Perfect Quantum State Revivals: Designing Arbitrary Potentials with Specified Energy Levels

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It is known that there exist a limited number of analytic potentials with the unusual property that any bound quantum state therein will be periodic in time. This is known as a perfect quantum state revival. Examples of such potentials are the infinte well, quantum harmonic oscillator and the Pöschl-Teller potentials; here, we present a general method of designing such potentials. A key requirement is that their energy eigenvalues have integer spacings (up to a prefactor). We first analyze the required conditions which permit quantum state revivals for potentials in general, and then we use techniques of iterated Hamiltonian intertwining to construct potentials exhibiting perfect quantum revivals. Our method can readily be extended to multiple dimensions.

I. INTRODUCTION

When a quantum wavepacket moving in a potential is launched, it spreads out, explores the potential, and in general never comes back to the exact initial state. However, under certain circumstances the wavepacket does recover to the initial state, and this almost magical effect is called *quantum state revival* [1]. This happens in general if the energy of the wavepacket is restricted to a narrow interval [2], and the character of the energy spectrum determines the revival time. Wavepacket revivals have been investigated in different contexts, both theoretically and experimentally [3, 4], including relativistic systems [5], and other related effects such as fractional revivals and superrevivals can be observed too; a comprehensive review can be found in Ref. [1].

At the same time, there exist certain quantum systems where the energy spectra meet particular requirements that allow exact wavepacket revival for *any* wavepacket whose energy is restricted to the discrete part of the spectrum. There are only a few known such systems. Two of them are familiar to any student of quantum mechanics, namely the infinite potential well and the harmonic oscillator, and others include exactly solvable Hamiltonians, in particular Pöschl-Teller potentials, both in the hyperbolic and trigonometric versions [6, 7]. The question then arises as to whether other potentials with perfect quantum state revivals exist or not. Here we show that the answer to this question is positive and that there is a vast range of such potentials. To construct them, we employ the intertwining method, and demonstrate quantum state revivals on a number of examples.

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II. QUANTUM STATE REVIVALS

Suppose we have a quantum particle in a static 1D potential $V(x)$ governed by the Schrödinger equation $\mathcal{H}\psi(x,t) = E\psi(x,t)$ where the Hamiltonian $\mathcal{H} = -\frac{1}{2}\frac{d^2}{dx^2} + V(x)$ and we work in units in which the mass of the particle and Planck's constant $\hbar$ are equal to unity. To find conditions under which an arbitrary state $\psi(x,t)$ will be periodic in time, we expand the initial wavefunction $\psi(x,0)$ at time $t = 0$ as a superposition $\psi(x,0) = \sum_n c_n\psi_n(x,0)$ of the eigenstates $\psi_n(x,0)$ of the Hamiltonian. As the wavepacket propagates within the potential, each component acquires a phase factor $\varphi_n(t) = -E_n t$ which depends on its eigenenergy E_n, so the state at time t becomes

$$\psi(x,t) = \sum_n c_n e^{-iE_n t}\psi_n(x,0)\,. \tag{1}$$

For the wavepacket at some moment $t = T_{\rm rev}$ (the revival time) to be equivalent to that at $t = 0$, all the exponential terms $e^{-iE_n t}$ have to be in phase at the time $T_{\rm rev}$. This occurs when the energy levels E_n have integral spacing, with a common real factor a and real offset b,

$$E_n = aN_n + b\,, \tag{2}$$

and the only requirement on N_n is that it be an integer. The state revival period can then easily be identified as $T_{\rm rev} = 2\pi/a$; at this moment $\psi(x,T_{\rm rev}) = e^{-\frac{2\pi i b}{a}}\psi(x,0)$, so the wavepacket is revived perfectly up to an unimportant global phase factor. A useful measure of the wavepacket revivals is the autocorrelation function $A(t) = \langle\psi(0)|\psi(t)\rangle$. For a perfect revival, $|A(T_{\rm rev})| = 1$.

It may happen that the Hamiltonian has a spectrum that is partly discrete and partly continuous. If the discrete levels obey the rule (2) and a wavepacket is created only from the eigenstates corresponding to the discrete levels, then the quantum revivals will still occur.

The required energy spacing of (2) is satisfied for the infinite potential well, the quantum harmonic oscillator and its isospectral variants [8, 9], and the Pöschl-Teller potentials, so these systems exhibit perfect wavepacket revivals. In the following we show how to design other potentials with such properties by engineering their energy levels to have suitable patterns through the intertwining technique [8, 10, 11]. This enables us to obtain an unlimited set of potentials where perfect revivals occur.

III. CONSTRUCTING POTENTIALS WITH DESIRED SPECTRA

The technique begins with a known potential $V_0(x)$ and associated Hamiltonian,

$$\mathcal{H}_0 = -\frac{1}{2}\frac{d^2}{dx^2} + V_0(x)\,. \tag{3}$$

A new potential $V_1(x)$ shall be constructed such that it has all the energy levels that $V_0(x)$ has, plus an additional level that shall lie below all those of $V_0(x)$. The total number of bound states of $V_1(x)$ is thus one greater than in $V_0(x)$. If $V_0(x)$ is one of the potentials whose discrete levels obey the contraints of (2) and we then add a new energy level below its ground state at the right energy, a new potential retaining the quantum state revival property will result. It is then possible to iterate the intertwining procedure with $V_1(x)$ as the starting point to add another energy level below its ground state, and then continue to create as many new energy levels as desired. In the standard intertwining procedure as described in [8, 10–12], an operator A_1^+ is sought which connects two Hamiltonians $\mathcal{H}_0$ and $\mathcal{H}_1$ such that

$$A_1^+\mathcal{H}_1 = \mathcal{H}_0 A_1^+\,. \tag{4}$$

In the first-order intertwining, which we shall limit ourselves to, the operator A_1^+ has the following form:

$$A_1^\pm = \frac{1}{\sqrt{2}}\left(\pm\frac{d}{dx} + U_1(x)\right) \tag{5}$$

The function $U_1(x)$ is to be determined, and $A_1^- = (A^+)^\dagger$. We take note that $\mathcal{H}_0$ and $\mathcal{H}_1$ can be factored, where $\mathscr{E}$ is an undetermined constant:

$$\mathcal{H}_0 = A_1^+A_1^- + \mathscr{E}\,, \qquad \mathcal{H}_1 = A_1^-A_1^+ + \mathscr{E}\,. \tag{6}$$

If we now take E_1 as an eigenenergy of $\mathcal{H}_1$ and ψ_1 as its normalized eigenfunction, then $A_1^-A_1^+\psi_1 = (E_1 - \mathscr{E})\psi_1$. Multiplying from the left by A_1^+ results in $A_1^+A_1^-(A_1^+\psi_1) = (E_1-\mathscr{E})(A_1^+\psi_1)$, equivalently

$$\mathcal{H}_0(A_1^+\psi_1) = E_1(A_1^+\psi_1). \tag{7}$$

If $A_1^+\psi_1 \neq 0$ then E_1 is also an eigenvalue of $\mathcal{H}_0$. The norm $\|A_1^+\psi_1\| = \sqrt{\langle A_1^+\psi_1|A_1^+\psi_1\rangle}$ is just $\sqrt{E_1-\mathscr{E}}$ and hence the normalized eigenfunction is

$$\underset{\text{(normalized)}}{\psi_0} = \frac{A_1^+\psi_1}{\|A_1^+\psi_1\|} = \frac{1}{\sqrt{E_1-\mathscr{E}}}(A_1^+\psi_1). \tag{8}$$

This is typically the situation unless E_1 is the ground state of $\mathcal{H}_1$, and this ensures the potentials have otherwise identical spectra. The interesting condition occurs when $A_1^+\psi_1 = 0$ which implies that $A_1^-A_1^+\psi_1 = 0$ and $\mathcal{H}_1\psi_1 = \mathscr{E}\psi_1$. Then the eigenvalue $E_1 = \mathscr{E}$ but ψ_0 is unnormalizeable since the norm in the denominator of (8) is zero. We then have a situation where the new potential $V_1(x)$ associated with Hamiltonian $\mathcal{H}_1$ is expected to have a ground state at energy $E_1 < E_0$ which $V_0(x)$ lacks. As for $U_1(x)$, by combining (3), (5), and (6), it can be shown that it must satisfy a Ricatti equation,

$$U_1'(x) + U_1^2(x) = 2[V_0(x) - \mathscr{E}]\,, \tag{9}$$

where prime denotes derivative with respect to x. The new potential is

$$V_1(x) = V_0(x) - U_1'(x) = U_1^2(x) - V_0(x) + 2\mathscr{E}\,, \tag{10}$$

where Eq. (9) has been used. A general solution of Eq. (9) is given in [8] and [13] and has the form

$$U_1(x) = \frac{d}{dx}\left[\ln\psi_0(x)\right]\,. \tag{11}$$

The unnormalized wavefunction $\psi_0(x) = A_1^+\psi_1$ pertains to the original function $V_0(x)$ and can be found by solving the Schrödinger equation (7) that we rewrite as

$$\psi_0''(x) + 2[\mathscr{E} - V_0(x)]\psi_0(x) = 0\,. \tag{12}$$

It is convenient to split $\psi_0(x)$ into even and odd parts, $\psi_0(x) = \psi_{\text{even}}(x) + \alpha\psi_{\text{odd}}(x)$, with the appropriate initial conditions $\psi'_{\text{even}}(0) = 0, \psi_{\text{even}}(0) = 1$ and $\psi_{\text{odd}}(0) = 0, \psi'_{\text{odd}}(0) = 1$. Since the new energy level $\mathscr{E}$ lies below the ground state of $V_0(x)$, the requirement that $\mathscr{E} < E_0$ will ensure that $\psi_{\text{even}}(x)$ be free of zeros. If $\psi_{\text{odd}}(x)$ is included in the solutions of $\psi_0(x)$, care must be taken that α is not too large, otherwise the zero always present in $\psi_{\text{odd}}(x)$ would show up in $\psi_0(x)$.

For a practical calculation of the new potential $V_1(x)$, it is more convenient to work with $U_1(x)$ rather than $\psi_0(x)$ because one avoids the astronomically large values that the unnormalizable $\psi_0(x)$ contains. To solve Eq. (9), we also need to employ the initial condition $U_1(0) = \alpha$ that follows from the initial conditions for ψ_0; the choice $\alpha = 0$ yields a symmetric potential $V_1(x)$ and in the following we will restrict ourselves to this case.

Equation (10) provides the desired potential V_1 with the added level $\mathscr{E}$ and it is suitable for numerical calculations. The entire procedure can be iterated to add arbitrarily many new energy levels, provided that each new level be below all of the previously added levels; refer to Algorithm 1 for a complete description. We have tested the algorithm to construct several potentials (see below), and it can easily be used to add hundreds of levels.

Algorithm 1: Procedure for Creating Potential with m New Energy Levels

Data:
- Starting potential $V_0(x)$
- An array of desired new levels E of size m in descending order

Result: New potential $V_m(x)$
for $i \leftarrow 1$ **to** m **do**
  Solve the equation for $U_i(x)$:
  $U_i' + U_i^2 + 2[E_i - V_{i-1}(x)] = 0$,
  where $U_i(0) = 0$.
  Create potential $V_i(x)$:
  $V_i(x) \leftarrow U_i^2(x) - V_{i-1}(x) + 2E_i$.
return $V_m(x)$;
Output the final potential containing additional m energy levels.

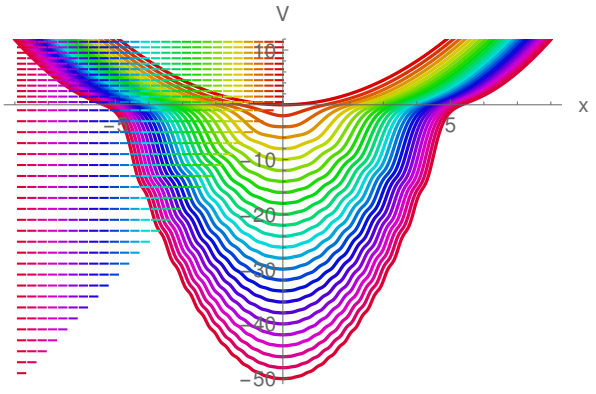

FIG. 1. Biperiodic potentials. Starting from the harmonic potential $x^2/2$ (the uppermost curve, red), $N = 25$ additional energy levels with spacing of 2 at values $-1, -3, \ldots, -49$ have been added successively using the scheme in Algorithm 1, creating 25 new potentials. The numerically calculated spectra of each potential are shown towards the left in matching colors; the leftmost column shows the spectrum of the lowermost potential. The fact that the levels of various potentials match is apparent.

The intertwining method works, by its nature, in the "top to bottom" direction: we start from a given potential and add new energy levels below its existing levels. However, it might be more natural to work "bottom to top", starting with the ground state and then adding sequentially higher energy levels. Fortunately, the method is suitable for this purpose too: We first list as many levels as desired that follow a required "bottom to top" pattern in our new potential. We then choose any suitable initial potential, typically the harmonic oscillator or a constant potential; we then invert our list of desired levels and add them in reverse order. Numerical simulations reveal that when many levels are added, then the choice of the initial potential becomes unimportant and does not influence the resulting potential on the relevant energy interval.

IV. CONSTRUCTING POTENTIALS WITH PERFECT REVIVALS

In the following we introduce several examples of energy level structure that yield perfect quantum state revivals, and construct the corresponding potentials. We also demostrate the revivals using the autocorrelation function and quantum carpets.

A. Biperiodic oscillator

Let's start from the harmonic oscillator potential with energy levels $E_n = n + \frac{1}{2}$ for $n \geq 0$ and sequentially add an additional N levels $E = -1, -3, -5, \ldots, -(2N-1)$. Employing the semiclassical relation between separation of quantum energy levels and classical period of oscillations [1], we find that this would correspond to a classical biperiodic oscillator that has frequency $\omega_1 = 1$ for $E > 0$ and $\omega_2 = 2$ for $E < 0$; such a potential was described for instance in Ref. [14]. The resulting quantum potentials and energy levels are shown in Fig. 1 for N ranging from 1 to 25; the levels were numerically calculated from the constructed potentials and they match the desired levels with a high precision. The quantum potentials closely resemble their classical counterpart (see Fig. 1(a) in Ref. [14]). We can conveniently represent the time evolution and revival of the wavepacket by the quantum carpet [1], which is analogous to the Talbot carpet in optics [15]. Fig. 2 shows examples of quantum carpets that represent the time evolution of wavepackets for the biperiodic potential with $N = 100$ added levels, where the bi-periodicity is apparent. Fig. 3(a) shows the magnitude of the autocorrelation function $|A(t)|$ and confirms perfect wavepacket revivals.

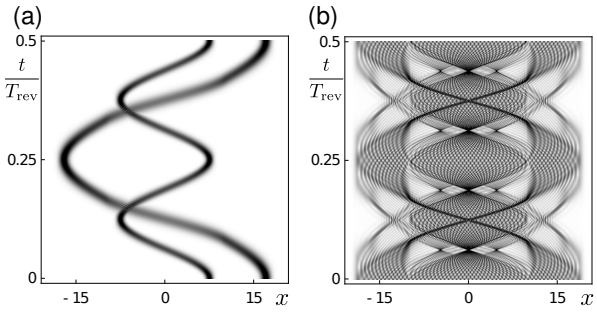

FIG. 2. Quantum carpets for the biperiodic potential where 100 levels with spacing of 2 were added to the original harmonic potential $x^2/2$. The absolute value of the wavefunction is indicated by the brightness (larger value corresponds to a darker color), the vertical axis represents time proportional to $T_{\rm rev} = 4\pi$. Two initial wavepackets were used: (a) superposition of two Gaussian states and (b) a shifted cosine oscillating between 0 and 1. It is apparent, especially in (a), that the period of the lower-amplitude Gaussian component is equal to half of the period of the higher-amplitude component.

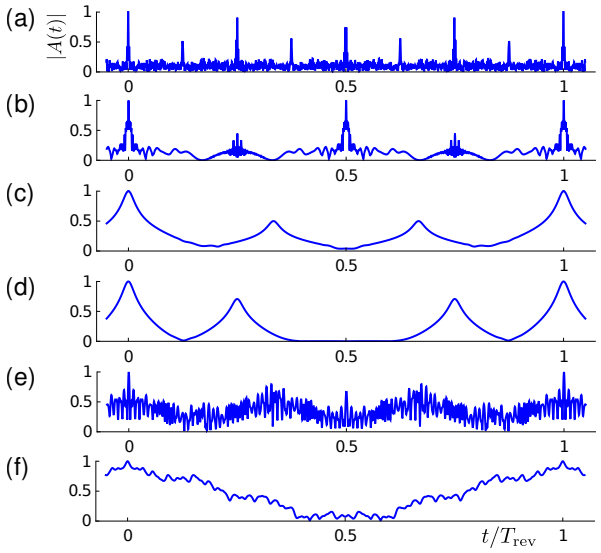

FIG. 3. The plots show the magnitude of the autocorrelation function, $|A(t)|$, for the following quantum carpets: (a) the carpet from Fig. 2(b), (b) the carpet from Fig. 5(b), (c-d) the carpets from Fig. 6(e-f), respectively, (e,f) the carpets from Fig. 7(b,d), respectively.

B. Reverse biperiodic oscillator

Consider now a situation similar to the previous one: we again start from the harmonic oscillator potential with energy levels $E_n = n + \frac{1}{2}$ for $n \geq 0$ and sequentially add an additional N levels $E = 0, -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, -(N-1)/2$, this time with a smaller separation than the original levels. The resulting quantum potentials and energy levels are shown in Fig. 4.

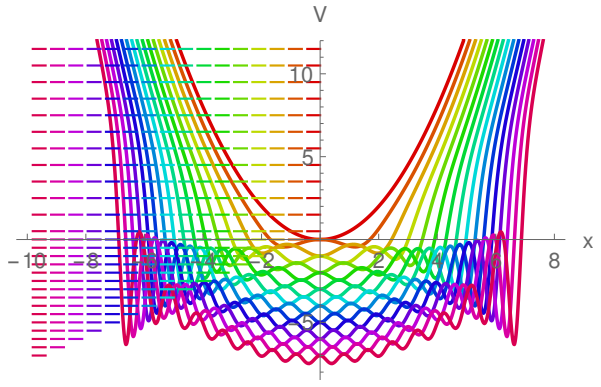

FIG. 4. Same as Fig. 1, but for reverse biperiodic potentials. $N = 15$ levels were added successively with spacing of $1/2$ at values $0, -1/2, -1, \ldots, -7$. Fewer added levels than in Fig. 1 were used for the sake of clarity of the plot.

Classically, this spectrum would correspond to a biperiodic oscillator with frequency $\omega_1 = 1$ for $E > 0$ and $\omega_2 = 1/2$ for $E < 0$. It turns out, however, that such a classical potential does not exist. Indeed, when solving the inverse problem of finding a classical potential $V(x)$ from a given dependence $T(E)$ of period on energy [16], one first obtains the function $x(V)$ that has to be inverted to get $V(x)$; since the resulting function is decreasing here, this is impossible. Such a non-existence of the classical counterpart is related to the oscillations of the quantum potentials apparent in Fig. 4; the particle tunnels between local potential minima, which is not possible classically. Quantum carpets for the reverse biperiodic potential with $N = 100$ added levels are shown in Fig. 5, the bi-periodicity is apparent again. The autocorrelation function is shown in Fig. 3(b).

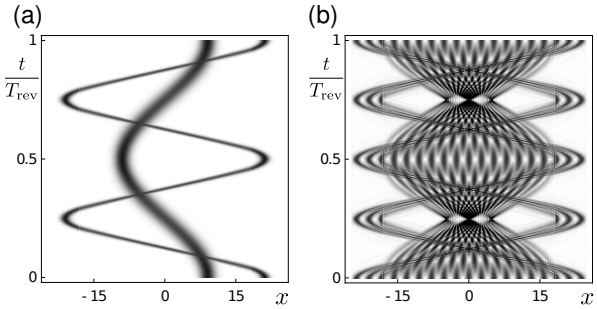

FIG. 5. Quantum carpets for the reverse biperiodic potential where 100 levels with spacing of $1/2$ were added to the original harmonic potential $x^2/2$. The initial conditions are similar as in Fig. 2. The biperiodicity is apparent again, this time the period of the lower-amplitude component equals twice the period of the higher-amplitude component. The revival time is $T_{\rm rev} = 4\pi$.

C. Potentials with alternating gaps between levels

As the next example, let us construct a potential with the gap between the subsequent levels alternating betwen two values. Starting from the harmonic oscillator, we construct $N = 100$ additional levels following this pattern.

The resulting potentials are illustrated in Fig. 6(a-b) for two particular cases of gap patterns between the levels. Fig. 6(c-f) then shows the corresponding quantum carpets exhibiting an interesting quantum interference effects. The autocorrelation functions are shown in Fig. 3(c,d).

D. Potentials with prime and Fibonacci spectra

To demonstrate the power of the method, we present two more examples, namely potentials whose spectra are given by prime and Fibonacci numbers. Denoting the $n^{\rm th}$ prime number by p_n, we start from the constant potential at the level p_{N+1} for some $N \in \mathbb{N}$ and add sequentially N levels $p_N, p_{N-1}, \ldots, p_1$. Fig. 7 (a–b) shows the resulting potential for $N = 50$ and an example of the quantum carpet; the autocorrelation functions is shown in Fig. 3(e).

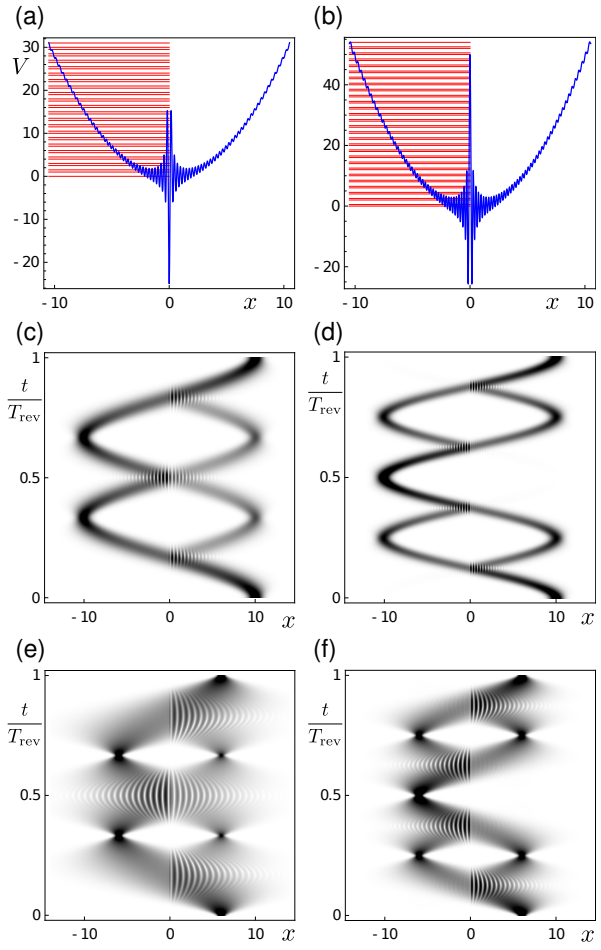

FIG. 6. Potentials with alternating gaps between the energy levels. The gaps alternate (a) between the values 1 and 1/2 and (b) between the values 1/2 and 3/2. The potentials were shifted by suitable constants to produce zero ground state energy. The spectrum is shown by the red lines. (c-d) show a quantum carpet for the potentials in (a-b), respectively, for an initial Gaussian wavepacket. (e-f) the same as (c-d), but for a narrower initial wavepacket. The potentials seem to act like diffraction gratings for the incoming wavepackets, splitting them into two and then rejoining them into one again. In both cases, the revival time is $T_{\rm rev} = 4\pi$.

Next we repeat the same procedure for Fibonacci numbers F_n instead of primes; however, the first Fibonacci number 1 has to be omitted to avoid repeating level values, so the designed level values are $F_2, F_3, \ldots, F_N$. The resulting potential has a regular structure; it is depicted along with its quantum carpet in Fig. 7 (c–d) for $N = 13$; the autocorrelation function is shown in Fig. 3(f).

V. MULTIDIMENSIONAL CASE

Our method can be readily extended to multiple dimensions. Suppose we have designed potentials $\{V_k(x); k = 1, \ldots, n\}$, all with the property (2) with common values of a and b. Consider now a combined potential $V(x_1, \ldots, x_n) = \sum_{k=1}^{n} V_k(x_k)$, where x_k are Cartesian coordinates of the n-dimensional Euclidean space. The energy levels of a particle moving in the potential V will be in the form of sums of energy levels of the individual potentials V_k, as can easily be shown via separation of variables in the n-dimensional Schrödinger equation. Therefore the energy levels in the potential V will also have the property (2) and will lead to perfect revivals of n-dimensional wavepackets.

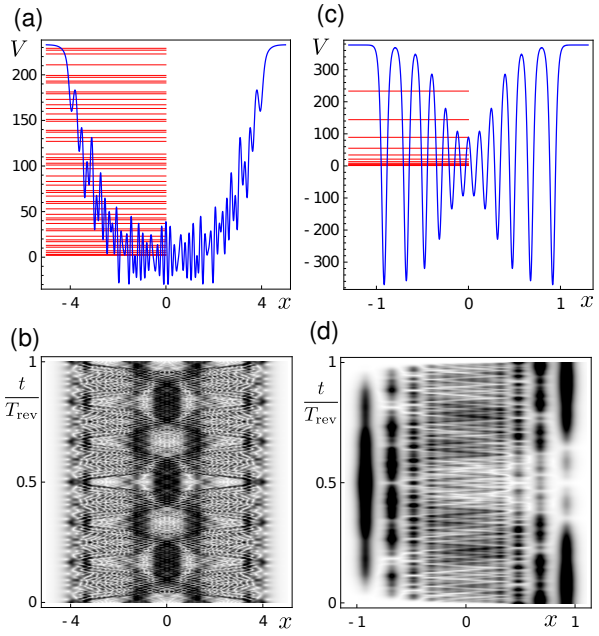

FIG. 7. (a) Potential with spectrum given by the first 50 prime numbers $(2, 3, 5, \ldots, 229)$ and (b) its quantum carpet. (c) Potential with spectrum given by the 12 Fibonacci numbers $1, 2, 3, 5, 8, \ldots, 233$ and (d) its quantum carpet. In both cases, the revival time is $T_{\rm rev} = 2\pi$.

VI. CONCLUSIONS

We have presented a general method for desiging potentials that exhibit perfect quantum state revivals. The examples given demonstrate the power and flexibility of the method; there exist endless possibilities for other energy level patterns that obey Rule (2); constructing their corresponding quantum potentials leads to a multitude of potentials that exhibit perfect wavepacket revivals. The intertwining method described above and applied in Algorithm 1 can also be used to quickly and easily design potentials that have any desired energy level positions and is thus a powerful inverse design technique that may possibly have applications beyond what we have considered here. Since separation of variables allows the construction of n-dimensional potentials from individual 1-dimensional potentials, the spectral engineering of any desired potential is conceivable with the algorithm pre-

sented. A possible extension of our work could include designing potentials for relativistic quantum revivals [5].

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