absolute instrument).

Thanks to the mechanical-optical analogy, the problem of designing AIs is very similar to the problem of finding classical bound systems that are maximally superintegrable. A superintegrable system is an integral system which admits more integrals of motion than degrees of freedom. It is known that maximal superintegrability leads to closed orbits in bound mechanical systems; in this case there must be $2n - 1$ independent integrals of motion in a system of $n$ degrees of freedom. For example, in 3D spherically-symmetric systems, the Bertrand theorem [19] states that only two potentials are able to give closed orbits for particles with any energy $E$, the Newtonian potential and the isotropic harmonic oscillator, and both of these systems possess five constants of motion; in the optical case, these systems correspond to the Eaton lens and the Luneburg lens, respectively.

Similarly, other maximally superintegrable bound classical mechanical systems also form families of absolute optical instruments. Some excellent references which introduce such mechanical systems and describe how they can be found are [20, 21, 22, 23, 24, 25, 26], with a thorough recent review given by Miller [27]. Potentials such as the Winternitz potential [28] yield rich families of AIs that have not been discussed in optics literature before, for example.

The mechanical-optical analogy, together with the feature of fixed energy, opens a vast new field of investigation: we can adapt the well-developed methods of theoretical mechanics, in particular the Hamilton-Jacobi theory, to the situation when the energy is set to a single value, and then find new absolute instruments by finding potentials that give closed trajectories for this energy only. So far, to the best of our knowledge, this interesting theoretical problem has not been investigated, and we hope it will attract the attention of the mathematical community.

# 3 Separable Hamilton-Jacobi equation leading to AIs

Due to the Liouville-Arnold theorem, a system is maximally superintegrable if and only if the Hamiltonian (expressed in action-angle variables) is a linear combination of action variables with integer coefficients, as explained clearly in [24]. We thus start with the Hamilton-Jacobi equation for a particle with mass $m$ moving in a potential $V(\vec{r})$. Since the Hamiltonian $H = p^2/(2m) + V(\vec{r})$ does not depend on time, we can write the action as $S = S_0 - Et$, where $E$ is the conserved value of energy and $S_0$ is the Hamilton's characteristic function (in the following we will call it just "action"). The Hamilton-Jacobi equation for $S_0$ is

$$\frac{(\nabla S_0)^2}{2m} + V(\vec{r}) = E\,. \tag{4}$$

We will assume that Eq. (4) it is fully separable in an orthogonal curvilinear coordinate system $(q_1, q_2, q_3)$. This assumption allows us to express the action as a sum $S_0 = S_1(q_1)+S_2(q_2)+S_3(q_3)$, and although separability is not a requirement for superintegrability in general [29], it is likely necessary for superintegrability in systems analogous to AIs [30, 31, 32, 33, 34, 35] although we are unaware of a suitable proof. After separation, the Hamilton-Jacobi equation gets the form

$$\frac{1}{2m}\sum_{i=1}^{3}\frac{1}{h_i^2}\left(\frac{\mathrm{d}S_i}{\mathrm{d}q_i}\right)^2 + V(q_1, q_2, q_3) = E\,, \tag{5}$$

where $h_i$ are Lamé coefficients. Separating Eq. (5), we get equations for $S_i$ in the form

$$S_i = \int \frac{\mathrm{d}S_i}{\mathrm{d}q_i}\,\mathrm{d}q_i = \int p_i(q_i, E, \alpha, \beta)\,\mathrm{d}q_i\,, \tag{6}$$

where $\alpha, \beta$ are separation constants and $p_i = \partial S_i/\partial q_i$ are the canonical momenta associated with the coordinates $q_i$. Their particular functional dependence $p_i(q_i, E, \alpha, \beta)$ follows from Eq. (5). We also define the action variables $J_1, J_2, J_3$ corresponding to full oscillations of the coordinates $q_1, q_2, q_3$, respectively:

$$J_i(E, \alpha, \beta) = \frac{1}{2\pi}\oint p_i\,\mathrm{d}q_i = \frac{1}{\pi}\int_{q_{i,-}}^{q_{i,+}} p_i\,\mathrm{d}q_i\,. \tag{7}$$

Here $q_{i\pm}$ denote the turning points of $q_i$, i.e., the values for which $p_i$ turns to zero. In the case of a coordinate that does not oscillate forwards and backwards between the turning points but rather changes monotonously (as in the case of the polar coordinate $\varphi$ for a particle orbiting the origin), the corresponding action variable would instead be $J_i = 1/(2\pi)\int_{q_{i-}}^{q_{i+}} p_i\,\mathrm{d}q_i$ and $q_{i\mp}$ would correspond to the endpoints of one full cycle of the coordinate $q_i$. We will refer to such a coordinate as a "cyclic" coordinate[1].

The knowledge of the three functions $J_i(E,\alpha,\beta)$ enables in principle eliminating the variables $\alpha,\beta$ and expressing the energy as a function of $J_1, J_2, J_3$, i.e., $E = E(J_1, J_2, J_3)$. Then we can calculate the frequencies of oscillations in each coordinate as [17]

$$\omega_i = \left(\frac{\partial E}{\partial J_i}\right)_{J_j,J_k}, \tag{8}$$

where the partial derivative is taken with the other two $J_j, J_k$ fixed. Only if these frequencies are commensurable (their ratios are rational) will we get closed orbits. This can be expressed by the relation

$$\frac{\omega_1}{b_1} = \frac{\omega_2}{b_2} = \frac{\omega_3}{b_3} \equiv \omega \tag{9}$$

between the frequencies, where $b_i \in \mathbb{N}$ and the greatest common divisor of $b_1, b_2, b_3$ is unity; $\omega$ is the frequency of the motion as a whole; the corresponding period is $T = 2\pi/\omega$.

To calculate the partial derivatives (8), we employ the Jacobian matrix of the transformation $(E,\alpha,\beta) \to (J_1, J_2, J_3)$ and its inverse,

$$M = \frac{\partial(J_1,J_2,J_3)}{\partial(E,\alpha,\beta)}, \qquad M^{-1} = \frac{\partial(E,\alpha,\beta)}{\partial(J_1,J_2,J_3)}. \tag{10}$$

We see that the first line of $M^{-1}$ contains precisely the desired frequencies $\omega_1, \omega_2, \omega_3$. To express them, we invert the matrix $M$ using the method of minors. This yields the frequencies

$$\omega_1 = (M^{-1})_{11} = \frac{M_{22}M_{33} - M_{23}M_{32}}{\det M}, \tag{11}$$

$$\omega_2 = (M^{-1})_{12} = \frac{M_{13}M_{32} - M_{12}M_{33}}{\det M}, \tag{12}$$

$$\omega_3 = (M^{-1})_{13} = \frac{M_{12}M_{23} - M_{22}M_{13}}{\det M}. \tag{13}$$

The conditions (9) then yield the following three conditions for derivatives of the actions:

$$\frac{\partial(J_i/b_j + J_j/b_i)}{\partial\alpha}\frac{\partial J_k}{\partial\beta} = \frac{\partial(J_i/b_j + J_j/b_i)}{\partial\beta}\frac{\partial J_k}{\partial\alpha}, \tag{14}$$

where the triple indices $(i,j,k)$ are cyclic permutations of $(1,2,3)$. Note that in these conditions there are no derivatives with respect to energy $E$. This is advantageous and also natural because we are interested in closed trajectories for a fixed value of energy, so any energy dependence of the quantities is irrelevant.

If we define the "total action" as

$$J \equiv b_1 J_1 + b_2 J_2 + b_3 J_3, \tag{15}$$

the conditions (14) can be rewritten as

$$\frac{\partial J}{\partial\alpha}\frac{\partial J_i}{\partial\beta} = \frac{\partial J}{\partial\beta}\frac{\partial J_i}{\partial\alpha}, \quad i = 1,2,3. \tag{16}$$

[1]We put the word "cyclic" into quotation marks to distinguish it from the usual meaning of cyclic coordinates—the ones on which the Lagrangian or Hamiltonian do not depend explicitly.

In the following we show that these conditions imply that both of the derivatives $\frac{\partial J}{\partial \alpha}$ and $\frac{\partial J}{\partial \beta}$ are equal to zero. To do that, we suppose for a moment that the contrary is true, i.e., that at least one of the derivatives $\frac{\partial J}{\partial \alpha}, \frac{\partial J}{\partial \beta}$ is nonzero. Then it would follow from equations (16) that also the other one has to be nonzero and so have to be all the other derivatives in equations (16). Then we could divide each equation by the respective $\frac{\partial J}{\partial \beta}\frac{\partial J_i}{\partial \beta}$ and obtain

$$\frac{\frac{\partial J_1}{\partial \alpha}}{\frac{\partial J_1}{\partial \beta}} = \frac{\frac{\partial J_2}{\partial \alpha}}{\frac{\partial J_2}{\partial \beta}} = \frac{\frac{\partial J_3}{\partial \alpha}}{\frac{\partial J_3}{\partial \beta}}\,, \tag{17}$$

which could be rewritten using an identity (common in thermodynamics),

$$\left(\frac{\partial A}{\partial B}\right)_C \left(\frac{\partial B}{\partial C}\right)_A \left(\frac{\partial C}{\partial A}\right)_B = -1\,, \tag{18}$$

into

$$\left(\frac{\partial \alpha}{\partial \beta}\right)_{J_1} = \left(\frac{\partial \alpha}{\partial \beta}\right)_{J_2} = \left(\frac{\partial \alpha}{\partial \beta}\right)_{J_3}\,. \tag{19}$$

This would imply that the action variables are functions of one another, or, equivalently, they all are functions of one common function $\gamma(\alpha, \beta)$. Taking this $\gamma$ as a alternative separation variable along with another, independent function $\delta(\alpha, \beta)$, it would follow that all $J_1, J_2, J_3$ are functions of only one separation variable, which is an unphysical situation. Therefore the assumption that some of the derivatives $\frac{\partial J}{\partial \alpha}$ and $\frac{\partial J}{\partial \beta}$ is nonzero is wrong, and we are left with the only other possibility—both of the derivatives must be zero:

$$\frac{\partial J}{\partial \alpha} = \frac{\partial J}{\partial \beta} = 0\,. \tag{20}$$

We have come to one of the key results of this paper: for a mechanical system that has closed trajectories for a given energy $E$, the total action $J$ (for this energy) cannot depend on the separation constants $\alpha, \beta$. Similarly, by the mechanical-optical analogy, the condition (20) is also the key property of absolute optical instruments for which we define the quantities $J_i, J, \omega_i$, etc., analogously using the relation (3).

### 3.1 The case of closed trajectories for different energies

Consider now the situation that is of great interest in mechanics, namely that the trajectories are closed for not just one, but for a range of energies. Then the condition (20) will hold for a range of $E$, and hence the total action $J(E, \alpha, \beta)$ can be a function of $E$ only, i.e., $J = f(E)$. Denoting the inverse function by $g$, we can write

$$E = g(J) = g(b_1 J_1 + b_2 J_2 + b_3 J_3)\,, \tag{21}$$

and we can easily express the three periods,

$$\omega_i = \frac{\partial E}{\partial J_i} = b_i g'(J)\,, \tag{22}$$

where prime denotes a derivative. Combining this with Eq. (9), we find that $\omega = g'(J)$, so $\omega$ is a function of $J$. But since $J$ is a function of $E$, it follows that $\omega$ is a function of energy, independent of $\alpha, \beta$. We arrive at an important result: in potentials that have the focusing property for a range of energies, the time period is the same for all trajectories with the same energy. We can verify this for two well-known examples: in the Hooke potential it is indeed so because the period is even equal for all energies; for the Newtonian potential it is also true because there the period depends on the length of the main axis of the elliptic orbit (Kepler's third law), which in turn depends on the energy only.

This means that in potentials that have closed orbits for a range of energies, when particles are shot from a given point with the same velocity in different directions, it will take the same time to all of them to complete a closed path. Note that this is not true for potentials that do not give closed trajectories for different energies but just for one; for example, a particle with zero energy moving in the Maxwell's fisheye potential $V = -2/(1+r^2)^2$ clearly has a shorter travel time for a centered circular path than for a highly eccentric path. What is the same even for Maxwell's fisheye is the optical time [time traveled by a light ray in the corresponding optical medium $n(\vec{r})$] which, in contrast to mechanical time, is the same for different trajectories; we will show this in the next section.

# 4 Total action as the optical path length

Consider an absolute optical instrument. For the corresponding mechanical system, i.e., a particle moving in a potential $V$ for which the total action (15) satisfies the conditions (20), we define the quantity

$$I = \oint \vec{p}\,\mathrm{d}\vec{l} = \oint p\,\mathrm{d}l\,, \tag{23}$$

where $\vec{p}$ is the particle momentum vector, and the integral is taken over the whole closed trajectory of the particle. By the mechanical-optical analogy, Eq. (3), $I$ is proportional to the optical path length $s = \oint n\,\mathrm{d}l$ of the closed light ray corresponding to the particle trajectory because the magnitude of the momentum $\vec{p}$ is proportional to the refractive index $n$. At the same time, we can express the integral (23) using the coordinates $(q_1, q_2, q_3)$ as

$$I = \oint \sum_{i=1}^{3} p_i\,\mathrm{d}q_i\,, \tag{24}$$

where $p_i$ are the canonical momenta associated with the coordinates $q_i$ because the form $\sum_i p_i\,\mathrm{d}q_i$ is invariant with respect to the point transformation $(x, y, z) \to (q_1, q_2, q_3)$, which is a special case of the Mathieu transformation [36]. Now, interchanging summation and integration in Eq. (24), we can express $I$ as

$$I = \sum_{i=1}^{3} b_i \oint p_i\,\mathrm{d}q_i = 2\sum_{i=1}^{3} b_i \int_{q_{i,-}}^{q_{i,+}} p_i\,\mathrm{d}q_i\,. \tag{25}$$

Here we have taken into account that to complete a closed orbit the particle will require a time $T = 2\pi/\omega$, which, due to Eq. (9), contains $b_i$ periods of the coordinate $q_i$; therefore the integral over one cycle of the coordinate $q_i$ has to be taken $b_i$ times. Comparing now Eq. (25) with the actions $J_i$ according to Eq. (7), we see that

$$I = 2\pi \sum_{i=1}^{3} b_i J_i = 2\pi J\,. \tag{26}$$

This way, the quantity $I$ and hence also the optical path length $s$ of a closed ray along with it, is proportional to the total action defined in Eq. (15). From conditions (20) it then follows that for absolute instruments the optical lengths of the rays should not depend on the separation constants, i.e., they should be the same for all rays. This is in fact a very natural requirement: if rays form closed trajectories, there are infinitely many rays by which one can get from a point back to the same point again. Since the optical path lengths of all these rays should be stationary due to Fermat's principle, they must simply be equal. This also gives the conditions (20) a clear physical meaning.

Finally, we calculate the time period $T_{\text{ray}}$ needed for a ray to complete a closed path. This period can clearly be expressed in terms of the optical path length of the ray $s$ defined above as $T_{\text{ray}} = s/c$, where $c$ is the speed of light in vacuum. Note that this period has nothing to do with the period $T_{\text{particle}} = 2\pi/\omega$ of motion of the equivalent mechanical particle. This is because the "mechanical time" $t$ and "optical time" $\tau$ are quite different quantities [16] despite the close mechanical-optical analogy. Indeed, we can express the differentials of both times using the path element $\mathrm{d}l$ as

$$\mathrm{d}t = \frac{\mathrm{d}l}{v_{\text{particle}}} = m\,\frac{\mathrm{d}l}{p}\,, \qquad \mathrm{d}\tau = \frac{\mathrm{d}l}{v_{\text{light}}} = \frac{1}{c}\,n\,\mathrm{d}l\,. \tag{27}$$

Now, since the refractive index $n$ is proportional to the particle momentum $p$, we see that if $n$ grows then $\mathrm{d}\tau$ also grows while $\mathrm{d}t$ decreases. This way, we cannot expect any direct relation between the mechanical period $T_{\text{particle}}$ and optical period $T_{\text{ray}}$.

# 5 Spectrum of absolute instruments from the WKB method

An important characteristic of an optical device with bounded trajectories is the frequency spectrum of its eigenmodes. It has been shown by two different methods [15, 14] that for spherically-symmetric absolute instruments, the spectrum has distinct properties: it is equidistant and degenerate, at least approximately. The reason for this is simple: if light rays form closed loops in an AI, then after a certain time all light rays emitted from a given point of AI must return to the same point again. Motion of rays in an AI is therefore periodic, and the same can be expected for waves, at least at high frequencies. Now each wave in an AI can be decomposed into modes, each of which evolves harmonically with its frequency. If the total wave should repeat periodically, there should be some common period of all the modes. In other words, their angular frequencies should be multiples of some common fundamental frequency.

In the following we show that the above characteristic of the spectrum applies not only to spherically-symmetric AIs but also to the most general ones discussed in this paper. To do this, we employ the Wentzel-Kramers-Brillouin (WKB) method and the fact that the action variables $J$ are closely related to the quantized semiclassical phases.

For simplicity, consider a monochromatic scalar wave $\psi$ of wavenumber $k = \omega/c$ propagating in an AI with refractive index $n(\vec{r})$. It is governed by the Helmholtz equation

$$\Delta\psi + k^2 n^2 \psi = 0\,. \tag{28}$$

This equation can be solved approximately by the WKB method. To do that, we write $\psi$ in the form $\psi = \rho\exp(\mathrm{i}\phi)$, where $\phi$ is the phase, or eikonal, of the wave. Performing the derivatives, neglecting the term $\Delta\rho$ with respect to $\rho(\nabla\phi)^2$, and separating the real and imaginary parts, we get two equations. One of them is the eikonal equation

$$(\nabla\phi)^2 = k^2 n^2 \tag{29}$$

and the other one is the equation for the $\rho$,

$$2\,\nabla\rho\nabla\phi + \rho\Delta\phi = 0\,. \tag{30}$$

Clearly, Eq. (29) is very similar to the Hamilton-Jacobi equation (4). The two equations become identical when we make the identification described by Eq. (3) and in addition set $\phi/k = S_0/\gamma$.

Now, according to our assumption, Eq. (4) separates in the coordinate system $(q_1, q_2, q_3)$ and therefore so does Eq. (29). To solve it, we can follow the same procedure as we did for Eq. (4), i.e., write $\phi$ as a sum $\sum_i \phi_i(q_i)$ and solve an equation analogous to Eq. (6) for each $\phi_i$. In addition, however, for each coordinate $q_i$ we now have to match the solution in the classically allowed region to the solution in the classically prohibited region where the wave has to die out evanescently. As is well known [37], there is a semiclassical phase factor of $\pi/4$ associated with each turning point $q_{i,\pm}$. (In the case of a "cyclic" coordinate defined in Sec. 3, this factor is missing.) Therefore the total change of the phase $\phi_i$ between the turning points $q_{i,-}$ and $q_{i,+}$,

$$\Phi_i \equiv \int_{q_{i,-}}^{q_{i,+}} \frac{\mathrm{d}\phi_i}{\mathrm{d}q_i} \, \mathrm{d}q_i \,, \tag{31}$$

must be $\Phi_i = (N_i + 1/2)\pi$, where $N_i = 0, 1, 2, \dots$ On the other hand, thanks to the above described identification $\phi/k = S_0/\gamma$, we can express the phase change $\Phi_i$ between the turning points in terms of the action variable $J_i$ using Eqs. (6), (7) and (31). This gives the relation

$$\Phi_i = \frac{k\pi J_i}{\gamma} = (N_i + 1/2)\pi \,, \tag{32}$$

which, in fact, is a quantization condition for $k$. To see its meaning clearly, we employ the fact that for an AI the total action (15) is a constant as we have shown. Using the relations (3) and (26), we see that $J = \gamma \oint n \mathrm{d}l/(2\pi) = \gamma s/(2\pi)$. Inserting this into Eq. (32) and using Eq. (15), we find the wavenumber corresponding to the mode $(N_1, N_2, N_3)$ as

$$k_{N_1,N_2,N_3} = \frac{2\pi}{s} \sum_{i=1}^{3} b_i \left( N_i + \frac{1}{2} \right) \,. \tag{33}$$

This is the formula for the semiclassical spectrum of an absolute instrument. Since $b_i \in \mathbb{N}$, we see that the spectrum obeys the general patterns of AIs mentioned above. In a case where one of the coordinates $q_i$ is "cyclic", the term $1/2$ in the parentheses would be missing for that coordinate.

## 5.1 Wave period

Thanks to the fact that the eigenfrequencies $\omega_{N_1,N_2,N_3} = ck_{N_1,N_2,N_3}$ of the modes obey the rule (33), there exists some minimum time period $T_{\text{wave}}$ after which the phases of all modes will advance by an integer number of $2\pi$ (up to a possible common global phase). After this time, the wave in the AI will resume its original state, so the wave motion will be periodic as we expect. To find this period, we use the fact that the greatest common divisor of the numbers $b_i$ is unity. The period must then be simply $T_{\text{wave}} = 2\pi/(2\pi c/s) = s/c = T_{\text{ray}}$. This is exactly what we would expect – the period of repetition of the wave pattern is equal to the period of motion of rays in the absolute instrument.

## 5.2 Spectrum of spherically-symmetric AIs

We now compare the general result (33) for an AI spectrum with a previously derived formula for spectra of AIs with spherical symmetry. In Ref. [14] it was derived by the WKB method that the semiclassical wavenumbers of the eigenmodes satisfy the condition

$$k = \frac{1}{r_0 n(r_0)} \left[ \mu \left( N + \frac{1}{2} \right) + l + \frac{1}{2} \right] \,, \tag{34}$$

where $r_0$ is the radius of the trajectory of a circular ray, $\mu$ determines the angle $\delta$ swept by the radius vector between two radial turning points as $\delta = \pi/\mu$, and $N, l \in \mathbb{N}_0$. Let us check what our general formula (33) states in this case. We will assume that $\mu = P/Q$ with $P, Q$ integers. Then the angle swept by the radius vector between two radial turning points is $\delta = Q\pi/P$, so there are $P$ cycles in the radial direction per $Q$ angular cycles, hence in Eq. (9) applied to spherical coordinates $(r, \theta, \varphi)$ we have $b_r = P$ and $b_\theta = b_\varphi = Q$. The optical path length of the full orbit is then $s = 2\pi Q r_0 n(r_0)$ (we are taking the circular orbit that corresponds to $Q$ full rotations). Combining all this together and taking into account that the azimutal angle $\varphi$ does not have the factor 1/2 for the reason described above, Eq. (33) gives the spectrum

$$k = \frac{1}{Q r_0 n(r_0)}\left[P\left(N_r + \frac{1}{2}\right) + Q\left(N_\theta + \frac{1}{2}\right) + Q N_\varphi\right] = \frac{1}{r_0 n(r_0)}\left[\mu\left(N_r + \frac{1}{2}\right) + N_\theta + N_\varphi + \frac{1}{2}\right], \tag{35}$$

which is exactly Eq. (34) if we identify $N = N_r$ and $l = N_\theta + N_\varphi$. This shows that our general method exactly matches the previously known special case.

# 6 Applying the general method to different coordinate systems

We now apply the condition (20) in different situations. Some of them are already known such as the case of spherically-symmetric AIs [8, 9]; in other cases we get new absolute instruments. For simplicity, we will set the mass of the particle to unity, $m = 1$ till the end of this paper.

## 6.1 Cartesian coordinates

In the first example, let $(q_1, q_2, q_3)$ be the Cartesian coordinate system $(x, y, z)$. The Hamiltonian

$$H = \frac{p_x^2 + p_y^2 + p_z^2}{2} + V(x, y, z) = 0 \tag{36}$$

separates if the potential is in the form $V = V_x(x) + V_y(y) + V_z(z)$. This leads to the action variables

$$J_x = \frac{1}{\pi}\int_{x_-}^{x_+} \sqrt{2[\alpha - V_x(x)]}\,\mathrm{d}x \tag{37}$$

$$J_y = \frac{1}{\pi}\int_{y_-}^{y_+} \sqrt{2[\beta - V_y(y)]}\,\mathrm{d}y \tag{38}$$

$$J_z = \frac{1}{\pi}\int_{z_-}^{z_+} \sqrt{2[E - \alpha - \beta - V_z(z)]}\,\mathrm{d}z\,. \tag{39}$$

From Eq. (20) we then get $b_x J_x'(\alpha) - b_z J_z'(E - \alpha - \beta) = 0$, where the prime denotes derivative. Since this must hold for any $\alpha$ and $\beta$, we immediately see that $J_z'(E-\alpha-\beta)$ cannot depend on $\beta$, so it must be a constant and cannot depend on $E$ either. Evaluating then the frequencies according to Eqs. (11–13) using Eqs. (37–39), we find that $\omega_x = 1/(\mathrm{d}J_x/\mathrm{d}\alpha)$, $\omega_y = 1/(\mathrm{d}J_y/\mathrm{d}\beta)$, and $\omega_z = 1/(\partial J_z/\partial E)_{\alpha,\beta}$. This way, the frequencies $\omega_i$ depend neither on the energy nor the separation constants. This is an important conclusion that is specific to the 3D situation as it does not hold for 2D systems [38].

To find the possible forms of the potentials $V_{x,y,z}$, we write explicitly the period of oscillation in $x$,

$$T_x = \frac{2\pi}{\omega_x} = \sqrt{2}\int_{x_-}^{x_+} \frac{\mathrm{d}x}{\sqrt{\alpha - V_x(x)}} \tag{40}$$

(we write just the expression for the $x$ coordinate as the expressions are similar for the others). This equation is invertible [17], and is equivalent to Equation 4 in [38] or Equation (0.10) in [24] where it is extensively studied in the context of AIs and superintegrable systems, respectively. Upon inversion,

$$x_+(V_x) - x_-(V_x) = \frac{T_x}{\pi\sqrt{2}} \int_0^{V_x} \frac{\mathrm{d}\alpha}{\sqrt{V_x - \alpha}} = \frac{T_x\sqrt{2V_x}}{\pi}\,, \tag{41}$$

where we have used the fact that $T_x$ is constant. Eq. (41) provides considerable freedom. It is just necessary that the two branches $x_-(V_x)$ and $x_+(V_x)$ of the function inverse to $V_x(x)$ be non-increasing and non-decreasing, respectively, but otherwise one of them can be chosen arbitrarily and the other branch is then calculated from Eq. (41). For the symmetric case when $x_+(V_x) = -x_-(V_x)$ we get $V_x = \omega_x^2 x^2/2$, which corresponds to the harmonic oscillator. Choosing this form of potential for each of the coordinates $x, y, z$ yields the Lissajous lens [11]. Another solution where the potential is analytic, the Winternitz model [39], corresponds to the choice

$$V_x = \frac{1}{2}\,\omega_x^2 x^2 + \frac{k_x}{x^2}\,. \tag{42}$$

Choosing the potentials $V_x, V_y, V_z$ in the most general manner yields the generalized Lissajous lenses discussed in Sec. IV of Ref. [38].

Note also that whatever combination of potentials $V_x, V_y, V_z$ we choose, there is always an additional freedom to later choose any $E$ in the index of refraction $n$, as the energy $E$ will not affect the dependence of the action variables on the separation constants.

## 6.2 Spherical coordinates

We now analyze the situation when the Hamilton-Jacobi equation separates in spherical coordinates $(r, \theta, \varphi)$.

### 6.2.1 Spherically-symmetric absolute instruments

First consider a spherically-symmetric (central) potential $V(r)$ depending only on the radial coordinate. The Hamilton-Jacobi equation

$$\frac{1}{2}\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \frac{1}{2r^2}\left(\frac{\mathrm{d}S_\theta}{\mathrm{d}\theta}\right)^2 + \frac{1}{2r^2\sin^2\theta}\left(\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right)^2 + V(r) = E \tag{43}$$

gives the separated equations

$$\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi} = \alpha\,, \quad \left(\frac{\mathrm{d}S_\theta}{\mathrm{d}\theta}\right)^2 + \frac{\alpha^2}{\sin^2\theta} = \beta^2\,, \quad \frac{1}{2}\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \frac{\beta^2}{2r^2} + V(r) = E\,. \tag{44}$$

The separation variable $\alpha$ has the physical meaning of projection of the angular momentum to the $z$-axis. The action variables are then

$$J_r = \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2[E - V(r)] - \frac{\beta^2}{r^2}}\,\mathrm{d}r \tag{45}$$

$$J_\theta = \frac{1}{\pi}\int_{\arcsin(\alpha/\beta)}^{\pi-\arcsin(\alpha/\beta)} \sqrt{\beta^2 - \frac{\alpha^2}{\sin^2\theta}}\,\mathrm{d}\theta = \beta - |\alpha| \tag{46}$$

$$J_\varphi = \frac{1}{2\pi}\int_0^{2\pi} |\alpha|\,\mathrm{d}\varphi = |\alpha|\,. \tag{47}$$

Note the fixed limits and the factor $1/(2\pi)$ in Eq. (47), which corresponds to the "cyclic" coordinate $\varphi$ in the sense explained in Sec. 3. Now let us assume just as in Sec. 5.2 that there are $P$ radial cycles per $Q$ cycles in $\theta$ and $\varphi$, so $b_r = P$ and $b_\theta = b_\varphi = Q$. Eq. (20) then gives two conditions. The first one, $\partial(PJ_r + QJ_\theta + QJ_\varphi)/\partial\alpha = 0$, is satisfied automatically because $J_r$ does not depend on $\alpha$. From the second condition $\partial(PJ_r + QJ_\theta + QJ_\varphi)/\partial\beta = 0$ it then follows that

$$\frac{\partial J_r}{\partial \beta} = -\frac{1}{\pi}\int_{r_-}^{r_+} \frac{\beta\,\mathrm{d}r}{r^2\sqrt{2[E-V(r)]-\frac{\beta^2}{r^2}}} = -\frac{Q}{P}\,. \tag{48}$$

The integral in this equation expresses the turning angle $\delta$, i.e., the angle swept by the radius vector between two radial turning points if the particle has total angular momentum $\beta$ [17]. This way, Eq. (48) expresses the requirement that the turning angle is a rational multiple of $\pi$, which is exactly the well-known condition for trajectories in a central potential to be closed [17]. If the energy is fixed, there are infinitely many such potentials. A comprehensive analysis of the corresponding refractive indices and a general method how to generate them is given in Ref. [9]. On the other hand, if we require the condition (48) to hold for a range of energies, there are just two potentials that satisfy it [19], namely $V(r) = -Cr^{-1}$ and $V(r) = Cr^2$ (with $C > 0$).

Finally, we express $J_r$ explicitly. Integration of Eq. (48) gives $J_r = \text{const} - \beta Q/P$. It is not difficult to determine the integration constant. Eqs. (46) and (47) tell us that $J_\theta = J_\varphi = 0$ must hold when $\beta = 0$, and we know that $J = PJ_r + QJ_\theta + QJ_\varphi$. This fixes the constant to $J/P$ and we get $J_r = (J - \beta Q)/P$.

### 6.2.2 Modifying spherically-symmetric AIs

It turns out that when we have a central potential corresponding to an absolute instrument, it is possible to modify it by adding certain terms and the focusing properties will not be lost, i.e., we get again an AI. We have found this interesting property after inspiration from a paper by Evans [20]. To show this, suppose that the potential $V(r)$ satisfies the condition (48), and take a new potential

$$V' = V(r) + \frac{k_x^2}{2x^2} + \frac{k_y^2}{2y^2} + \frac{k_z^2}{2z^2} \tag{49}$$

with arbitrary $k_x, k_y, k_z > 0$. The Hamilton-Jacobi equation has now three additional terms compared to Eq. (43). Repeating the standard procedure, we find the action variables

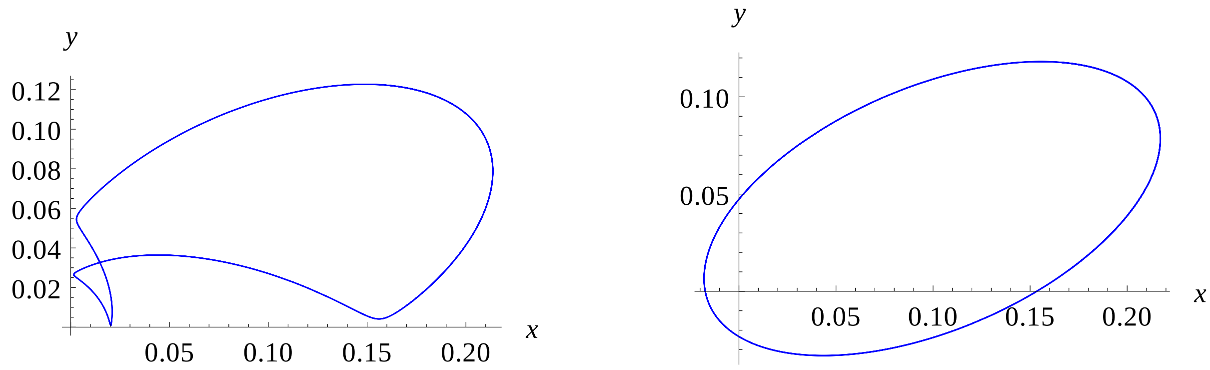
$$\begin{aligned}
J_r &= \frac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2[E-V(r)]-\frac{\beta^2}{r^2}}\,\mathrm{d}r = \frac{J-Q\beta}{P}\,, && (50)\\
J_\theta &= \frac{1}{\pi}\int_{\theta_-}^{\theta_+} \sqrt{\beta^2-\frac{k_z^2}{\cos^2\theta}-\frac{\alpha^2}{\sin^2\theta}}\,\mathrm{d}\theta = \frac{\beta-k_z-|\alpha|}{2}\,, && (51)\\
J_\varphi &= \frac{1}{\pi}\int_{\varphi_-}^{\varphi_+} \sqrt{\alpha^2-\frac{k_x^2}{\cos^2\varphi}-\frac{k_y^2}{\sin^2\varphi}}\,\mathrm{d}\varphi = \frac{|\alpha|-k_x-k_y}{2}\,, && (52)
\end{aligned}$$

where in Eq. (50) we have used the result from the previous section. Note that the coordinate $\varphi$ is no longer "cyclic".

It is now easy to check that

$$J' \equiv PJ_r + 2QJ_\theta + 2QJ_\varphi = J - Q(k_x + k_y + k_z)\,, \tag{53}$$

where $J$ refers to the total action for the potential $V(r)$ calculated in the previous section. If we now interpret $J'$ as the total action for the problem with the potential (49), we see that it satisfies the



(a) (b)

Figure 2: (a) The trajectory of a particle with energy $E = -4$ in the potential (49) with $V(r) = -1/r$ being the Newtonian potential and $k_x = 0.01, k_y = 0.005, k_z = 0$. The initial condition was chosen such that the trajectory lies in the plane $z = 0$. The soft reflection from the planes $x = 0$ and $y = 0$ is clearly seen. (b) For comparison we show the trajectory in a pure Newtonian potential $V(r) = -1/r$ with the same initial conditions.

conditions (20) because it does not depend on $\alpha, \beta$. Moreover, from Eq. (53) we see that the new coefficients from Eq. (9) are now $b'_r = P$ and $b'_\theta = b'_\varphi = 2Q$. This shows that adding the special terms to the central potential $V$ indeed preserves the focusing properties of the potential. However, thanks to doubling of the constants, $b'_\theta = 2b_\theta$, $b'_\varphi = 2b_\varphi$ compared to the potential $V$ there are now twice as many oscillations in the angles $\theta, \varphi$ per one oscillation in $r$ than in the previous case. Another difference is that the total action $J'$ is now smaller than the previous one $J$. A trajectory in the potential (49) with $V(r)$ taken as the Newtonian potential is shown in Fig. 2(a) and compared to the situation without the additional terms (Fig. 2(b)).

An interesting situation corresponds to the limit $k_x, k_y, k_z \to 0$. Then the effect of the additional terms is present only in the immediate neighbourhood of the planes $x = 0$, $y = 0$ and $z = 0$ where there are infinite potential barriers. This is equivalent to an effect of three mutually orthogonal plane mirrors placed in these planes. The focusing property of the potential $V'$ is then not surprising – compared to the potential $V$, the motion of the particle is now simply "reflected" in the three mirrors, so it occurs (or "is imaged") just in a single octant of space.

Of course, we could also modify the potential $V$ by adding fewer than three special terms discussed above, which would correspond to one or two of the $k_i$ in Eq. (49) being zero. In this case a similar analysis could be made and it would be found again that we get an absolute instrument, now with still different frequency constants $b''_r, b''_\theta, b''_\varphi$. We leave this analysis to the reader.

## 6.3 Rotational parabolic coordinates

Next we apply our method to the case of rotational parabolic coordinates $\sigma, \tau, \varphi$ that are related to Cartesian coordinates by

$$x = \sigma\tau\cos\varphi\,, \quad y = \sigma\tau\sin\varphi\,, \quad z = \frac{\tau^2 - \sigma^2}{2}\,. \tag{54}$$

The Hamiltonian

$$H = \frac{p_\sigma^2 + p_\tau^2}{2(\sigma^2 + \tau^2)} + \frac{p_\varphi^2}{2\sigma^2\tau^2} + V(\sigma, \tau) \tag{55}$$

yields the Hamilton-Jacobi equation

$$\left(\frac{\mathrm{d}S_\sigma}{\mathrm{d}\sigma}\right)^2 + \left(\frac{\mathrm{d}S_\tau}{\mathrm{d}\tau}\right)^2 + \left(\frac{1}{\tau^2} + \frac{1}{\sigma^2}\right)\left(\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right)^2 + 2(\sigma^2 + \tau^2)\,V(\sigma, \tau) = E\,. \tag{56}$$

Being inspired by the Hamilton-Jacobi equation for the radial part of a harmonic oscillator in polar coordinates, we take the potential in the form

$$V = \frac{B(\sigma^2 - \tau^2) - 2A}{\sigma^2 + \tau^2} = -\frac{A + Bz}{r} \tag{57}$$

where $A \geq 0$ and $B$ are constants. For $B = 0$, $V$ reduces to the Newtonian potential. We substitute Eq. (57) into Eq. (56) and find the action variables

$$J_\sigma = \frac{1}{\pi}\int_{\sigma_-}^{\sigma_+} \sqrt{2A + \beta + 2(E - B)\sigma^2 - \frac{\alpha^2}{\sigma^2}}\, \mathrm{d}\sigma = \frac{2A + \beta}{4\sqrt{-2(E - B)}} - \frac{|\alpha|}{2}, \tag{58}$$

$$J_\tau = \frac{1}{\pi}\int_{\tau_-}^{\tau_+} \sqrt{2A - \beta + 2(E + B)\tau^2 - \frac{\alpha^2}{\tau^2}}\, \mathrm{d}\tau = \frac{2A - \beta}{4\sqrt{-2(E + B)}} - \frac{|\alpha|}{2}, \tag{59}$$

$$J_\varphi = |\alpha|\,. \tag{60}$$

In evaluating the integrals for $J_\sigma$ and $J_\tau$ we have assumed that $E < -|B|$. The total action is then

$$J = b_\sigma J_\sigma + b_\tau J_\tau + b_\varphi J_\varphi \tag{61}$$

$$= \left(b_\varphi - \frac{b_\sigma}{2} - \frac{b_\tau}{2}\right)|\alpha| + \left(\frac{b_\sigma}{\sqrt{-2(E - B)}} - \frac{b_\tau}{\sqrt{-2(E + B)}}\right)\frac{\beta}{4} + \frac{b_\sigma A}{2\sqrt{-2(E - B)}} + \frac{b_\tau A}{2\sqrt{-2(E + B)}}\,. \tag{62}$$

The conditions (20) then require that the factors in front of $\beta$ and $\alpha$ are zero, which results in the conditions

$$b_\varphi = \frac{b_\sigma + b_\tau}{2}\,, \tag{63}$$

$$E = -B\,\frac{b_\sigma^2 + b_\tau^2}{b_\sigma^2 - b_\tau^2}\,. \tag{64}$$

Keeping aside the well-known case when $V$ reduces to the Newtonian potential (this corresponds to $B = 0$ and $b_\sigma = b_\tau = b_\varphi$), let us discuss the case of $B \neq 0$. The condition $E < -|B|$ implies that if $B > 0$, we must have $b_\sigma > b_\tau$, and if $B < 0$, we must have $b_\sigma < b_\tau$.

Eq. (64) then reveals very interesting properties of the potential (57): any combination of the coprime natural numbers $b_\sigma$ and $b_\tau$, $b_\sigma \neq b_\tau$, corresponds to a certain energy value for which we get an absolute instrument. It is not hard to see that such energies form a dense set on the interval $(-\infty, -|B|)$. This way, we have found a potential that has focusing properties for infinitely many energies, and the character of the motion is different for different energies. The trajectories for some of them are shown in Fig. 3. Moreover, the potential (57) can even be modified by adding $k_x^2/x^2 + k_y^2/y^2$ to it (but not $k_z^2/z^2$), which does not destroy the properties of the AI. The reason is similar as in the case of spherical coordinates in the previous section.

## 6.4 Bispherical coordinates

Next we take the bispherical coordinates $(\sigma, \tau, \varphi)$ related to the Cartesian ones by

$$x = \frac{\sin\sigma}{\cosh\tau - \cos\sigma}\cos\varphi \tag{65}$$

$$y = \frac{\sin\sigma}{\cosh\tau - \cos\sigma}\sin\varphi \tag{66}$$

$$z = \frac{\sinh\tau}{\cosh\tau - \cos\sigma}\,. \tag{67}$$

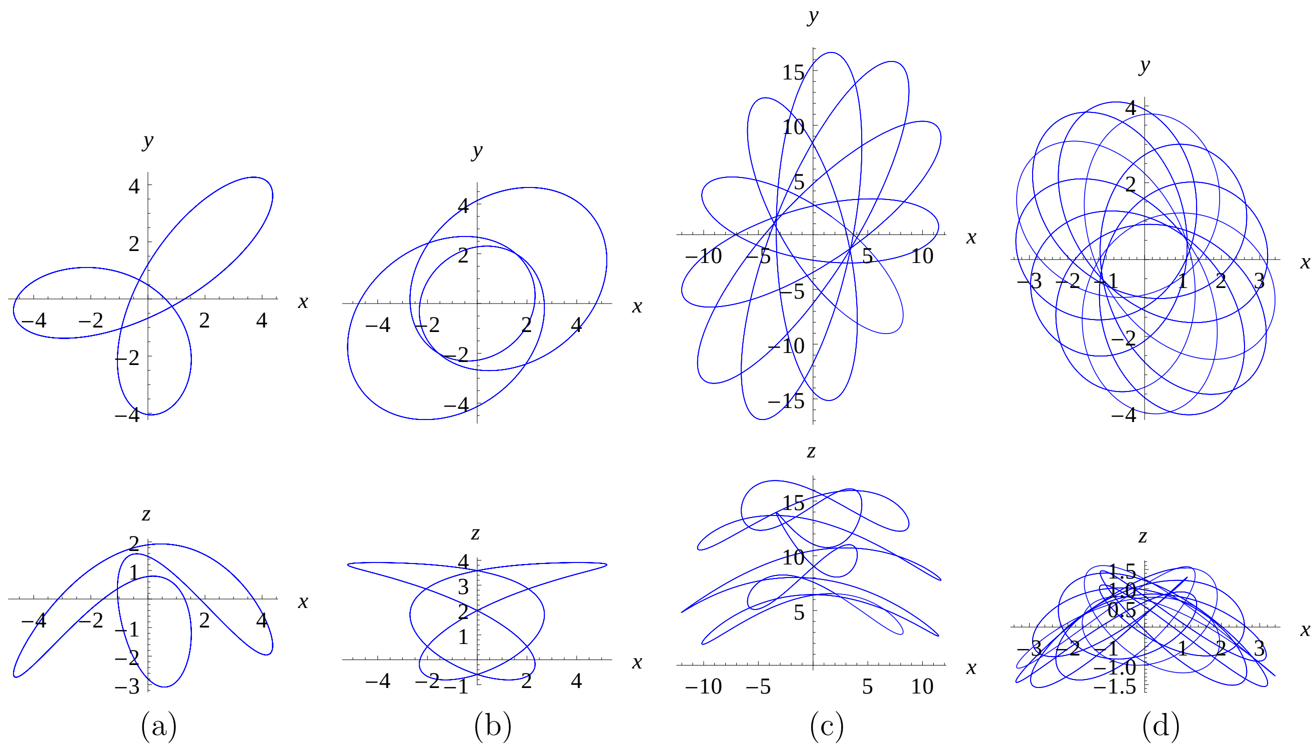



Figure 3: Projections of trajectories into planes $xy$ and $xz$ in the potential (57) with $A = 1$ and $B = 0.1$ and different energies: (a) $E = -5B/4$ (with $b_\sigma = 3$, $b_\tau = 1$, $b_\varphi = 2$), (b) $E = -5B/3$ (with $b_\sigma = 4$, $b_\tau = 2$, $b_\varphi = 3$), (c) $E = -61B/60$ (with $b_\sigma = 11$, $b_\tau = 1$, $b_\varphi = 6$), and (d) $E = -109B/60$ (with $b_\sigma = 13$, $b_\tau = 7$, $b_\varphi = 10$).

It is easy to show that for the potential

$$V = \frac{1}{2} f(\tau)(\cosh\tau - \cos\sigma)^2 \tag{68}$$

the Hamilton-Jacobi equation separates for a fixed energy $E = 0$. Indeed, in this case we have the Hamiltonian

$$H = \frac{(\cosh\tau - \cos\sigma)^2}{2}\left(p_\sigma^2 + p_\tau^2 + \frac{p_\varphi^2}{\sin^2\sigma} + f(\tau)\right) = 0 \tag{69}$$

and the Hamilton-Jacobi equation becomes

$$\left(\frac{\mathrm{d}S_\sigma}{\mathrm{d}\sigma}\right)^2 + \frac{1}{\sin^2\sigma}\left(\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right)^2 + \left(\frac{\mathrm{d}S_\tau}{\mathrm{d}\tau}\right)^2 + f(\tau) = 0\,. \tag{70}$$

Repeating the usual procedure with separation constants $\beta \geq 0$ and $\alpha$, we find the action variables

$$J_\varphi = |\alpha| \tag{71}$$

$$J_\sigma = \frac{1}{\pi}\int_{\sigma_-}^{\sigma_+}\sqrt{\beta^2 - \frac{\alpha^2}{\sin^2\sigma}}\,\mathrm{d}\sigma = \beta - |\alpha| \tag{72}$$

$$J_\tau = \frac{1}{\pi}\int_{\tau_-}^{\tau_+}\sqrt{-\beta^2 - f(\tau)}\,\mathrm{d}\tau\,. \tag{73}$$

If we put $b_\sigma = b_\varphi$, the condition $\partial J/\partial\alpha = 0$ is satisfied automatically. The second condition $\partial J/\partial\beta = 0$ then implies

$$\frac{\partial J_\tau}{\partial\beta} = -\frac{1}{\pi}\int_{\tau_-}^{\tau_+}\frac{\beta\,\mathrm{d}\tau}{\sqrt{-\beta^2 - f(\tau)}} = -\frac{b_\sigma}{b_\tau}\,. \tag{74}$$

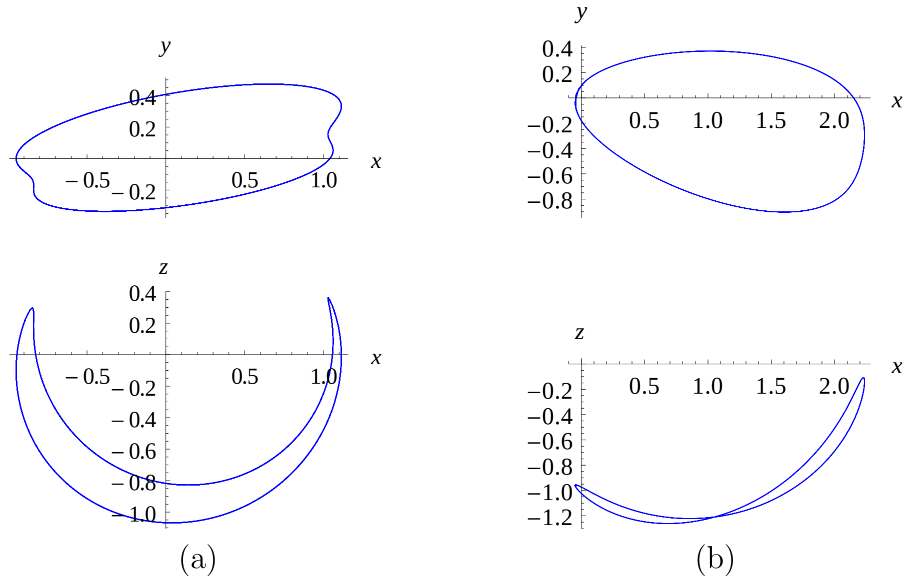



Figure 4: Projections of trajectories into planes $xy$ and $xz$ in the potential (68) for two different functions $f(\tau)$: (a) $f(\tau) = \mathrm{e}^{4\tau} - 2\mathrm{e}^{2\tau}$, which corresponds to transmutation of the Hooke potential (Luneburg lens) in spherical inversion, and (b) $f(\tau) = 2(\mathrm{e}^{2\tau} - \mathrm{e}^{\tau})$, which corresponds to transmutation of the Newtonian potential (Eaton lens). Each trajectory lies on a sphere because before the transmutation, it would lie in a plane, and spherical inversion transforms planes into spheres.

Compare now this equation with Eq. (48). Setting $\tau = \ln r$, $f = 2r^2 V$, $E = 0$, $b_\sigma = Q$ and $b_\tau = P$, we can transform the equations to one another. This means that whenever we find some AI that separates in spherical coordinates as described in Sec. 6.2 (call this $\mathrm{AI}_{\text{spherical}}$), we can find its counterpart that separates in bispherical coordinates (call this $\mathrm{AI}_{\text{bispherical}}$). This remarkable relation can be explained in a natural way. It can be shown that the $\mathrm{AI}_{\text{bispherical}}$ is related to the $\mathrm{AI}_{\text{spherical}}$ by a transmutation [16] by spherical inversion. The spherical inversion is a conformal map (the only non-trivial 3D one), so it transforms isotropic refractive index profiles again to isotropic ones. Fig. 4 shows trajectories in the potential (68) for two different functions $f(\tau)$ corresponding to transmutation of the Hooke potential and the Newtonian potential.

Moreover, the transmutation argument can further be extended to arbitrary AIs, also the ones without spherical symmetry. This way, for any absolute instrument we can find infinitely many of its partners by transmuting it by spherical inversions with different centres and radii.

A special case of the potential (68) corresponds to the choice $f = -1/\cosh^2\tau$, which gives $V = -2/(1+r^2)^2$. Remarkably, we obtain the same potential even when we perform the above described transmutation (i.e., $f/(2r^2) = -2/(1+r^2)^2$ as well). Both of these potentials correspond to Maxwell's fisheye which is a transmutation of itself in spherical inversion.

Finally, we put the potential (68) into the form suitable for raytracing. Direct calculation shows that

$$V = \frac{1}{2}\,\frac{1}{\left(\frac{1+r^2}{2}\right)^2 - z^2}\, f\left[\operatorname{arcsinh}\frac{z}{\sqrt{\left(\frac{1+r^2}{2}\right)^2 - z^2}}\right]. \tag{75}$$

## 6.5 Superintegrable potentials as AIs

As we have mentioned, a superintegrable system is a system with more integrals of motion than degrees of freedom. The additional integrals restrict the trajectory in the phase space, so such systems are good candidates for absolute instruments. In the following we discuss separately the cases of maximally and minimally superintegrable systems.

### 6.5.1 Maximally superintegrable potentials

In a maximally superintegrable system with $n$ degrees of freedom, there are $2n-1$ independent integrals of motion, so the motion is restricted to a 1D manifold in the phase space. Therefore maximal superintegrability leads to closed orbits in bound systems, which immediately gives AIs.

A great deal of literature exists on superintegrable mechanical systems. For example, Evans [20] has presented a complete list of all 3D superintegrable systems with integrals that are linear or quadratic polynomials in the momenta. With the restriction to quadratic integrals, there are 11 orthogonal coordinate systems in which the Hamilton-Jacobi equation separates. Evans presents a table of maximally superintegrable potentials with 5 integrals of motion, which we label $V_{1-5}$ in line with his table ordering in Ref. [20]; in three of these cases, $V_1$, $V_2$, and $V_5$, the trajectories are closed and they directly correspond to absolute optical instruments for any energy. Although the trajectories are not closed in $V_3$ and $V_4$, modifications to these potentials are possible which break the restriction to quadratic integrals but which then result in new AIs. For example, we have found that the term $k_0 r^2$ can be added to $V_3$ which prevents $J_r$ from diverging.

We present the action variables for potentials $V_1$, $V_2$, and our modified $V_{3,\text{modified}} = V_3 + k_0 r^2$ in Table 1. ($V_5$ differs from $V_1$ in an insignificant way so we omit it for brevity.) We see that indeed for each case the total action does not depend on the separation variables as required by Eq. (20). Note that the potentials $V_1$ and $V_2$ are special cases of AIs discussed in Sec. 6.2.2.

### 6.5.2 Minimally superintegrable potentials

Evans [20] also presents a table of minimally superintegrable systems in 3D with 4 integrals of motion; in these systems, the trajectories are not closed in general, but are restricted to 2D surfaces in the phase space. Moreover, in all of these minimally superintegrable systems there is a great deal of freedom since each contains a function $F$ of some combination of coordinates to be chosen freely. This raises an interesting question: is it possible to choose some special form of $F$ and/or a fixed value of energy such that the trajectories become closed? The answer it that in many cases it is so, which allows us to create new absolute instruments. In the following we show several examples.

First, we show this procedure for a potential from Table II of Ref. [20]:

$$V_9 = -\frac{k}{\sqrt{x^2+y^2}} + \frac{k_1 x}{y^2\sqrt{x^2+y^2}} + \frac{k_2}{y^2} + F(z)\,, \tag{76}$$

which separates in cylindrical polar coordinates. The Hamilton-Jacobi equation is

$$\frac{1}{2}\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \frac{1}{2r^2}\left(\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right)^2 + \frac{1}{2}\left(\frac{\mathrm{d}S_z}{\mathrm{d}z}\right)^2 + V_9 = E \tag{77}$$

and the separated equations for this potential are

$$\frac{1}{2}\left(\frac{\mathrm{d}S_r}{\mathrm{d}r}\right)^2 + \frac{k}{r} + \frac{\alpha}{2r^2} + \beta = E\,, \quad \left(\frac{\mathrm{d}S_\varphi}{\mathrm{d}\varphi}\right)^2 + \frac{2k_2}{\sin^2\varphi} + \frac{2k_1\cos\varphi}{\sin^2\varphi} = \alpha\,, \quad \frac{1}{2}\left(\frac{\mathrm{d}S_z}{\mathrm{d}z}\right)^2 + F(z) = \beta\,. \tag{78}$$

Table 1: Maximally superintegrable potentials based on Evans [20]

| Potential and separation coordinates | Action Variables | Notes |
|---|---|---|
| $V_1 = k(x^2+y^2+z^2)+\frac{k_1}{x^2}+\frac{k_2}{y^2}+\frac{k_3}{z^2}$ <br> Cartesian | $J_x = \frac{\alpha}{2\sqrt{2k}} - \sqrt{\frac{k_1}{2}}$ <br> $J_y = \frac{\beta}{2\sqrt{2k}} - \sqrt{\frac{k_2}{2}}$ <br> $J_z = \frac{E-\alpha-\beta}{2\sqrt{2k}} - \sqrt{\frac{k_3}{2}}$ <br> $J = \frac{E}{2\sqrt{2k}} - \sqrt{\frac{k_1}{2}} - \sqrt{\frac{k_2}{2}} - \sqrt{\frac{k_3}{2}}$ | Special case of Sec. 6.1 and Sec. 6.2.2; $b_i = 1$ |
| $V_2 = -\frac{k}{r}+\frac{k_1}{x^2}+\frac{k_2}{y^2}$ <br> spherical | $J_r = -\beta - \frac{k}{\sqrt{-2E}}$ <br> $J_\theta = \beta - \alpha$ <br> $J_\varphi = \frac{\alpha}{2} - \sqrt{\frac{k_1}{2}} - \sqrt{\frac{k_2}{2}}$ <br> $J = -\sqrt{2k_1} - \sqrt{2k_2} + \frac{k}{\sqrt{-2E}}$ | Special case of Sec. 6.2.2; $b_r = b_\theta = 1, b_\phi = 2$ |
| $V_3 = \frac{k_1 x}{y^2\sqrt{x^2+y^2}} + \frac{k_2}{y^2} + \frac{k_3}{z^2}$ <br> $V_{3,\text{modified}} = V_3 + k_0 r^2$ <br> spherical | $J_r = -\frac{1}{2}\beta + \frac{E}{2\sqrt{2k_0}}$ <br> $J_\theta = \frac{\beta-\alpha+\sqrt{2k_3}}{2}$ <br> $J_\varphi = \alpha - \sqrt{\frac{k_2-k_1}{2}} - \sqrt{\frac{k_2+k_1}{2}}$ <br> $J = -\sqrt{\frac{k_2-k_1}{2}} - \sqrt{\frac{k_2+k_1}{2}} +$ <br> $\sqrt{2k_3} + \frac{E}{\sqrt{2k_0}}$ | See Sec. 6.5.1; $b_r = b_\theta = 2$, $b_\varphi = 1$ |

The action variables are then evaluated by the residue method:

$$J_r = \frac{1}{\pi}\int_{r_-}^{r_+}\sqrt{2\left(E-\beta-\frac{k}{r}\right)-\frac{\alpha}{r^2}}\,\mathrm{d}r = \sqrt{\alpha}+\frac{k}{\sqrt{2(\beta-E)}} \tag{79}$$

$$J_\varphi = \frac{1}{\pi}\int_{\varphi_-}^{\varphi_+}\sqrt{\alpha-\frac{2k_2}{\sin^2\varphi}-\frac{2k_1\cos\varphi}{\sin^2\varphi}}\,\mathrm{d}\varphi = -\sqrt{\alpha}+\frac{\sqrt{k_2+k_1}}{\sqrt{2}}+\frac{\sqrt{k_2-k_1}}{\sqrt{2}} \tag{80}$$

$$J_z = \frac{1}{\sqrt{2}\pi}\int_{z_-}^{z_+}\sqrt{\beta-F(z)}\,\mathrm{d}z\,. \tag{81}$$

We immediately notice that $E$ and $\beta$ are together in the same term under the square root in $J_r$, but $J_z$ is nominally a function only of $\beta$. For the total action $J = b_r J_r + b_\varphi J_\varphi + b_z J_z$ to be independent of $\beta$, $E$ must therefore be a constant, and we can put $E = 0$ without loss of generality. The condition $\partial J/\partial\alpha = 0$ implies $b_r = b_\varphi$, and the condition $\partial J/\partial\beta = 0$ implies

$$\int_{z_-}^{z_+}\frac{\mathrm{d}z}{\sqrt{\beta-F(z)}} = \frac{b_r k\pi}{2b_z\beta^{3/2}}\,. \tag{82}$$

This equation can be inverted by the inverse Abel transformation, similar to the case of Cartesian coordinates in Section 6.1. This way we find

$$z_+(F)-z_-(F) = \frac{b_r k}{2b_z}\int_{\beta_0}^{F}\frac{\beta^{-3/2}\,\mathrm{d}\beta}{\sqrt{F-\beta}} = \frac{b_r k\sqrt{F-\beta_0}}{b_z F\sqrt{\beta_0}}\,, \tag{83}$$

where $\beta_0 > 0$ is the value of $F$ at its minimum; we can choose this parameter. Similarly as in the case of Cartesian coordinates in Section 6.1, there is a great deal of freedom in choosing $F(z)$. One possible choice is to require that the function be symmetric, $z_+(F) = -z_-(F)$. We then obtain

$$F(z) = \frac{\gamma^2-\sqrt{\gamma^4-16\gamma^2\beta_0^2 z^2}}{8\beta_0 z^2}\,, \tag{84}$$

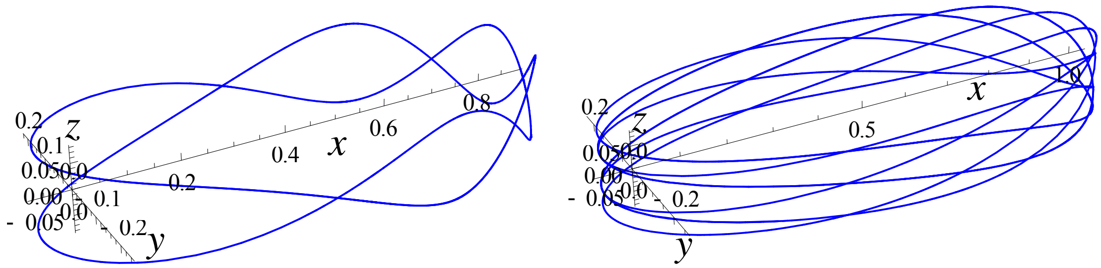
where we have denoted $\gamma = b_r k/b_z$ for brevity. The function $F$ is defined only on the interval $[-z_{\max}, z_{\max}]$, where $z_{\max} = \gamma/(4\beta_0)$, and reaches the maximum of $2\beta_0$ at $z = \pm z_{\max}$. This way, the planes $z = \pm z_{\max}$ form boundaries of the device. It may happen that for certain initial conditions the particle will reach one of the boundaries and escape, so it would no longer form closed trajectories. However, if $z$ stays inside the boundaries, the trajectories will be closed. Fig. 5 shows the trajectories for several choices of the constants $b_r, b_z$.

A similar procedure can be applied to potentials $V_{10}$, $V_{12}$, and $V_{13}$ from Table II of Ref. [20], with the following results for $F_{10}$ and $F_{12}$, where here $\beta_0$ is a rational number:

$$F_{10}(\theta) = \frac{\alpha_0}{(\cos\beta_0\theta+\sin\beta_0\theta)^2} \tag{85}$$

$$F_{12}(\varphi) = \pm\frac{1}{2}\csc^2(2\varphi)\sqrt{(\cos(4\varphi)+1)\left(4\alpha_0^2+4\alpha_0\sqrt{\alpha_0-k_1}\sqrt{\alpha_0+k_1}+k_1^2\cos(4\varphi)-3k_1^2\right)} + \csc^2(2\varphi)\left(\sqrt{\alpha_0-k_1}\sqrt{\alpha_0+k_1}+\alpha_0\right) \tag{86}$$

For $F_{12}(\varphi)$, the negative branch should be chosen when $\varphi$ is small, and the positive branch can chosen after the branch cut for larger values of $\varphi$. $F_{13}(z)$ also has a closed algebraic solution, but is many pages long and thus is not shown here. Similar to $F(z)$ in $V_9$, $F_{13}(z)$ depends on $E$ (as well as $k$, $k_1$, $k_2$, and $\alpha_0$) and only works for rays within a certain $z$ range.




(a) (b)

Figure 5: Trajectories in the potential (76) with $F(z)$ according to Eq. (84) with the parameters $k = 1$, $k_1 = k_2 = 0$, $\beta_0 = 0.9$, $E = 0$ and (a) $b_r = b_\varphi = 2$, $b_z = 5$ and (b) $b_r = b_\varphi = 6$, $b_z = 7$. The part of $V$ depending on $x, y$ is the Newtonian potential, so the projections of the trajectories into the plane $xy$ are the Kepler ellipses. This motion is combined with oscillations in the $z$ direction.

We summarize the action variables for the minimally superintegrable potentials of Evans [20] for an AI in Table 2. ($V_8$ and $V_{11}$ are very similar to $V_7$ and $V_{10}$, respectively, and are omitted from the table.) In each case, the action variable depending on the function $F$ can be expressed using the conditions (20), and the resulting equation can be inverted to find the specific form of $F$. This way, each of these minimally superintegrable systems will form an AI. Some of them will even work for different energies, for example $V_{10}$ or $V_{12}$.

# 7 Conclusion

We have analyzed general properties of focusing potentials and absolute optical instruments by separating the Hamilton-Jacobi equation. We have defined the total action that has a simple interpretation in the optical case as the optical path length of a closed ray. Using methods of theoretical mechanics, we have derived a central result of this paper, namely that the total action should not depend on the separation variables for the trajectories to be closed. This result has a nice interpretation in the optical case where it corresponds to Fermat's principle of stationary time. We have also employed the WKB method for finding the general properties of spectra of absolute instruments; these properties perfectly agree with the ones derived previously for special cases of AIs.

Then, enforcing the condition that the total action must be independent of the separation constants in various coordinate systems at a single energy $E$, we have found numerous absolute optical instruments. In particular, we have applied our theory to potentials with spherical symmetry, including their modifications, and to Cartesian coordinates to confirm the previously known AIs. Working with bispherical coordinates, we have found a class of new AIs that turned out to be transmutations via the spherical inversion of the spherically-symmetric AIs. This idea was then extended to any AI that can be transmuted by inversion to obtain another AI. In the rotational parabolic system, a remarkable new AI was constructed in Sec. 6.3 that gives different types of closed trajectories for different energies as shown in Fig. 3. From superintegrable potentials already known in classical mechanics, we have identified those which can form absolute optical instruments due to the additional freedom from choosing a free function in them and/or fixing the energy which had not been reported as AIs before.

Nonetheless, the theory is still incomplete; although trying, we have been unable to identify new AIs in prolate spherical coordinates, for example, and we have not yet examined some other coordinate systems in which the Hamilton Jacobi equation separates. We suspect that in higher order orthogonal coordinate systems, such as cyclidic coordinates, additional AIs may be identified. Indeed, the identification of superintegrable mechanical systems is an active area of research. In this

Table 2: Minimally superintegrable potentials based on Evans [20]

| Potential and separation coordinates | Action Variables | Notes |
|---|---|---|
| $V_6 = F_6(r) + \frac{k_1}{x^2} + \frac{k_2}{y^2} + \frac{k_3}{z^2}$<br>spherical | $J_\phi = \dfrac{\lvert\alpha\rvert - \sqrt{2k_1} - \sqrt{2k_2}}{2}$<br>$J_\theta = \dfrac{\beta - \lvert\alpha\rvert - \sqrt{2k_3}}{2}$<br>$J_r = \dfrac{1}{\pi}\int_{r_-}^{r_+} \sqrt{2E - 2F_6(r) - \dfrac{\beta^2}{r^2}}\ \mathrm{d}r$ | $F_6(r)$ analyzed in Sec. 6.2.2 and Ref. [9]. |
| $V_7 = k(x^2+y^2) + \frac{k_1}{x^2} + \frac{k_2}{y^2} + F_7(z)$<br>Cartesian | $J_x = \dfrac{\alpha}{2\sqrt{2k}} - \sqrt{\dfrac{k_1}{2}}$<br>$J_y = \dfrac{\beta}{2\sqrt{2k}} - \sqrt{\dfrac{k_2}{2}}$<br>$J_z = \dfrac{1}{\pi}\int_{z_-}^{z_+} \sqrt{2[E - \alpha - \beta - F_7(z)]}\ \mathrm{d}z$ | $F_7(z)$ analyzed in Section Sec. 6.1 (special case). |
| $V_9 = -\frac{k}{\sqrt{x^2+y^2}} + \frac{k_2}{y^2} + \frac{k_1 x}{y^2\sqrt{x^2+y^2}} + F(z)$<br>cylindrical | $J_r = \sqrt{\alpha} + \dfrac{k}{\sqrt{2(\beta - E)}}$<br>$J_\theta = -\sqrt{\alpha} + \dfrac{\sqrt{k_2+k_1}}{\sqrt{2}} + \dfrac{\sqrt{k_2-k_1}}{\sqrt{2}}$<br>$J_z = \dfrac{1}{\sqrt{2}\pi}\int_{z_-}^{z_+} \sqrt{\beta - F(z)}\ \mathrm{d}z$ | $F(z)$ analyzed in Sec. 6.5.2 |
| $V_{10} = k(x^2+y^2+z^2) + \frac{k_3}{z^2} + \frac{F_{10}(x,y)}{x^2+y^2}$<br>cylindrical | $J_z = \dfrac{\beta}{2\sqrt{2k}} - \dfrac{\sqrt{k_3}}{2}$<br>$J_\theta = \dfrac{1}{\pi}\int_{\theta_-}^{\theta_+} \sqrt{\alpha - 2F_{10}(\theta)}\ \mathrm{d}\theta$<br>$J_r = \dfrac{E-\beta}{2\sqrt{2k}} - \dfrac{\sqrt{\alpha}}{2}$ | $F_{10}(\theta)$ analyzed in Sec. 6.5.2 |
| $V_{12} = -\frac{k}{r} + \frac{k_1 z}{r(x^2+y^2)} + \frac{F_{12}(x,y)}{x^2+y^2}$<br>spherical | $J_r = -\sqrt{\beta} + \dfrac{k}{\sqrt{-2E}}$<br>$J_\theta = \sqrt{\beta} - \dfrac{\sqrt{\alpha - k_1}}{\sqrt{2}} - \dfrac{\sqrt{\alpha + k_1}}{\sqrt{2}}$<br>$J_\varphi = \dfrac{1}{\pi}\int_{\varphi_-}^{\varphi_+} \sqrt{2(\alpha - F_{12}(\varphi)}\ \mathrm{d}\varphi$ | $F_{12}(\varphi)$ analyzed in Sec. 6.5.2 |
| $V_{13} = \frac{k}{R} + \frac{k_1\sqrt{R+y}}{R} + \frac{k_2\sqrt{R-y}}{R} + F_{13}(z)$<br>parabolic cylindrical | $J_\sigma = i\dfrac{8\beta E + 8Ek - 8(\beta+k) + k_2^2}{16(-\alpha+E)^{3/2)}}$<br>$J_\tau = i\dfrac{-8\beta E + 8Ek + 8(\beta-k) + k_1^2}{16(-\alpha+E)^{3/2}}$<br>$J_z = \dfrac{1}{\pi}\int_{z_-}^{z_+} \sqrt{2(\beta - F_{13}(z))}\ \mathrm{d}z$ | $R = \sqrt{x^2+y^2}$ $F_{13}(z)$ analyzed in Sec. 6.5.2 |

work we tied this body of literature to classical and wave optics, so that future advances in mechanics would be immediately applicable here as well.

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