Omni-directional cloaking with ideal lenses

Johannes Courtial, 1 Tomáš Tyc, 2 Stephen Oxburgh, 1 Jakub Bélín, 1 Euan N. Cowie, 1 and Chris D. White 3

1 School of Physics & Astronomy, College of Science & Engineering, University of Glasgow, Glasgow G12 8QQ, United Kingdom
2 Institute of Theoretical Physics and Astrophysics, Masaryk University, Kotlarska 2, 61137 Brno, Czech Republic
3 Centre for Research in String Theory, Queen Mary University of London, 327 Mile End Road, London E1 4NS, United Kingdom

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We present the ideal-lens cloak, a simple structure of idealised thin lenses that acts as an omni-directional, ray-optical (and ray-optically perfect), transformation-optics device. The ideal-lens cloak distorts the interior physical space similarly for all viewing directions. If interior physical-space positions are mapped to the exterior, the ideal-lens cloak becomes a new kind of cloak that we call an abyss cloak. We support our results using raytracing simulations. The ideal-lens cloak represents a breakthrough in our quest to build macroscopic, white-light, omni-directional, TO devices. A realisation in which the ideal lenses are approximated by actual lenses should be relatively easy to manufacture.

Introduction. Transformation optics (TO) [1, 2] is a relatively new, and highly active, research field. It uses the mathematics of differential geometry to describe novel metamaterials, capable of bending light in interesting ways. A much-publicised example device is the invisibility cloak, which steers incident light around a central volume, such that the outgoing light rays appear undeviated. Any object in the central region is thus rendered invisible, as is the cloak itself if absorption by the metamaterial is neglected. One decade on, however, very significant difficulties still need to be overcome before metamaterial structures that allow white-light, macroscopic cloaking [3] (or indeed any type of TO) become feasible.

These deficiencies have motivated a number of alternatives to metamaterials, including multidirectional prism structures [4], arrangements of lenses and mirrors [5], and even digital cloaks [5] based on the ideas of integral photography [6]. Particularly relevant to this study are the “Rochester cloak”, a combination of four lenses that can hide objects placed in certain places when viewed from a small range of directions [7], and our own theoretical proposals for omnidirectional cloaks consisting of planar, microstructured telescope windows [8–10]. The latter can be understood as involving pixelated approximations of generalised lenses, or glenses [11], which have two focal lengths instead of one (as in a lens). In Refs [8, 9], both focal lengths of each glens are (differently) infinite, as is the distance of the optic axis. In the second [10], all glens focal lengths are finite, but equal for some glenses such that these are actually ideal thin lenses.

Here we present an omni-directional TO device that consists entirely of ideal thin lenses. Arguably, our device has cloaking characteristics, so we refer to it as ideal-lens cloak. It should be relatively simple to create an experimental realisation in which the ideal thin lenses are approximated by actual lenses. Such an experimental realisation would avoid all shortcomings (e.g. Ref. [12]) related to the pixellation of telescope windows. Our design demonstrates that omni-directional TO, rather than relying on metamaterials, can be realised with just a few lenses.

Imaging in TO devices. Consider light rays that intersect at a given point P inside a TO device, said to be in physical space, and which may be contrasted with the point P’ where the straight-line continuations of the outside segments intersect (Fig. 1). The latter is said to be in electromagnetic (EM) space, and the “T” in TO
refers to the coordinate transformation that maps the two spaces. The above implies that if a small object is placed at a physical-space position $P$, it appears to be located at the corresponding EM-space position $P'$ when seen from outside the TO device.

We define a TO device by the existence of a unique mapping from physical space to EM space [13]. As ideal lenses image every position to a corresponding position, any physical-space position is mapped to a unique EM-space position by any given combination of lenses. But in the ideal-lens cloak, the same physical-space positions can be seen from outside the cloak through different combinations of lenses, and it is by no means obvious that every one of these lens combinations images every point to the same position. For example, Fig. 1 is drawn such that $P$ is imaged to $P'$ irrespective of whether the relevant light rays travelled from $P$ to the outside through lens $L_{01}$ or through the combination of lenses $L_{12}$ and $L_{02}$. In a TO device, this must hold for all lens combinations encountered along any path to the outside, and we will show below that this is indeed the case in the ideal-lens cloak of figure 2. We show this using two theorems called the loop imaging theorem and the edge-imaging theorem which we derive below, and which we also use to calculate the focal lengths of the lenses.

Consider a generic lens structure whose interior physical space comprises polyhedral cells numbered 1, 2, ..., $N$ (Fig. 1), and whose exterior we call cell 0. The lens $L_{ij}$ at the boundary of cells $i$ and $j$ performs optical imaging from cell $i$ to cell $j$ according to a mapping that we will denote $c_{ij}$; the imaging from cell $j$ to cell $i$ is given by the mapping $c_{ij}^{-1}$. As the inside of each cell is empty, light rays therein travel in straight lines. Consequently, points that lie on the same straight line must be mapped to points that lie on another straight line, hence the mappings $c_{ij}$ between neighboring cells are collinear [14].

To find the image $P'$ of a point $P$ in some cell (say cell $i$) formed by rays that passed successively through lenses $L_{ij}, L_{jk}, \ldots, L_{m0}$ (the last lens being at the outer surface of the lens structure), we simply combine the corresponding mappings so that $P' = c_{0m} \cdots c_{kj} c_{ji} P$. Clearly, the lens structure will be a TO device if and only if the position $P'$ is independent of the lens combination by which we can reach the exterior of the device (cell 0) from cell $i$. If this is the case, then we can define a unique mapping $C_i$ from that cell to the exterior of the device, cell 0, according to $C_i = c_{0m} \cdots c_{kj} c_{ji}$, and as cell $i$ is an arbitrary cell, we can do this for any cell. The mapping $C_i$ determines where points in that cell appear when observed from the outside; it is therefore the mapping of cell $i$ of physical space to EM space.

We see that the lens structure will be a TO device if and only if the combined mapping $c_{0m} \cdots c_{kj} c_{ji}$ does not depend on the path $i \rightarrow j \rightarrow k \rightarrow \cdots \rightarrow m \rightarrow 0$ by which we reach cell 0 from cell $i$. In other words,

$$c_{0m} \cdots c_{kj} c_{ji} = c_{0m'} \cdots c_{kj'} c_{ji'}$$

must hold, where $i \rightarrow j' \rightarrow k' \rightarrow \cdots \rightarrow m' \rightarrow 0$ is another path from cell $i$ to cell 0. Thanks to the property $c_{ij} = c_{ji}^{-1}$, we can rewrite this equation as

$$c_{0m} \cdots c_{kj} c_{ji} c_{ij} \cdots c_{m0} = I,$$

where $I$ is the identity map. The LHS of Eq. (2) describes the mapping of cell 0 to itself via a path through cells $m', \ldots, k', j', i, j, k, \ldots, m$ which, as we see, must be the identity mapping. The path, which forms a closed loop that starts and ends in the same cell, here cell 0, satisfies what we call the loop-imaging condition: the combination of all optical elements encountered along the loop images every position back to itself.

By choosing the two paths from cell $i$ to cell 0 such that the final segments of both parts are identical, i.e. by choosing paths in which $m = m'$, and multiplying Eq. (2) by $c_{0m}^{-1}$ from the left and by $c_{m0}^{-1}$ from the right, we see that the loop $m \rightarrow \cdots \rightarrow k' \rightarrow j' \rightarrow i \rightarrow j \rightarrow k \rightarrow \cdots \rightarrow m$ must also satisfy the loop-imaging condition, and the same result can be obtained similarly for any cell and any closed loop in any TO device. We call this result the loop-imaging theorem.

The loop-imaging theorem holds specifically for closed loops encircling just a single edge in the lens structure (the equivalent statement in 2D is about closed loops encircling an individual vertex, of which Fig. 1 shows an example), which are the smallest loops that satisfy the loop-imaging condition non-trivially. Moreover, by combining several such loops into one it is easy to see that if all closed loops encircling any single edge yield the identity mapping, then any closed loop will yield the identity mapping as well. This, in turn, implies that the mapping from physical space to EM space is unique, and that the device therefore satisfies the definition of a TO device.

We define the edge-imaging condition as the requirement that, for every one of the edges in a structure, successive imaging by all lenses meeting at the edge yields the identity mapping. In this way, we arrive at a simple formulation of the edge-imaging theorem: A lens structure is a TO device if and only if it satisfies the edge-imaging condition. The edge imaging theorem applies equally to structures composed of lenses [11].

Ideal-lens cloak. Fig. 2 shows a model of the proposed ideal-lens cloak, where all external and internal faces bounded by red tubes denote the presence of a lens with a suitable focal length. The lenses divide the interior space of the device into seven tetrahedral cells; the space between the lenses is empty.

We use equations derived from the loop-imaging theorem applied to the edges in the ideal-lens cloak to calculate the focal lengths of all lenses from the parameters defined in Fig. 2 and the focal length of the bottom
FIG. 2. Model of the omni-directional ideal-lens cloak. Each face, enclosed by red tubes, constitutes a lens (16 in total). The lenses form a structure that can be understood as three nested tetrahedra that share a base (an equilateral triangle with vertices $V_1$, $V_2$ and $V_3$, centred at $N_0$, and with circumradius $R$) but have different heights, respectively $h_1$ (the tetrahedron with fourth vertex $N_1$), $h_2$ (fourth vertex $N_2$), and $h$ (the outermost tetrahedron with fourth vertex $N_3$). Black spheres are centred on the positions $N_0$ to $N_3$, which are the positions of the nodal points of the lenses. (The nodal point of a thin lens coincides with the lens’s (optical) centre; we use here nodal point instead of centre to avoid confusion between the optical centre and the aperture centre, which are different for all but the base lens.) The vertical line through the nodal points is an axis of three-fold symmetry.

FIG. 3. Raytracing simulations of the ideal-lens cloak. (a) Physical-space structure, where each face bounded by red tubes is a lens. A white sphere is placed inside the structure. (b) Visual appearance of the device (with idealised lenses, made visible with a transmission coefficient of 0.9). The appearance is now that of electromagnetic space, where the inner vertices appear shifted from their physical-space positions, and the white sphere appears distorted, displaced, and de-magnified. Focal lengths of the lenses are chosen such that the height of the bottom tetrahedron in EM space is 1/5 of its physical-space height. The simulations were performed using an extended version of our custom raytracer Dr TIM [15]. Note that Dr TIM is not capable of correctly simulating the shadows of objects inside TO devices [10], which is why we usually switch off such shadows, as was done here for the shadow of the white sphere.

One may demonstrate the action of the cloak using raytracing simulations. Fig. 3(a) shows a model of the physical-space structure, similar to that shown in Fig. 2, together with additional objects placed outside the cloak and a white sphere placed inside the bottom tetrahedron. In Fig. 3(b), the model is replaced with the corresponding ideal-lens cloak. Objects external to the cloak appear invariant upon looking through it, as expected given that all exterior points are mapped to themselves. Points inside the cloak, however, are in general imaged to a different position, and so the inside of the cloak appears distorted. Specifically, both inner vertices appear lower, and the white sphere in the bottom tetrahedron appears lower, distorted, and smaller. We have further verified the functionality of our ideal-lens cloak by performing numerous raytracing simulations (not shown) with different parameters and different view directions.

**Abyss cloak.** In the cloak of Fig. 2, the focal lengths can be chosen such that physical-space positions inside the cloak get mapped to EM-space positions outside the cloak. For example, it is easy to choose the focal length of the base lens such that the position of the nodal point $N_1$ is imaged to the other side of that lens, i.e. below the cloak and therefore outside it. Any point object placed at a position $P$ sufficiently close to that vertex will then also be imaged to a position $P'$ outside the cloak. However, the image can be visible only when light rays have actually passed through the object inside the cloak, which is clearly only possible if the image is seen in the same direction as part of the cloak (Fig. 4(a)). For larger objects, only the part of the image that is seen in the same solid angle as the cloak is visible.

The other frames of Fig. 4 show raytracing simulations of an ideal-lens cloak with the physical-space structure shown in Fig. 3(a) but with a choice of focal lengths such
that the lower inner vertex is imaged to the same distance from the base lens, but on the opposite side of the base lens. The cloak also contains the white sphere shown in Fig. 3(a), identically placed. In Fig. 4(b), the camera is positioned such that the image of the white sphere is seen in a different direction than the cloak, and so it is invisible. In Fig. 4(c), the camera position is chosen such that the image of the white sphere is seen behind the cloak, and so it is now visible. From above, it appears as if the sphere is placed inside an invisible abyss that opens at the cloak. For this reason, we call it an abyss cloak.

Discussion. The main result of this study is that it is possible to realise omni-directional TO devices using structures of idealised, thin, lenses. Unfortunately, there are no known perfect physical realisations of ideal thin lenses, and so while the ideal-lens cloak is perfect in many ways, a real-lens cloak will not be. Imperfections include imperfect imaging, chromatic errors, and Fresnel losses, but on the plus side such devices will be very significantly easier and cheaper to manufacture than comparable metamaterial devices.

Our next step is to realise such a real-lens cloak experimentally. We would like to build a demonstrator that has a significant effect, i.e. in which EM space and physical space are quite different, but this requires lenses with focal lengths close to zero. However, the smaller the focal length of a lens, the worse its imaging quality. There will therefore be a trade-off between achieving a significant effect and good quality.

We believe that the ideal-lens cloak proposed in this paper is only the first in a whole series of omni-directional TO devices made from lenses. It will be possible to design other devices; for example, two or more ideal-lens cloaks can be combined, e.g. one nested inside the other. More interestingly, however, it should be possible to construct lens structures with completely different geometries that act as TO devices. More calculations are required for such cases, but the loop imaging theorem provides the requisite tool for doing this.

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* Johannes.Courtial@glasgow.ac.uk

[13] Note that this mapping can be many-to-one, as several physical-space positions can map to the same EM-space position.