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A simple model explaining super-resolution in absolute optical instruments

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Abstract

We develop a simple, one-dimensional model for super-resolution in absolute optical instruments that is able to describe the interplay between sources and detectors. Our model explains the subwavelength sensitivity of a point detector to a point source reported in previous computer simulations and experiments (Miñano 2011 New J. Phys. 13 125009; Miñano 2014 New J. Phys. 16 033015).

Perfect imaging with positive refraction [1] has been subject to considerable controversy [2-17] that has given important insight into the matter. Much of the controversy has centred on the role of detection in perfect imaging-the perfect transfer of the electromagnetic field from object to image is only possible if the image is detected. Another important point was noticed in a computer simulation [18] and a subsequent experiment [19] by Miñano et al: at specific resonance frequencies of the instrument a point detector is sensitive to displacements of a point source with an accuracy that is significantly better than the diffraction limit. No physical explanation for this feature has been found yet. Here we develop a simple model that captures both issues, the role of the detection and the role of the resonance, which allows us to deduce both physical explanations and analytic expressions for the sensitivity.

Perfect-imaging devices with positive refraction are absolute optical instruments [20] with closed loops of rays [21, 22]. The archetype of such instruments is Maxwell's fish eye [23] where light goes in circles and where all light circles originating from any given point intersect at a corresponding image point. Luneburg [24] discovered a geometrical picture that explains the properties of Maxwell's fish eye: the refractive-index profile of Maxwell's device appears to light as the surface of a sphere in two-dimensional (2D) and hypersphere in threedimensional (3D) space; light propagates in the medium of the fish eye as if it were confined to spherical surfaces. The geodesics on the sphere appear as the circles of light (by stereographic projection), object and image correspond to antipodal points on the sphere where geodesics intersect. Let us consider the simplest case of perfect imaging, the one-dimensional (1D) sphere: the circle (figure 1). Imagine that light is confined to a circle, say a fibre loop or ring resonator. Here light can go in only two directions, to the right or to the left. An 'image' is formed when the two rays meeting have the same phase, which happens when both are antipodal. Light is coupled in and out of the circle by two 1D channels that represent the source and the detector. These 1D channels are idealizations of the cables used for injecting and extracting radiation in the simulation [18] and experiment [19]. Clearly, this 1D system represents a rather primitive model, but it is going to reproduce the findings of the experiment [19], the model is simple, but not too simple.

We are going to show how a point detecter is able to sense minute displacements of a point source. Note that this is not imaging in the traditional sense of taking the image of a source distribution all at once, but rather it corresponds to scanning, as follows. Suppose that the detector is moved across the imaging region of the device.

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The detector would only produce a signal when it is close to the imaging point of the point source. The resolution is the distance from the actual imaging point where the detector begins to fire. We will show that, for light at the resonance frequency of the instrument, the resolution of the scan is infinitely fine. Well-known and widely-used examples of scanning methods that beat the diffraction limit are near-field scanning optical microscopy [25] and fluorescence microscopy [26]. In contrast to the former, in the case considered here the detector is placed far away from the source, and in contrast to the latter, only linear optics is used. The idea of Miñano *et al* [18] of turning an absolute optical instrument into a super-resolving scanning device may thus find

a place in the arsenal of methods for breaking the diffraction limit. Here we explain how it works. The first important conceptual point we make concerns the sources and detectors. In the context of the controversy on perfect imaging, finding a simple, practical model for detectors has been a long-outstanding problem [9, 15, 17, 27–30]. Usually a source was assumed to be a predetermined current that generates electromagnetic waves, a detector was modelled as a drain. Active drains are produced by predetermined currents just like sources; passive drains are supposed to react to the in-coming electromagnetic radiation and absorb it. In truth, both sources and detectors are neither fully active nor fully passive. A detector is a dynamical system, it responds to the electromagnetic field by absorbing radiation, and then reacts back to the field: it is both passive and active. Also a realistic source is not simply a predetermined current that generates the field: the source must be able to receive radiation not captured by the detector, because otherwise an equilibrium between in- and out-going radiation is impossible. The simplest source is an atom in an excited state, emitting light, the simplest detector is an atom in the ground state, absorbing light. In the experiment [19] the source was a cable where microwave radiation was injected from a synthesiser, the detector was a cable connected to a vector analyser. Source and detector are essentially the same, what distinguishes them are the initial conditions: in a source radiation is injected with a predetermined flux, in a detector radiation is not injected, although it could be, if, for example, the detecting atom were excited or the detector cable connected to a synthesiser. We thus need to consider sources and detectors as identical physical systems with different initial conditions.

Describing sources and detectors is particularly easy in our 1D model: both are identical vertices where an external channel is connected to the circle (figure 2). We model them as specific linear multiports [31, 32] using the following arguments. Consider one vertex, say the source. The vertex has three in-going modes, the mode incident through the external channel and one clockwise- and one counter-clockwise-propagating mode in the device. The vertex turns these three in-going modes into three out-going modes: it is a six-port. As the out-going radiation must be proportional to the in-going radiation, the six-port is required to be a linear device. We denote





the complex amplitude of the incident radiation in the external channel by a_0 and the amplitudes of the clockwise- and counter-clockwise-propagating waves in the circle by a_{\pm} ; together they constitute the amplitude vector $\mathbf{a} = (a_0, a_+, a_-)^T$. To denote the out-going modes we use primes: $\mathbf{a}' = (a'_0, a'_+, a'_-)^T$. We require that the vertex performs a linear transformation:

$$a' = Va. \tag{1}$$

As a consequence of energy conservation, a Hamiltonian for the mode transformation must exist, from which follows that *V* is unitary [31]. In order to deduce the specific form of *V* we make use of the semantics and the symmetries of our system. At the source, light is coupled in with a certain efficiency and then distributed equally to the two waves in the device. At the detector, light is captured equally from the two waves and coupled out with a certain efficiency, assumed to be the same as for the source. Let us mentally separate the in- and out-coupling from the equal distribution and gathering, by writing *V* as

$$V = S T S^{-1} \tag{2}$$

with

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}.$$
 (3)

The orthogonal matrix *S* describes the equal distribution of an amplitude from an intermediate channel to a_{\pm} . As intermediate channel we have chosen the second component of the amplitude vector. The $1/\sqrt{2}$ terms guarantee that the sum of the intensities of the outgoing amplitudes is equal to the modulus squared of the initial amplitude. By the same token, the inverse matrix S^{-1} describes the equal gathering of the clockwise- and counter-clockwise-propagating waves. The intermediate channel is coupled to the external channel via the matrix *T* that depends on the coupling efficiency. We know that *V* must be unitary, so *T* should be a 2D unitary transformation between the external and the intermediate channel. For simplicity, we assume *T* to be real. For perfect coupling *T* describes a flip between the two channels, for imperfect coupling an incomplete flip:

$$T = \begin{pmatrix} -\sin\alpha & \cos\alpha & 0\\ \cos\alpha & \sin\alpha & 0\\ 0 & 0 & 1 \end{pmatrix}.$$
 (4)

The angle α parameterizes the coupling efficiency; for $\alpha = 0$ we get a perfect flip and hence perfect coupling, for general α the cosine of α describes the transmissivity τ and sin α the reflectivity ρ . It turns out to be wise to parametrize the coupling as

$$g = \sqrt{2} \tan\left(\frac{\pi}{4} - \frac{\alpha}{2}\right),\tag{5}$$

where we get for the transmissivity and reflectivity

$$\tau = \cos \alpha = \frac{2\sqrt{2}g}{2+g^2}, \quad \rho = \sin \alpha = \frac{2-g^2}{2+g^2}.$$
 (6)

In terms of g we find for the vertex matrix

$$V = \begin{pmatrix} \frac{g^2 - 2}{g^2 + 2} & \frac{2g}{g^2 + 2} & \frac{2g}{g^2 + 2} \\ \frac{2g}{g^2 + 2} & \frac{2}{g^2 + 2} & -\frac{g^2}{g^2 + 2} \\ \frac{2g}{g^2 + 2} & -\frac{g^2}{g^2 + 2} & \frac{2}{g^2 + 2} \end{pmatrix}.$$
(7)

The form (7) of the vertex matrix reveals an important property: V is symmetric

$$V = V^T.$$
 (8)

As V is an orthogonal matrix, V^T must be the inverse of V, and thus

$$V = V^{-1}.$$
(9)

The in-coupling vertex is also the out-coupling vertex, sources and detectors are fundamentally the same.

Let us briefly discuss two limiting cases, g = 0 and $g = \infty$. For g = 0 we obtain V = diag(-1, 1, 1), the incident radiation is perfectly reflected with the reflected radiation changing sign, while the modes inside are not changed at all. They are shielded from the external channel: the case g = 0 corresponds to zero coupling where the vertex acts as a perfect mirror. For $g = \infty$ we obtain

$$V = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix},$$

the incident field is also reflected, but without changing sign, while the waves inside the circle are reflected with change of sign: the case $g = \infty$ describes a perfect scatterer. Perfect coupling corresponds to $\alpha = 0$ and so to the case $g = \sqrt{2}$. In order to find a direct interpretation for *g* we derive from equations (1) and (7) the relations

$$a'_{+} + a_{-} = a_{+} + a'_{-}, \quad a_{0} + a'_{0} = g(a'_{+} + a_{-}).$$
 (10)

We may interpret relations (10) as conditions on the amplitudes of the fields and in this way obtain an interpretation for *g* as follows.

Consider the field in the circle around the source point. As coordinate on the circle we chose the angle ϑ where the source sits at $\vartheta = 0$. The field shall oscillate ν times along the circle, ν corresponds to the wavenumber with respect to the angle ϑ . If ν is integer the light is resonant. On the right of the source point (figure 2) the clockwise-propagating mode is outgoing with amplitude a'_+ and the counter-clockwise mode is incident with amplitude a_- . On the left of the source the clockwise-propagating mode is incident with a_+ and the counterclockwise mode is outgoing with a'_- . Hence we can write the complex field ψ as

$$\psi = \begin{cases} a'_{+}e^{i\nu\vartheta} + a_{-}e^{-i\nu\vartheta} & \text{for } \vartheta \ge 0, \\ a_{+}e^{i\nu\vartheta} + a'_{-}e^{-i\nu\vartheta} & \text{for } \vartheta \le 0. \end{cases}$$
(11)

Relations (10) show that the field is required to be continuous at the source. We also see that the intensity $|a_0 + a'_0|^2$ in the external channel is g^2 times larger than the field intensity inside. For example, for perfect incoupling $g = \sqrt{2}$, so the incident channel must provide twice the intensity of the field inside, as one would expect, because the incident radiation propagates away in two directions. The parameter *g* thus describes the ratio between external and coupled field amplitude, which is a useful parameterization of the coupling.

Having developed a simple model for both the source and the detector, we can now combine it with the propagation in the device (figure 1). The detector is shifted by δ from the antipodal position (it sits at $\vartheta = \pi + \delta$ while the source sits at $\vartheta = 0$). Waves propagating in positive direction from the source thus experience a phase shift of ν ($\pi + \delta$) followed by a phase shift of ν ($\pi - \delta$) back to the source, for waves propagating in negative direction the phase shift is ν ($\pi - \delta$) followed by ν ($\pi + \delta$). The total phase of each round trip is $2\pi\nu$, a constant, which is the property of an absolute optical instrument [20–22] that we use. We describe the propagations with the help of the matrices

$$U_{\pm} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\nu(\pi\pm\delta)} & 0 \\ 0 & 0 & e^{i\nu(\pi\pm\delta)} \end{pmatrix}.$$
 (12)

The light coupled in at the source propagates to the detector,

$$\begin{pmatrix} a_0' \\ b_+ \\ b_- \end{pmatrix} = U_+ V \begin{pmatrix} a_0 \\ a_+ \\ a_- \end{pmatrix},$$
(13)

where a'_0 denotes the light reflected back to the source and the b_{\pm} are the wave amplitudes incident at the detector. There the light is partly coupled out, with amplitude a'_1 , partly reflected back to the source

$$\begin{pmatrix} a_1' \\ a_+ \\ a_- \end{pmatrix} = U_- V \begin{pmatrix} a_1 \\ b_+ \\ b_- \end{pmatrix}.$$
(14)

We require that light with unity amplitude is incident at the source and that no light enters through the detector

$$a_0 = 1, \quad a_1 = 0.$$
 (15)

The transmission coefficient t is given by

$$t = |a_1'|^2. (16)$$

Equations (13)–(15) with definitions (7) and (12) establish six inhomogeneous linear equations for the six variables a_{\pm} , b_{\pm} , a'_0 and a'_1 with a unique solution. Solving this system we obtain for the transmission coefficient

$$t = \frac{16g^4 \cos^2 \nu \delta \, \sin^2 \nu \pi}{4g^4 \sin^2 2\nu \pi + \left(\left(g^4 + 4 \right) \sin^2 \nu \pi - g^4 \sin^2 \nu \delta \right)^2}.$$
 (17)

Figure 3 shows the transmission as a function of ν for various coupling strengths *g* and offsets δ . Let us discuss the most relevant limiting cases. For perfect alignment, $\delta = 0$, we obtain from our result (17)

$$t_0 \equiv t \mid_{\delta=0} = \frac{16g^4}{16g^4 + \left(g^4 - 4\right)^2 \sin^2 \nu \pi}.$$
(18)

For perfect alignment the transmission is periodic in ν , it reaches unity at resonance where ν is integer, and we obtain for the integral

$$T = \int_0^1 t_0 \, \mathrm{d}\nu = \frac{4g^2}{4+g^4}.$$
(19)

For perfect coupling, $g = \sqrt{2}$, the transmission (18) is unity for all ν , but for different coupling parameters the total transmission (19) lies below unity. The device behaves like a typical Fabry–Perot resonator [20], it spectrally distributes the transmission such that at resonance it always reaches unity. This Fabry–Perot feature of the device is completely expected. The surprising feature of the perfect-imaging device appears for $\delta \neq 0$ where source and detector are misaligned. We directly see from our result (17) that

$$t = 0 \quad \text{for } \delta \neq 0 \text{ and } \nu \in \mathbb{N}.$$
(20)

Exactly at resonance, the Fabry–Perot transmission curve drops to zero for $\delta \neq 0$. To deduce a measure for the width of the dip we calculate the second derivative of *t* at the resonance (the first derivative vanishes, as zero is obviously a minimum of $t = |a_1|^2$). We find

$$\frac{1}{2} \frac{\partial^2 t}{\partial \nu} \bigg|_{\nu \in \mathbb{N}} = \frac{16 \,\pi^2 \cos^2 \nu \delta}{g^4 \sin^4 \nu \delta} \sim \frac{\pi^2}{(g\nu \delta/2)^4} \quad \text{for small } \delta \neq 0.$$
(21)

The smaller the displacement δ of the detector the sharper is the dip. However, regardless how small the displacement is, at resonance occurs a step change between alignment and displacement. This ultrasensitive behaviour can be used to measure, with a fixed detector, small displacements of the source, as only the relative angle between source and detector matters. Irrespective whether our model is of direct practical relevance, it represents the simplest toy model for the super-resolution in absolute optical instruments. The diffraction limit of imaging [20] would suggest a resolution of $\nu \delta/2 \sim 1$, here the resolution is in-principle unlimited for finite wavenumber. Miñano *et al* observed the characteristic transmission dips for a 2D system in a computer simulation [18] and then in an experiment [19]; let us call them *Miñano dips*. Here we have captured this characteristic feature in a simple formula⁵.

 $^{^{5}}$ In practice [18, 19] the Miñano dips were shifted by about 10^{-3} with respect to the exact resonance, which is presumably due to the finite size of the sources and detectors used.



Figure 3. Transmission through the device. We plot the transmission *t* from equation (17) as a function of wavenumber ν and for various displacements δ (light grey: $\delta = 0.1$, grey: $\delta = 0.05$, black: $\delta = 0.01$). The transmission curves follow Fabry–Perot resonances that depend on the coupling coefficient *g*. Near the resonances where ν is integer the transmission sharply drops to zero. The smaller the displacement the narrower the dip, as described by equation (21). At exact resonance, the transmission changes abruptly between 0 for $\delta \neq 0$ and 1 for $\delta = 0$.

Moreover, using our simple 1D model for perfect imaging, we can also identify the physical mechanism behind the Miñano dips. At resonance and for misaligned source and detector, the incident radiation would build up an infinite field inside the device, unless, in the stationary regime, it is prevented from entering. For $\delta \neq 0$ and $\nu \in \mathbb{N}$ we obtain from the solution of equations (13)–(15)

$$a_{+} = a'_{+} = a_{-}^{*} = a_{-}^{'*} = \frac{i e^{-\nu\delta}}{g \sin \nu\delta}, \quad a'_{0} = 1, \quad a_{1} = 0.$$
 (22)

With these coefficients we get for the field (11)

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$$\psi = \frac{2\sin\left[\nu\left(\delta - \vartheta\right)\right]}{g\sin\nu\delta}.$$
(23)

The incident radiation is reflected, without changing sign, while a standing wave with finite amplitude is formed inside the device. The incident plus the reflected amplitude amounts to $a_0 + a'_0 = 2$, which, according to relations (10), is *g* times the field amplitude ψ (0) at the source, as equation (23) shows. Similarly, the standing wave described by equation (23) has a node at the detector, as there the total out-coupled field $a_1 + a'_1$ is zero. We thus see how for $\delta \neq 0$ the standing wave formed inside the device adjusts itself at equilibrium such that further radiation is prevented from entering, which reduces the transmission to zero, causing the characteristic Miñano dips.

Now consider the case of perfect alignment, $\delta = 0$. We obtain from the solution of equations (13)–(15) the amplitudes

$$a_{+} = a_{-} = \frac{\tau}{\sqrt{2}} e^{2i\nu\pi} \rho \eta, \quad a_{+}' = a_{-}' = \frac{\tau}{\sqrt{2}} \eta,$$
$$a_{0}' = \left(e^{2i\nu\pi} - 1\right) \rho \eta, \quad a_{1}' = \tau^{2} e^{i\nu\pi} \eta$$
(24)

in terms of the reflectivity ρ and transmissivity τ according to equation (6) and the coefficient η describing Fabry–Perot multiple reflections:

$$\eta = \frac{1}{1 - e^{2i\nu\pi}\rho^2} = \sum_{m=0}^{\infty} e^{2mi\nu\pi}\rho^{2m}.$$
(25)

The total transmission (19) we can understand as $\tau^4 \sum_{m=0}^{\infty} \rho^{4m}$, the product of the transmissions τ^2 at source and detector times the multiple reflections in the device. We get for the field (11)

$$\psi = \frac{\tau}{\sqrt{2}} \left(e^{i\nu|\vartheta|} + e^{2i\nu\pi}\rho \ e^{-i\nu|\vartheta|} \right) \eta.$$
(26)

The exp (i $\nu |\vartheta|$) and exp ($-i\nu |\vartheta|$) are the Green functions of wave propagation with two sources, one at $\vartheta = 0$ and one at $\vartheta = \pi$, describing waves running from source to detector and vice versa. The field thus consists of running waves (and their multiple reflections). No standing wave is formed, because it does not need to be formed: for perfect alignment the radiation can run through the device; it is not accumulated at resonance. In the case of perfect coupling we get

$$\psi = \frac{\mathrm{e}^{\mathrm{i}\nu|\vartheta|}}{\sqrt{2}},\tag{27}$$

which is the 1D equivalent of the propagating wave in Maxwell's fish eye that performs perfect imaging [1]. The amplitude is reduced by $\sqrt{2}$, because the incident radiation is distributed to two partial waves, one running to the right, the other to the left.

To summarize, we have developed a simple 1D model for super-resolution in absolute optical instruments that has allowed us to describe the interplay between source and detector. The key innovation of our model is the description of sources and detectors as linear dynamical systems. We used a simple 1D model that we believe can be extended to 2D or 3D systems. Our model has captured some of the most characteristic features of super-resolution with positive refraction: we have found analytic expressions for the Miñano dips [18, 19] and their physical explanation.

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