Magnifying absolute instruments for optically homogeneous regions

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We propose a class of magnifying absolute optical instruments with a positive isotropic refractive index. They create magnified stigmatic images, either virtual or real, of optically homogeneous three-dimensional spatial regions within geometrical optics.

Most optical instruments, including a simple lens or sophisticated camera lenses, have various types of aberrations. There exist, however, optical instruments that are free of aberrations and provide sharp (stigmatic) images of all points in some 3D region of space within geometrical optics; such devices are called absolute instruments [1]. A prototype of an absolute instrument is Maxwell's fish eye, a device designed in 1854 by J. C. Maxwell [2]. It uses positive refractive index and images stigmatically the whole space. Another type of absolute instrument is based on materials with a negative refractive index. It was proposed by J. Pendry in 2000 [3] and later realized experimentally [4]. Remarkably, both Maxwell's fish eye and Pendry's lens are not limited by diffraction and provide sub-wavelength resolution [3–7].

Among absolute instruments there is a class of a particular interest, namely devices whose object and image spaces are optically homogeneous regions, i.e., regions with a uniform refractive index. Until recently, the only known such devices were plane mirrors [1]. This has been changed by a recent excellent work of J. C. Miñano [8] who proposed several new absolute instruments imaging homogeneous regions and also showed that some well-known optical devices such as Eaton lens or Luneburg lens [9] are in fact absolute instruments as well. All of these devices have unit magnification, giving an image of the same size as the original object, and no magnifying absolute instrument for homogeneous regions has been known. We proposed a magnifying absolute instrument recently [10] based on a numerically found refractive index with certain special properties, but our colleague Klaus Bering has later shown analytically that such an index in fact does not exist [11].

Here we present several magnifying absolute instruments that provide stigmatic images of homogeneous regions of 3D space with an arbitrary magnification. They are all based on the same idea and provide either real or virtual images. This is the first proposal of a magnifying absolute instrument for homogeneous regions that employs isotropic materials with positive refractive index. We will explain our idea first on a particular example of a magnifying absolute instrument resembling Eaton lens [12], and then proceed to other devices.

Our device consists of two distinct regions (see Fig. 1). The first region (we will call it region I) is a sphere of unit
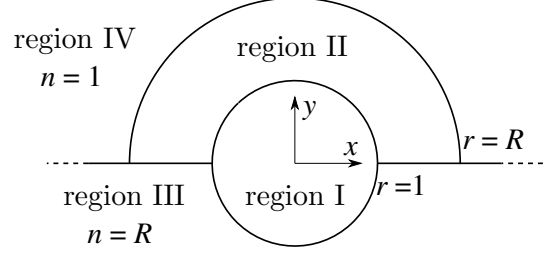

FIG. 1: Regions in the magnifying device based on Eaton lens. Regions III and IV that lie outside the device are the optically homogeneous object and image space.

radius, the second region (II) occupies the space between two hemispheres with the radii 1 and $R > 1$, respectively, lying in the half-space $y > 0$. Both regions are filled with a spherically symmetric refractive index that we will denote by $n_{\mathrm{I}}(r)$ and $n_{\mathrm{II}}(r)$, respectively. The indices are chosen such that $n_{\mathrm{I}}(1) = n_{\mathrm{II}}(1) = R$ and $n_{\mathrm{II}}(R) = 1$. The medium surrounding the lens is composed of two parts as well. In the region $r \geq 1, y < 0$ (region III) the refractive index is equal to R while in the region $r \geq R, y > 0$ (region IV) the refractive index is equal to unity. Thus the index of the lens matches that of the surrounding medium, apart from the annulus $1 < r < R, y = 0$, and also the indexes at the border between regions I and II match each other.

The refractive index $n_{\mathrm{II}}(r)$ is chosen such that a light ray incident from region IV to region II is bent towards the center, eventually crossing the border between regions II and I. There is a large variety of refractive index profiles that achieve this, one option is to choose $n_{\mathrm{II}}(r) = [1 + c(r-1)(R-r)]R/r$ with $c > 0$ sufficiently large, which we have also used in our ray tracing simulations with $c = 1$. We then design the refractive index $n_{\mathrm{I}}(r)$ in region I such that the light ray coming from region II is bent further and leaves region I for region III in exactly opposite direction than was the original direction of the ray in region IV, see Fig. 2. The performance of the device is thus similar to the performance of Eaton lens; we can therefore call it "magnifying Eaton lens". As we shall see, the difference is that the impact parameter of the outgoing ray is R times smaller than the impact pa-
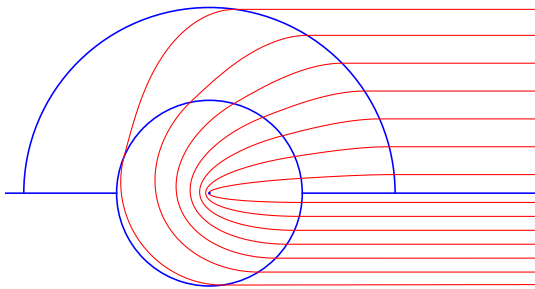
FIG. 2: Magnifying Eaton lens with $R = 2$. Light rays incident on it from region IV are changed into rays in region III propagating in the opposite direction, with impact parameters reduced by the factor of R.
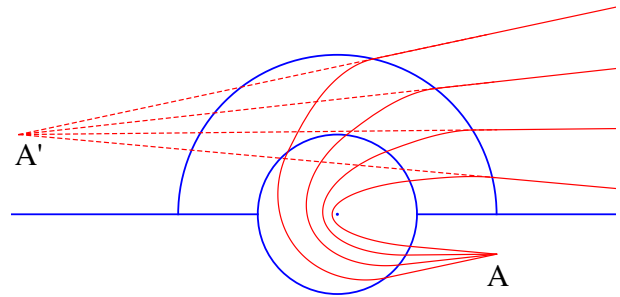

FIG. 3: Image formation in magnifying Eaton lens. Light rays emerging from a point A at $\vec{r}_A$ in region III, after passing through the lens, propagate in region IV as if they originated from a point A$'$ at $\vec{r}_{A'} = -R\vec{r}_A$. The image is virtual and is magnified R-times.

rameter of the incoming ray, and this fact is responsible for the lens magnification.

To design the refractive index $n_I(r)$, we employ the standard method for solving the inversion problem [9, 13]. We will characterize rays in the lens by the quantity $L = nr \sin \alpha$ analogous to mechanical angular momentum, where α is the angle between the radius vector and the ray [14]. Angular momentum is conserved and motion of a particle is planar in central potentials, and the same holds for light ray in a spherically symmetric refractive index. Consider a ray propagating in the plane xy horizontally (i.e., in the direction of x-axis) in region IV and entering region II of the lens at point A. Since $n = 1$ in region III, the angular momentum L is equal to the impact parameter of the incoming ray. The polar angle h swept by the ray in region II before entering region I (say, at point B) can be calculated by the expression [15]

$$h(L) = L \int_1^R \frac{\mathrm{d}r}{r\sqrt{[rn_{II}(r)]^2 - L^2}}. \tag{1}$$

$h(L)$ is at the same time the change of the ray direction during propagation in region II from point A to B. This is because the product nr is the same at both points A and B, so is $L = nr \sin \alpha$, and therefore the angle α between the ray and radius vector is the same at B as is in A.

The scattering angle χ (change of ray direction) corresponding to motion in region I must therefore be

$$\chi(L) = \pi - h(L), \tag{2}$$

which ensures that the total change of ray direction during motion in regions II and I is π. Solving the inversion problem, we then arrive at the following implicit equation for the refractive index $n_I(r)$ [9, 13]:

$$n_I(r) = R \exp \left[\frac{1}{\pi} \int_{rn_I(r)}^R \frac{\chi(L)\mathrm{d}L}{\sqrt{L^2 - [rn_I(r)]^2}} \right]. \tag{3}$$

This way the refractive index is expressed analytically, although not explicitly. The refractive index in regions I and II is shown in Fig. 4.

Now we have to show that the device we have just described is indeed an absolute magnifying instrument, i.e., it provides stigmatic image of some 3D region of space. First we note that from conservation of L it follows that the impact parameter of the outgoing ray in region II is R-times smaller than the impact parameter of the incoming ray in region IV. Second we note that although we considered a horizontal ray in the xy-plane in our construction, the lens will have a similar effect on most other rays too, i.e., rays incident on it from region IV will be changed to rays moving in the opposite direction in region III and having R-times smaller impact parameters. This is caused by the spherical symmetry of the refractive index in regions I and II. There will also be rays for which this does not happen, namely the ones that at some point cross the interface between regions II and III, but still for an infinite number of rays the lens does the job it is designed for.

Now consider a collection of rays emerging from some point A at radius vector $\vec{r}_A$ in region III and incident on region I of the lens. As we have seen, these rays will be transformed by the lens into rays propagating in region IV, each parallel to the original ray in region III. Therefore the lines on which these outgoing rays lie intersect at the point A$'$ with radius vector $\vec{r}_{A'} = -R\vec{r}_A$, which this way becomes the virtual image of the point A (see Fig. 3), and the magnification of the device is clearly equal to R.

As can be shown, the refractive index in region I diverges for $r \to 0$. To avoid this singularity, we can modify the lens by utilizing the fact that Luneburg lens [9] equipped with a spherical mirror on its surface has the same effect on the incoming rays [8] as Eaton lens. Imagine we place a mirror on the part of the interface between regions I and II, allowing the rays in region I to be reflected before re-entering region II. We again require that the outgoing rays move in the opposite direction with respect to the incoming rays, but now rays with small impact parameters do not have to make a rapid turn near the origin at it was the case with magnifying Eaton
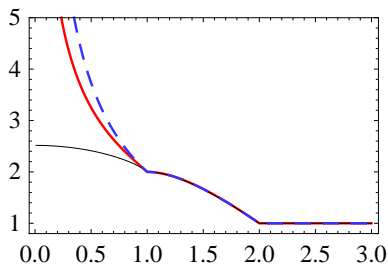
FIG. 4: Refractive index in regions I and II of magnifying
Eaton lens (thick red), Luneburg lens (thin black) and mag-
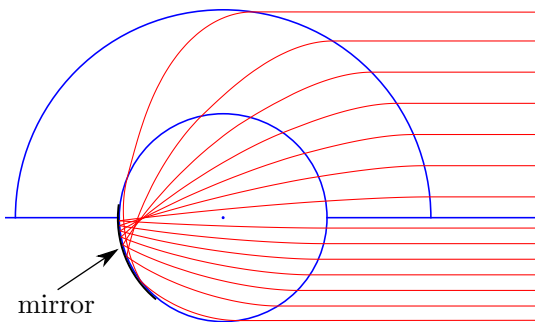nifying invisible sphere (dashed blue) with $R = 2$.

FIG. 5: Magnifying Luneburg lens with $R = 2$. A spherical
mirror covers part of the interface between regions I and II.
The effect on rays coming from region IV is the same as that
of magnifying Luneburg lens and therefore it also provides a
magnified virtual image.

lens, and therefore the index will not need a singularity
there. It can be shown by simple geometrical consider-
ations that in this case the required scattering angle in
region I corresponding to the ray segment between point
of entrance to region I and the point of incidence on the
mirror is

$$\chi(L) = \arcsin \frac{L}{R} - \frac{h(L)}{2}, \qquad (4)$$

which, after substitution to Eq. (3) leads to the refractive
index shown in Fig. 4 for $R = 2$. Ray tracing in the
magnifying Luneburg lens is shown in Fig. 5.

Another interesting magnifying device can be derived
by our method from the invisible sphere described in [13].
Here we require again that the rays leaving the lens prop-
agate parallel to their original direction, but this time go
forwards instead of backwards. The scattering angle in
this case is $\chi(L) = 2\pi - h(L)$. If the lens should work
well for rays with the direction close to x-axis, the border
between regions III and IV now has to be in the yz plane
and similarly region II now lies at $x > 0$ instead of $y > 0$.
The virtual image of a point A is now formed at point
A$'$ with $\vec{r}_{A'} = R\vec{r}_A$. Ray tracing in this lens is shown in
Fig. 6, the refractive index is in Fig. 4.
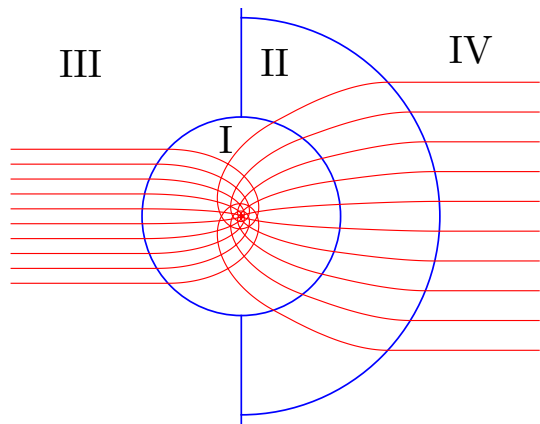
The last device we will discuss is a magnifying absolute

FIG. 6: Magnifying device derived from the invisible
sphere [13]. The rays make loops in region I and after leav-
ing the device, they propagate in the original direction. This
device again forms a virtual image.

instrument that provides real images. Now we have to
arrange regions I – IV in a somewhat different fashion,
see Fig. 7: region I is given by the condition $r > R$, so
it is the whole 3D space with the exception of the sphere
of radius R. Region II occupies the space between two
hemispheres with the radii 1 and $R > 1$, respectively,
lying in the half-space $x > 0$. Region III is the unit
hemisphere at $x > 0$ and region IV is a hemisphere of
radius R at $x < 0$. Refractive indices in regions II, III
and IV are as before. Light rays now enter the lens (re-
gion II of it) from region III, i.e., from the inside. After
having been bent in region II, they propagate in region I
and we require that when they enter region IV, they are
parallel to their original direction in region III, see Fig. 7.
To find the refractive index $n_I(r)$ that achieves this, we
have to solve an "outer" inversion problem instead of the
usual "inner" problem. This can be done by employ-
ing inversion in the sphere of radius R which transforms
the outer problem to the inner one. A careful analysis
of the scattering angle χ' in the transformed problem
reveals that $\chi'(L) = 4\arcsin(L/R) - h(L)$, which then
gives the transformed refractive index $n_I'(r')$ as a func-
tion of the transformed radius $r' = R^2/r$ with the help
of formula (3), but with the "R" omitted in front of the
exponential since we require $n_I'(R) = 1$. The index $n_I(r)$
can then be calculated as

$$n_I(r) = \frac{R^2}{r^2} n_I'(R^2/r), \qquad (5)$$

which follows from the equality of optical paths in the
original and transformed region I.

To see that this device indeed creates a real magnified
image, consider rays emerging from some point A at ra-
dius vector $\vec{r}_A$ in region III, see Fig. 8. The rays get to
region IV assuming their original direction and head to-
wards the point A$'$ at $\vec{r}_{A'} = R\vec{r}_A$. This point lies outside
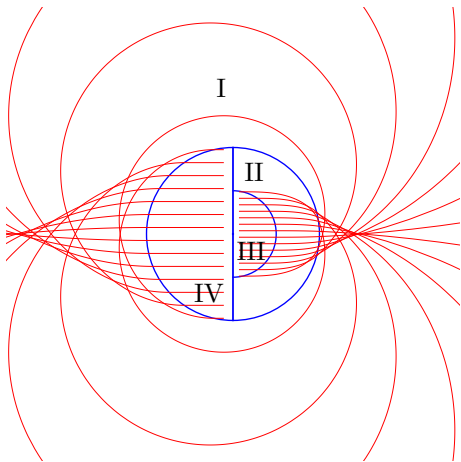of region IV and therefore the image is virtual. However,

FIG. 7: Inside-out magnifying instrument with the regions marked. The rays enter the lens from region III inside the lens (the object space) and after making loops in region I, which now extends to infinity, enter region IV (image space) in their original direction.
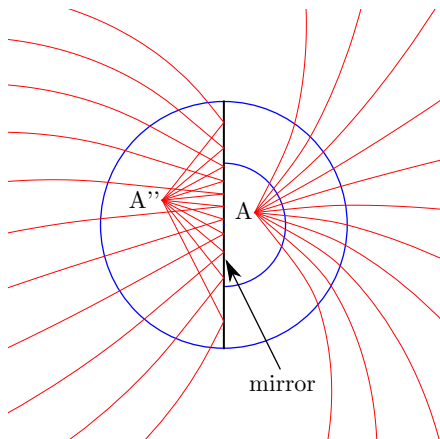

FIG. 8: Image formation on the instrument from Fig. 7. Rays originating at point A in region III and converging in region IV to a point outside this region are reflected by the mirror to the real image A'' of A.

we can take advantage of the fact that the rays are converging and place a mirror at the flat interface of region IV, see Fig. 8. This way the virtual image it turned into a real image at point A'', which is a mirror image of A' in the plane $x = 0$. Making the mirror double-sided, also rays emerging from the point A to the left will contribute to forming the image.

In conclusion, we have proposed several absolute optical instruments that create magnified stigmatic images of homogeneous 3D region. They are all designed by the same general idea. Especially appealing is the lens giving a real magnified image and the magnifying Luneburg lens with its moderate refractive index range; for R not too large, it should be possible to realize the latter for near infrared or even visible light using e.g. graded index structures in silicon [16] or diamond. Further research will reveal whether some of these devices, e.g. the lens giving the real image, could provide sub-wavelength resolution similarly to Maxwell's fish eye [5–7]. Magnifying absolute instruments could find their applications in various fields, for example in photolitography, but more importantly, our research has shown that such devices exist at all, something that was not clear until this date [1, 8].

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