Absolute optical instruments

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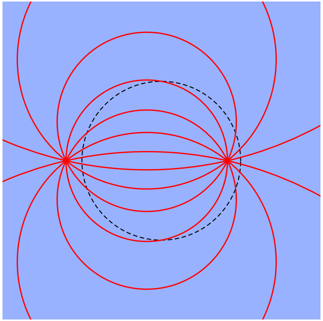
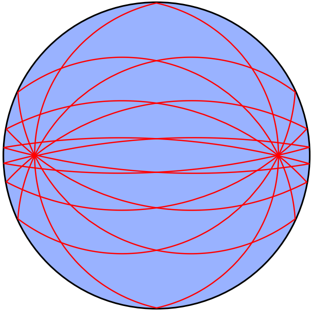
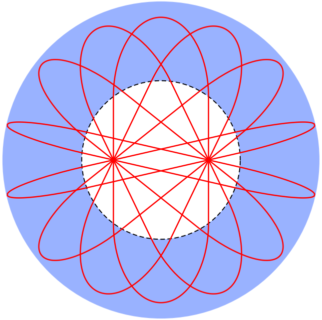
What is an absolute instrument?

Absolute instrument (AI) images a 3D region stigmatically (sharply)







Theory of AI within geometrical optics

[J. C. Maxwell, Camb. Dublin Math. J. 8, 188 (1854)]

[M. Born & E. Wolf, Principles of Optics]

[J. C. Miñano, Opt. Express 14, 9627 (2006)]

[T. Tyc, L. Herzánová, M. Šarbort, K. Bering, New J. Phys. 13, 115004 (2011)]

Magnifying AI:
[T. Tyc, Phys. Rev. A 84, 031801(R) (2011)]

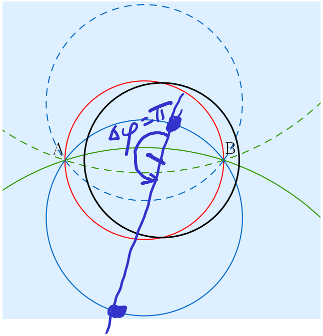
Maxwell's fish eye

Maxwell's fish eye – discovered by J. C. Maxwell in 1854

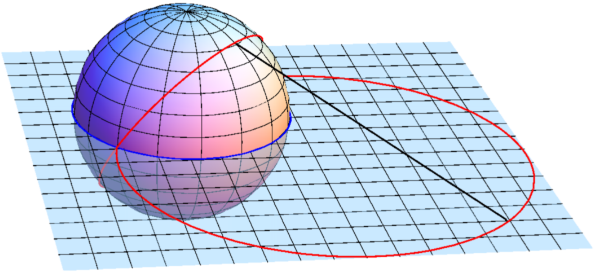
Spherically symmetric refractive index

$$n = \frac{2}{1 + r^2}$$

Ray trajectories are circles, every point of space has a sharp image



Stereographic projection leading to Maxwell's fish eye



Stereographic projectio is a conformal map $\Rightarrow$ isotropic magnification

Geometrical path on the sphere = optical path in the plane:

$$\mathrm{d}L = \frac{2}{1+r^2}\,\mathrm{d}l$$

Geodesics on the sphere are mapped to rays in the plane

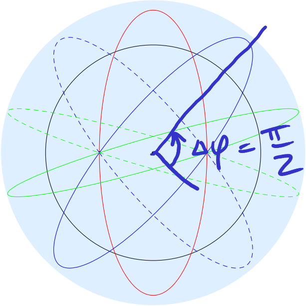
More examples of AI

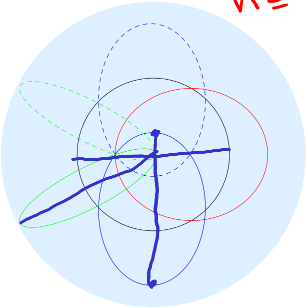
Luneburg index profile

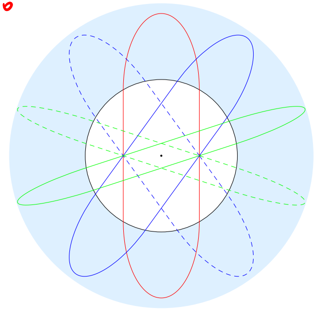
Eaton index profile

Miñano lens

$$n = \frac{\sqrt{2m(E-V(\vec{r}))}}{p_0}$$







$$n(r) = \sqrt{2 - r^2}$$

$$n(r) = \sqrt{\frac{2}{r} - 1}$$

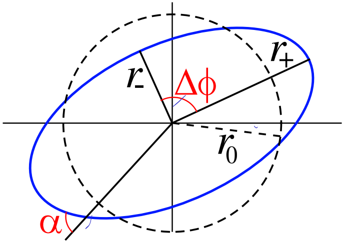
$$n(r) = \begin{cases} \sqrt{\frac{2}{r} - 1}, & r \leq 1 \\ 1, & r \geq 1 \end{cases}$$

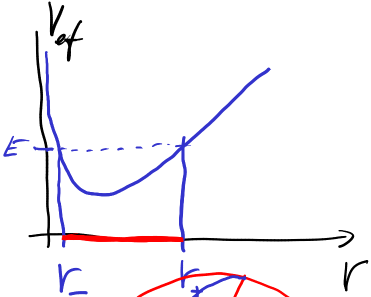
How to design AIs?

[T. Tyc, L. Herzánová, M. Šarbort, K. Bering, New J. Phys. 13, 115004 (2011)].

Problem similar to the inverse scattering problem

We assume radially symmetric index $n(r)$

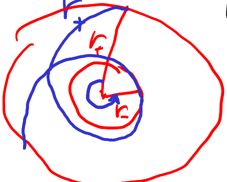




$$\Delta\varphi = \frac{P}{Q}\pi$$

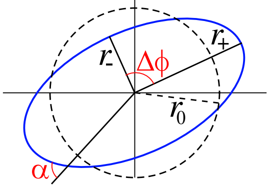
$$P, Q \in \mathbb{N}$$

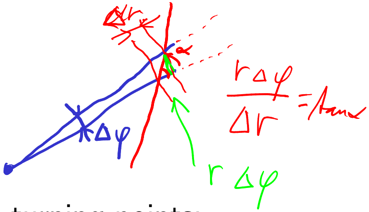
Conservation of angular momentum $L = rn \sin\alpha = \rho \sin\alpha$



$$rn(r) = \rho$$

$$\tan\alpha = r\,\frac{d\varphi}{dr} = \frac{L}{\sqrt{\rho^2(r) - L^2}},$$





Turning angle $\Delta\varphi$ – angle swept by radius vector between the turning points:

$$\Delta\varphi(L) = L \int_{r_-(L)}^{r_+(L)} \frac{\mathrm{d}r}{r\sqrt{\rho^2(r) - L^2}}$$

Changing variables from r to $x = \ln r$ gives

$$\Delta\varphi(L) = L \int_{x_-(L)}^{x_+(L)} \frac{\mathrm{d}x}{\sqrt{\rho^2(x) - L^2}} \qquad (1)$$

We want to find unknown $\rho(x)$ from the known $\Delta\varphi(L)$ – inverse problem

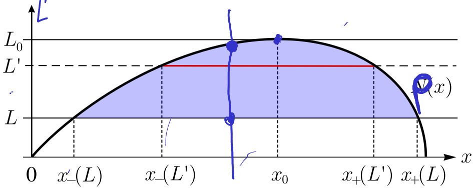
How to solve it? In a similar way as an unknown potential can be found from a known period of motion as a function of energy [Landau & Lifshitz, Mechanics]

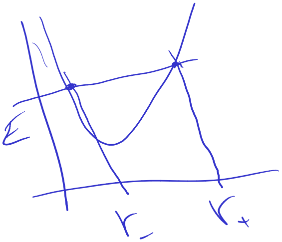
$$\int_L^{L_0} \frac{\Delta\varphi(L')\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}} = \int_L^{L_0} \int_{x_-(L')}^{x_+(L')} \frac{\mathrm{d}x}{\sqrt{\rho^2(x) - L'^2}} \frac{L'\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}}$$

$$= \int_{x_-(L)}^{x_+(L)} \int_L^{\rho(x)} \frac{L'\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}\sqrt{\rho^2(x) - L'^2}}\,\mathrm{d}x$$

$$= \int_{x_-(L)}^{x_+(L)} \left[\arcsin\sqrt{\frac{L'^2 - L^2}{\rho^2(x) - L^2}}\right]_{L'=L}^{L'=\rho(x)} \mathrm{d}x$$

$$= \int_{x_-(L)}^{x_+(L)} \left(\frac{\pi}{2} - 0\right) \mathrm{d}x = \frac{\pi}{2}\left(x_+(L) - x_-(L)\right) = \frac{\pi}{2} \ln\frac{r_+(L)}{r_-(L)}$$





$$T = 2\int_{x_1}^{x_2} \frac{dx}{V} = 2\int_{x_1}^{x_2} \frac{dx}{\sqrt{\frac{2}{m}(E-V)}} = \sqrt{2m}\int_{x_1}^{x_2} \frac{dx}{\sqrt{E-V}}$$

$L = mr\sin\alpha = \rho\sin\alpha$

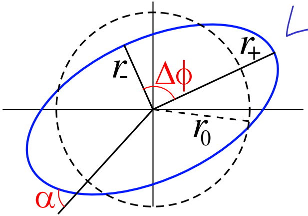
$L \le \rho$

$\rho \ge L$ $\quad \rho(x) \ge L$

We can rewrite this as

$$\ln \frac{r_+(L)}{r_-(L)} = \frac{2}{\pi} \int_L^{L_0} \frac{\Delta\varphi(L')\, dL'}{\sqrt{L'^2 - L^2}}$$

To get closed ray trajectories, $\Delta\varphi$ should be constant, $\Delta\varphi = \pi/\mu$, $\mu \in \mathbb{Q}$



$$\Delta\varphi = \frac{P}{Q}\pi$$

$$\mu = \frac{Q}{P}$$

Then we get

$$\ln \frac{r_+(L)}{r_-(L)} = \frac{2}{\mu} \operatorname{arcosh} \frac{L_0}{L}$$

$$\frac{L_0}{L} = \cosh\left(\frac{\mu}{2} \ln \frac{r_+}{r_-}\right) = \frac{1}{2}\left(e^{\frac{\mu}{2}\ln\frac{r_+}{r_-}} + e^{-\frac{\mu}{2}\ln\frac{r_+}{r_-}}\right)$$

which can be expressed as

$$\frac{L_0}{L} = \frac{1}{2}\left[\left(\frac{r_+(L)}{r_-(L)}\right)^{\mu/2} + \left(\frac{r_-(L)}{r_+(L)}\right)^{\mu/2}\right]. \qquad (2)$$

Now we define a function $f(r)$ such that

$$f(r) = \begin{cases} r_+(L(r)) & \text{for} \quad r \leq r_0 \\ r_-(L(r)) & \text{for} \quad r \geq r_0 \end{cases}$$

The function $f(r)$ is hence defined such that for a given lower turning point r_- it produces the upper turning point r_+ corresponding to the same angular momentum and vice versa:

$$r_\pm = f(r_\mp)$$

$$f(f(r)) = r$$

$$f(r_-) = f(f(r_+)) = r_+$$

$$\frac{L_0}{L} = \frac{1}{2}\left[\left(\frac{r_+(L)}{r_-(L)}\right)^{\mu/2} + \left(\frac{r_-(L)}{r_+(L)}\right)^{\mu/2}\right]$$

$$L = \frac{2L_0}{\left[\left(\frac{r_+}{r_-}\right)^{\mu/2} + \left(\frac{r_-}{r_+}\right)^{\mu/2}\right]}$$

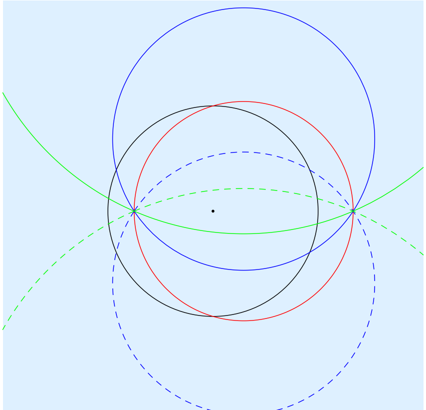
This yields the refractive index

$$n(r) = \frac{2L_0}{r\left[\left(\frac{r}{f(r)}\right)^{\mu/2} + \left(\frac{f(r)}{r}\right)^{\mu/2}\right]}$$

Examples of AIs

Maxwell's fish eye: $\Delta\varphi = \pi$, $\mu = 1$,

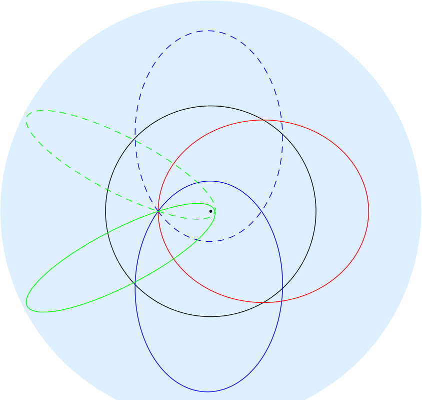
$$f(r) = 1/r, \quad n(r) = \frac{2}{1+r^2}$$



Examples of AIs

Eaton index profile: $\Delta\varphi = \pi$, $\mu = 1$,

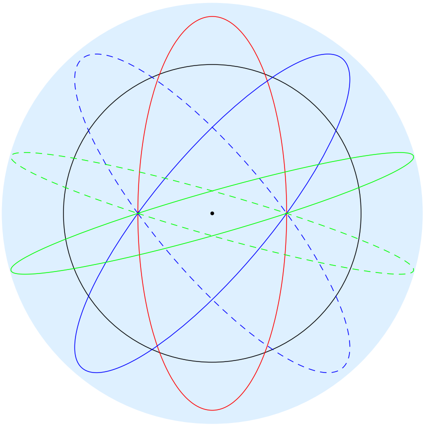
$$f(r) = 2 - r, \quad n(r) = \sqrt{\frac{2}{r} - 1}$$



Examples of AIs

Luneburg index profile: $\Delta\varphi = \pi/2$, $\mu = 2$,

$$f(r) = \sqrt{2 - r^2}, \quad n(r) = \sqrt{2 - r^2}$$



Examples of AIs

Maxwell's fish eye mirror: $\Delta\varphi = \pi/2$, $\mu = 2$,

$$f(r) = 1, \quad n(r) = \frac{2}{1 + r^2}$$