Gouy phase for full-aperture spherical and cylindrical waves

Tomáš Tyc
Faculty of Science and Faculty of Informatics, Masaryk University, Kotlarska 2, 61137 Brno, Czech Republic
(tomtyc@physics.muni.cz)

Received January 5, 2012; revised January 20, 2012; accepted January 20, 2012; posted January 23, 2012 (Doc. ID 161027); published February 28, 2012

We investigate the Gouy phase shift for full-aperture waves converging to a focal point from all directions in two and three dimensions. We find a simple interpretation for the Gouy phase in this situation and show that it has a dramatic effect on reshaping sharply localized pulses. © 2012 Optical Society of America

OCIS codes: 070.7345, 050.5080.

The Gouy phase shift [1–2] has been known for more than a century, and it still attracts a lot of interest in the optical community. Its origin has been explained in different ways and contexts; for an overview, see [3] and references therein.

The Gouy phase is a phase shift of a converging wave obtained, for instance, by focusing light with a lens when passing through the focus. It turns out that the phase change of the wave upon passing through the focal point is smaller by π compared to the situation if a plane wave were propagating instead. In other words, the local wavelength near the focus is slightly larger than \( \lambda_0 = 2\pi c/\omega \), the wavelength of a plane wave of the same frequency \( \omega \). One interpretation of this fact is that near the focus, the wavevector has nonnegligible transversal components and therefore its longitudinal component must be somewhat reduced, which increases the local wavelength [4].

The Gouy phase is usually discussed for light that propagates more or less in one direction, as when focusing light with a lens or creating a Gaussian beam. On the other hand, recently other geometries of light propagation have been studied in relation to perfect imaging devices such as Maxwell’s fish eye [5,6] or to 4π microscopy [7]. There, light arrives at the focal point from all spatial directions or leaves that point for all directions. In this Letter, we show that the Gouy phase exists in this situation as well and has a nice, straightforward interpretation. We also discuss the influence of the Gouy phase shift on light pulses passing through the focal point and show that it has a dramatic effect on the pulse shape in the two-dimensional (2D) case.

Our starting point is the three-dimensional (3D) wave equation

\[
c^2 \Delta \psi - \psi_{tt} = 0, \tag{1}
\]

which can be used for describing different types of waves, e.g., scalar or electromagnetic waves, in a homogeneous nondispersive medium. Here \( c \) is the speed of the waves. Now consider a monochromatic spherical wave of frequency \( \omega \) converging to the focal point at the origin of coordinates. This wave can be expressed as

\[
\psi_{in} = a \frac{\exp[-i k r - i \omega t]}{r}, \tag{2}
\]

with \( k = \omega/c \). However, Eq. (2) is not a solution of Eq. (1) at the very origin because the left-hand side diverges there. Therefore, if there is no absorber (drain) for the radiation at \( r = 0 \), Eq. (2) does not represent the full solution of the wave equation. This has a simple reason: the converging wave cannot just disappear at the focal point [6], but it changes there into a diverging wave \( \psi_{out} = b \exp[i k r - i \omega t]/r \), which must be superimposed with Eq. (2) to get the full solution. The only way to satisfy the wave equation at the origin is to set \( b = -a \), i.e., add the diverging wave with a phase shift \( \pi \), which gives the total wave

\[
\psi = \psi_{in} + \psi_{out} = -2 i k a \text{sinc} (k r) \exp(-i \omega t). \tag{3}
\]

The phase shift of \( \pi \) between the two waves at the origin can be interpreted as the Gouy phase. As we have seen, for spherical waves it ensures that there is no singularity in the field. On the other hand, the Gouy phase also prevents focusing light into a subwavelength spot because the central maximum of the sinc function in Eq. (3) is diffraction limited. To achieve subwavelength focusing, the diverging wave has to be eliminated by employing a suitable drain [5–9], which leaves only the converging wave with a much better localization.

To illustrate the Gouy phase in another way, let us look more closely at the wave in Eq. (3). Figure 1 shows two functions: one is \( \text{sinc} k x \), which represents the spherical wave in Eq. (3) [apart from the global factor

![Fig. 1. (Color online) Comparison of the function \( \text{sinc} k x \) (blue thick curve) and \( -\sin k x \) (red thin curve) describing a superposition of converging and diverging spherical waves and a plane wave, respectively, along an axis passing through the focal point. The phases match in the region \( x < 0 \), while they differ by \( \pi \) for \( x > 0 \), which demonstrates the Gouy phase shift.](image-url)
\(-2ika \exp(-i\omega t)\) along an \(x\) axis passing through the focal point. The second function is \((-\sin kx)\) and represents a plane wave of the same frequency propagating along the \(x\) axis whose phase was chosen to coincide with the phase of the sinc wave in the region \(x < 0\). Obviously, in the region \(x > 0\), the phases of the two waves differ: the sinc wave is delayed by \(\pi\) with respect to the plane wave. We see that a similar thing happens here as near the focus of a lens. This demonstrates the Gouy phase for spherical waves in an elegant way, by simply comparing the graphs of the functions \(\sin kx\) and \(\sin kx\). We can also see that the plane wave occurs on the length scale of the order of the wavelength, which is consistent with the case of the lens where the Gouy phase shift occurs on the scale of \(\lambda f^2/a^2\) with \(f\) denoting the focal distance and \(a\) the aperture \([10]\); in our case \(f\) and \(a\) can be considered to be of the same order.

A similar consideration can be made for 2D rotationally symmetric waves (or cylindrical waves in 3D). Similar to before, a converging wave (created, e.g., by the 2D Maxwell’s fish eye \([5]\)) turns into a diverging wave at the focal point. The converging and diverging waves described by Hankel functions are singular at the origin, but the combined wave satisfies the wave equation at the origin and is described by the Bessel function of the first kind \(J_0(kr)\). Now, similarly to the 3D case, we compare this wave with a plane wave of the same frequency. Figure 2 shows the two waves along an \(x\) axis passing through the focal point. The phase of the plane wave was chosen to coincide with the phase of the cylindrical wave in the region \(x^2 - \lambda = -2\pi/k\) with the help of the asymptotic formula \([11]\) for the Bessel function

\[J_0(kx) = \sqrt{\frac{2}{\pi kx}} \cos \left(kx - \frac{\pi}{4}\right). \tag{4}\]

Looking at Fig. 2 or using Eq. (4) again, we see that in the region \(x^2 > \lambda\), the cylindrical wave is delayed by \(\pi/2\) with respect to the plane wave, which is precisely the Gouy phase shift in two dimensions.

The last thing we will demonstrate is the effect of the Gouy phase on sharply localized spherical optical pulses converging to a focal point. For convenience we surround the focal point at the origin by a spherical (in 3D) or circular (in 2D) mirror of a unit radius. This makes the number of modes countable, which simplifies expressing the pulses as superpositions of the monochromatic modes. We will also set \(c = 1\) in Eq. (1), which can always be done by a suitable choice of units.

Let us start with the 3D case. For simplicity we will consider scalar waves now to avoid relatively complicated boundary conditions for the vector potential \([12]\). Because we are interested in spherically symmetric pulses only, the spatial parts of the relevant modes depend only on the radius and are again described by the sinc function as \(\psi_n(r) = \sin(k_n r)/r\), where the numbers \(k_n = nx\), \(n \in \mathbb{N}\) represent the frequencies of the standing waves that turn to zero at the mirror. Now consider a pulse emitted at \(t = 0\) from the origin expressed as the following superposition of the modes

\[\psi(r, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \sin k_n r \sin k_n t. \tag{5}\]

With the help of trigonometric identities and Fourier analysis, Eq. (5) can be rewritten as

\[\psi(r, t) = \frac{1}{\pi} \sum_{n=0}^{\infty} \{\cos[k_n(r - t)] - \cos[k_n(r + t)]\}, \tag{6}\]

\[\Delta(r) = \Delta(r - t) - \Delta(r + t), \tag{7}\]

where \(\Delta\) denotes the periodic delta function with period 2, \(\Delta(x) = \sum_{n=-\infty}^{\infty} \delta(x - 2\pi n)\).

We see from Eq. (7) that in the time interval \(t \in (0, 1)\) there is a delta peak propagating from the center to the spherical mirror at \(r = 1\) that is reflected from the mirror at time \(t = 1\), after which it propagates back to the center with the negative amplitude. The change of the sign is the result of the boundary condition at the mirror. After the converging pulse then reaches the center at \(t = 2\), its sign is flipped again and it turns into a positive diverging pulse. However, this time the sign change is due to the Gouy phase. The position of the peak can also be inferred directly from Eq. (6): it is the point where all the first or all the second cosine terms are in phase, i.e., when either \(r - t\) or \(r + t\) is an integer.

In two dimensions, the situation is more interesting. The rotationally symmetric modes are the Bessel functions \(J_0(k_n r)\), and the frequencies \(k_n\) are given by the boundary condition at the mirror, i.e., at \(r = 1\), as \(J_0(k_n) = 0\), so \(k_n\) is the \(n\)th zero of \(J_0\). Consider a pulse

\[\psi(r, t) = \sum_{n=1}^{\infty} \sqrt{2\pi k_n} J_0(k_n r) \sin \left(k_n t + \frac{\pi}{4}\right). \tag{8}\]

Using Eq. (4) and the approximate values of \(k_n\) derived from it,

\[k_n = \left(n - \frac{1}{4}\right) \pi, \tag{9}\]

we get after some manipulation...
A similar pulse is obtained if $\tau$ is not too small. Let us now investigate the sine terms that can be combined all in phase, namely at the point $r = \tau$, and their superposition forms a pulse localized around that point. But the shape of this pulse is now completely different from the pulse obtained previously by the cosine terms; instead of a $\delta$-like peak, we now have a double peak resembling the cotangent function [see Fig. 3(c)] and also the localization of the pulse is much worse. This dramatic change of the pulse shape is caused by the Gouy phase of $\pi/2$ that, for each monochromatic component of the pulse, converted a cosine in Eq. (10) into a sine in Eq. (11). A similar pulse reshaping has been observed experimentally [13].

In conclusion, we have analyzed the Gouy phase for rotationally and spherically symmetric waves. We have shown that for monochromatic waves, the Gouy phase has a simple meaning that can be visualized by comparing graphs for radially symmetric solutions of the wave equation with plane wave solutions. We have also shown that the Gouy phase has a dramatic effect on sharp rotationally symmetric pulses in 2D whose shape is changed completely after passing the focal point.

I thank Ulf Leonhardt, Martin Plesch, and Michal Lenc for discussions and acknowledge support of grant P201/12/G028 by the Grant Agency of the Czech Republic and the QUEST programme grant of the Engineering and Physical Sciences Research Council.

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