Absolute optical instruments and geodesic lenses

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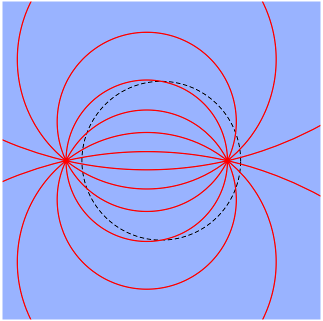
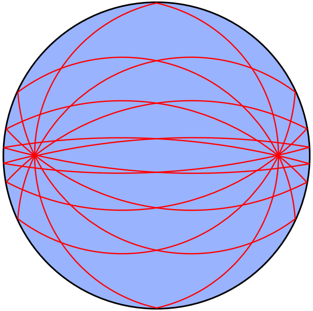
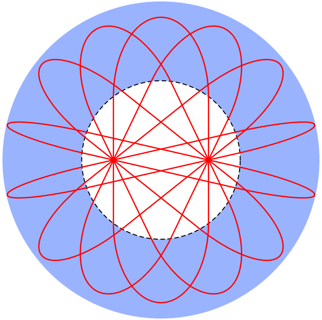
What is an absolute instrument?

Absolute instrument (AI) images a 3D region stigmatically (sharply)







Theory of AI within geometrical optics

[J. C. Maxwell, Camb. Dublin Math. J. 8, 188 (1854)]

[M. Born & E. Wolf, Principles of Optics]

[J. C. Miñano, Opt. Express 14, 9627 (2006)]

[T. Tyc, L. Herzánová, M. Šarbort, K. Bering, New J. Phys. 13, 115004 (2011)]

Magnifying AI:
[T. Tyc, Phys. Rev. A 84, 031801(R) (2011)]

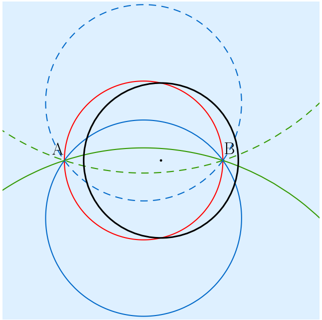
Maxwell's fish eye

Maxwell's fish eye – discovered by J. C. Maxwell in 1854

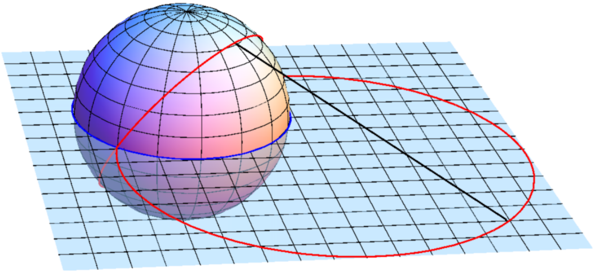
Spherically symmetric refractive index

$$n = \frac{2}{1 + r^2}$$

Ray trajectories are circles, every point of space has a sharp image



Stereographic projection leading to Maxwell's fish eye



Stereographic projectio is a conformal map $\Rightarrow$ isotropic magnification

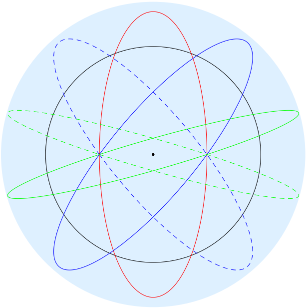
Geometrical path on the sphere = optical path in the plane:

$$\mathrm{d}L = \frac{2}{1+r^2}\,\mathrm{d}l$$

Geodesics on the sphere are mapped to rays in the plane

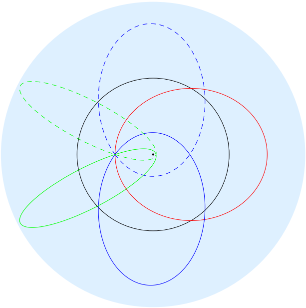
More examples of AI

Luneburg index profile



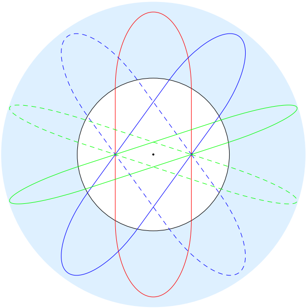
$$n(r) = \sqrt{2 - r^2}$$

Eaton index profile



$$n(r) = \sqrt{\frac{2}{r} - 1}$$

Miñano lens



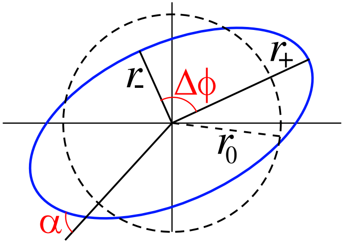
$$n(r) = \begin{cases} \sqrt{\frac{2}{r} - 1}, & r \leq 1 \\ 1, & r \geq 1 \end{cases}$$

How to design AIs?

[T. Tyc, L. Herzánová, M. Šarbort, K. Bering, New J. Phys. 13, 115004 (2011)].

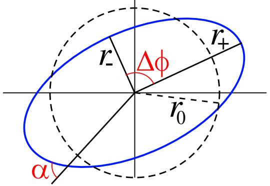
Problem similar to the inverse scattering problem

We assume radially symmetric index $n(r)$



Conservation of angular momentum $L = rn \sin\alpha = \rho \sin\alpha$

$$\tan\alpha = r\,\frac{\mathrm{d}\varphi}{\mathrm{d}r} = \frac{L}{\sqrt{\rho^2(r) - L^2}}\,,$$



Turning angle $\Delta\varphi$ – angle swept by radius vector between the turning points:

$$\Delta\varphi(L) = L\int_{r_-(L)}^{r_+(L)} \frac{\mathrm{d}r}{r\sqrt{\rho^2(r) - L^2}}$$

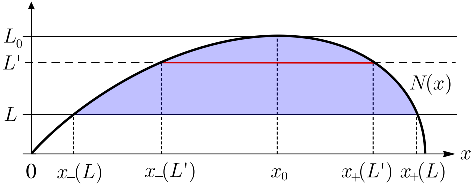
Changing variables from r to $x = \ln r$ gives

$$\Delta\varphi(L) = L\int_{x_-(L)}^{x_+(L)} \frac{\mathrm{d}x}{\sqrt{\rho^2(x) - L^2}} \qquad (1)$$

We want to find unknown $\rho(x)$ from the known $\Delta\varphi(L)$ – inverse problem

How to solve it? In a similar way as an unknown potential can be found from a known period of motion as a function of energy [Landau & Lifshitz, Mechanics]

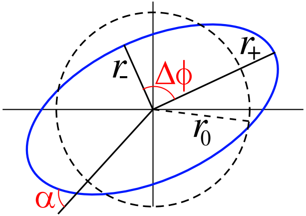
$$
\begin{aligned}
\int_L^{L_0} \frac{\Delta\varphi(L')\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}} &= \int_L^{L_0} \int_{x_-(L')}^{x_+(L')} \frac{\mathrm{d}x}{\sqrt{\rho^2(x) - L'^2}} \frac{L'\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}} \\
&= \int_{x_-(L)}^{x_+(L)} \int_L^{\rho(x)} \frac{L'\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}} \frac{\mathrm{d}x}{\sqrt{\rho^2(x) - L'^2}} \\
&= \int_{x_-(L)}^{x_+(L)} \left[\arcsin\sqrt{\frac{L'^2 - L^2}{\rho^2(x) - L^2}}\right]_{L'=L}^{L'=\rho(x)} \mathrm{d}x \\
&= \int_{x_-(L)}^{x_+(L)} \left(\frac{\pi}{2} - 0\right) \mathrm{d}x = \frac{\pi}{2}\left(x_+(L) - x_-(L)\right) = \frac{\pi}{2} \ln \frac{r_+(L)}{r_-(L)}
\end{aligned}
$$



We can rewrite this as

$$\ln \frac{r_+(L)}{r_-(L)} = \frac{2}{\pi} \int_L^{L_0} \frac{\Delta\varphi(L')\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}}$$

To get closed ray trajectories, $\Delta\varphi$ should be constant, $\Delta\varphi = \pi/\mu$, $\mu \in \mathbb{Q}$



Then we get

$$\ln \frac{r_+(L)}{r_-(L)} = \frac{2}{\mu} \operatorname{arcosh} \frac{L_0}{L}$$

which can be expressed as

$$\frac{L_0}{L} = \frac{1}{2}\left[\left(\frac{r_+(L)}{r_-(L)}\right)^{\mu/2} + \left(\frac{r_-(L)}{r_+(L)}\right)^{\mu/2}\right]. \tag{2}$$

Now we define a function $f(r)$ such that

$$f(r) = \begin{cases} r_+(L(r)) & \text{for} \quad r \leq r_0 \\ r_-(L(r)) & \text{for} \quad r \geq r_0 \end{cases}$$

The function $f(r)$ is hence defined such that for a given lower turning point r_- it produces the upper turning point r_+ corresponding to the same angular momentum and vice versa:

$$r_\pm = f(r_\mp)$$

$$\frac{L_0}{L} = \frac{1}{2}\left[\left(\frac{r_+(L)}{r_-(L)}\right)^{\mu/2} + \left(\frac{r_-(L)}{r_+(L)}\right)^{\mu/2}\right]$$

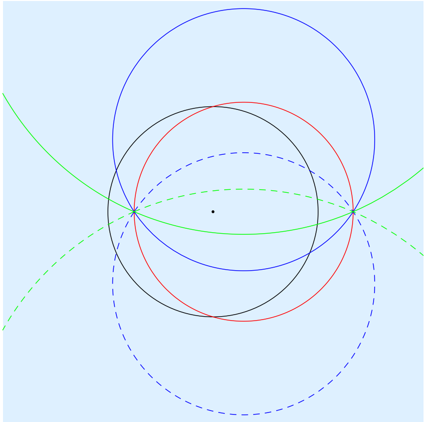
This yields the refractive index

$$n(r) = \frac{2L_0}{r\left[\left(\frac{r}{f(r)}\right)^{\mu/2} + \left(\frac{f(r)}{r}\right)^{\mu/2}\right]}$$

Examples of AIs

Maxwell's fish eye: $\Delta\varphi = \pi$, $\mu = 1$,

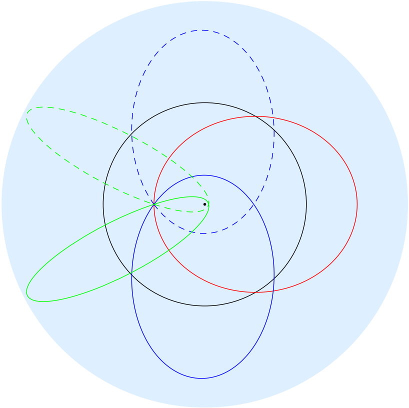
$$f(r) = 1/r, \quad n(r) = \frac{2}{1 + r^2}$$



Examples of AIs

Eaton index profile: $\Delta\varphi = \pi$, $\mu = 1$,

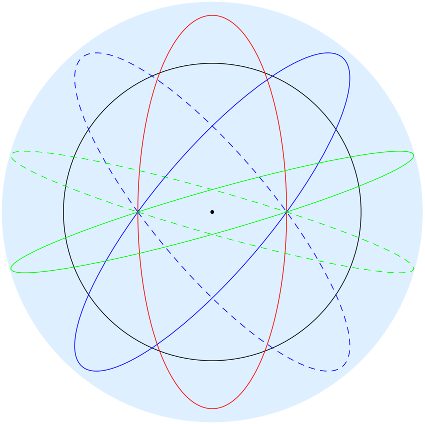
$$f(r) = 2 - r, \quad n(r) = \sqrt{\frac{2}{r} - 1}$$



Examples of AIs

Luneburg index profile: $\Delta\varphi = \pi/2$, $\mu = 2$,

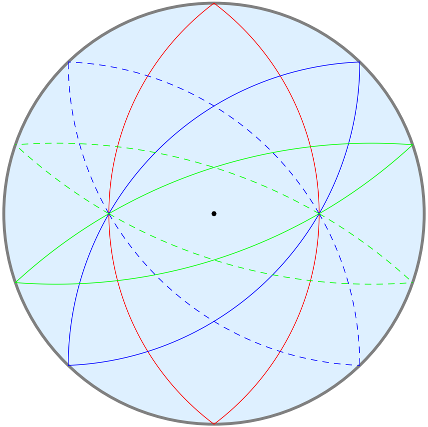
$$f(r) = \sqrt{2 - r^2}, \quad n(r) = \sqrt{2 - r^2}$$



Examples of AIs

Maxwell's fish eye mirror: $\Delta\varphi = \pi/2$, $\mu = 2$,

$$f(r) = 1, \quad n(r) = \frac{2}{1 + r^2}$$



Examples of AIs

Lens with $n = 1$ in an inner region: we require $n = 1$ at the lower turning point r_-

Also $\Delta\varphi = \pi/2$, $\mu = 2$

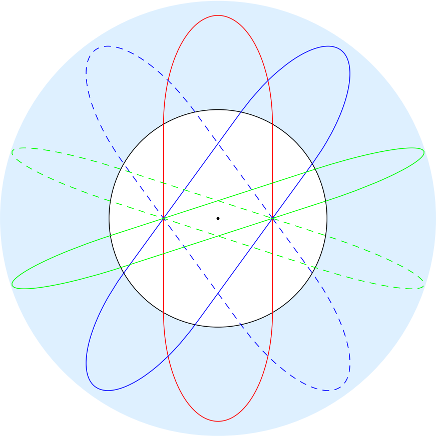
Then

$$r_+ = f(r_-) = 1 + \sqrt{1 - r_-^2}$$
$$r_- = f(r_+) = \sqrt{2r_+ - r_+^2}$$

which yields the Miñano lens (inside-out Eaton lens) index profile:

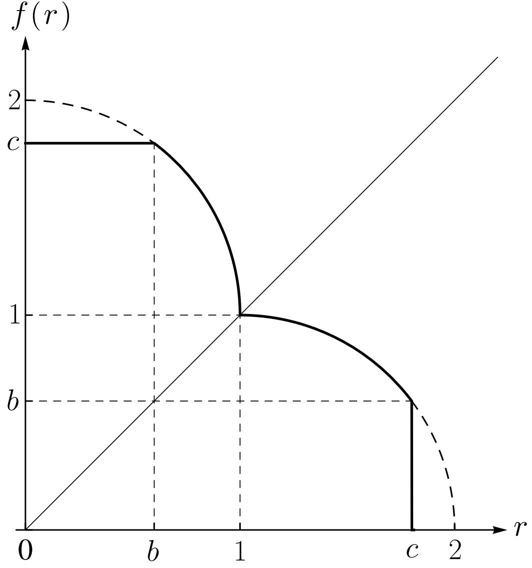
$$n(r) = \begin{cases} \sqrt{\frac{2}{r} - 1} & \text{for } r \leq 1 \\ 1 & \text{for } r \geq 1 \end{cases}$$

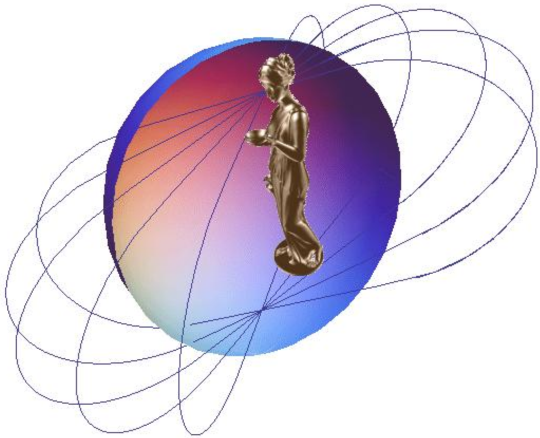
Examples of AIs

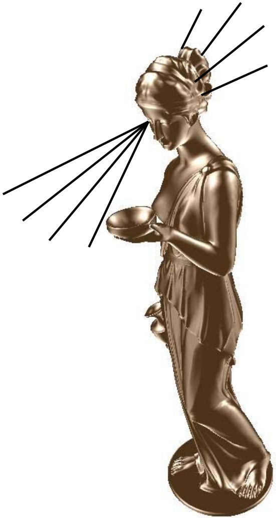


[J. C. Miñano, Opt. Express 14, 9627 (2006)]

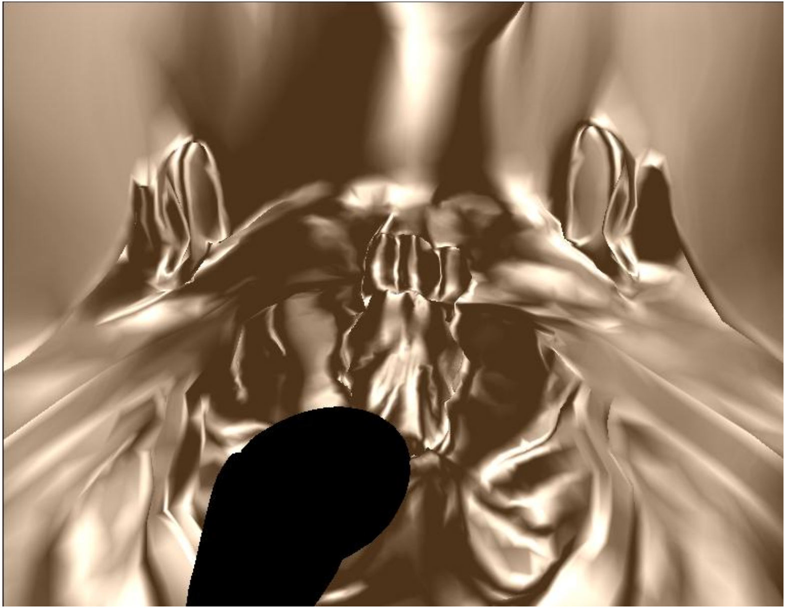
Function f for Miñano lens (inside-out Eaton lens):



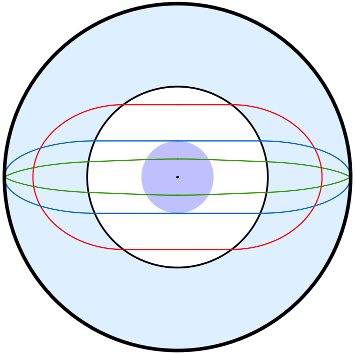


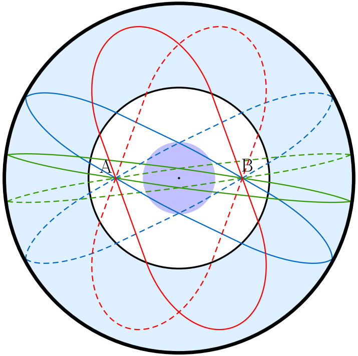


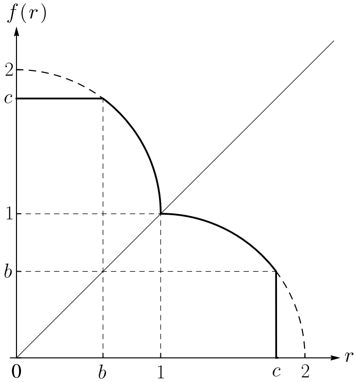
(visualisation by Aaron Danner)



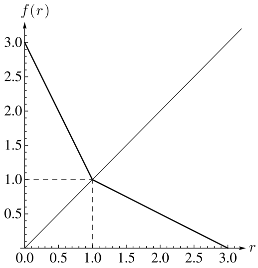
Examples of AIs

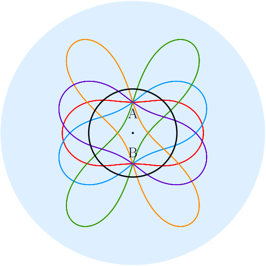


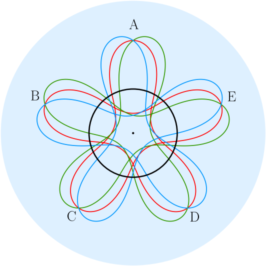


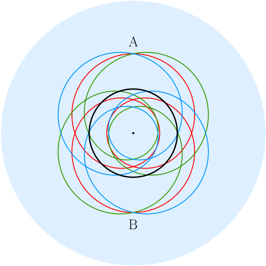


Examples of AIs

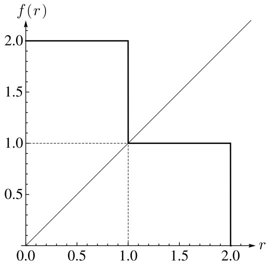


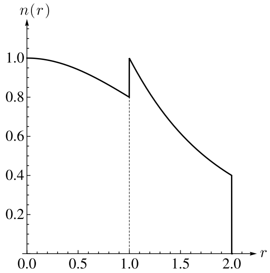


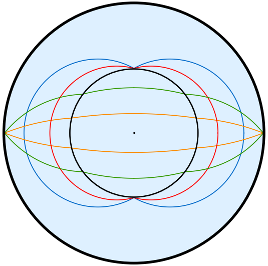


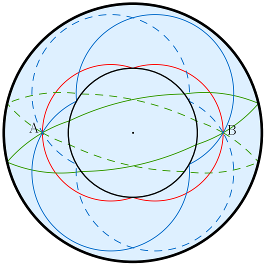


Examples of AIs

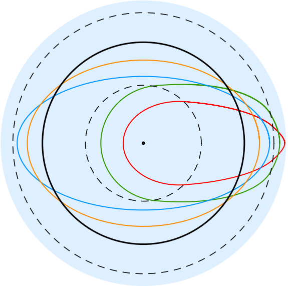


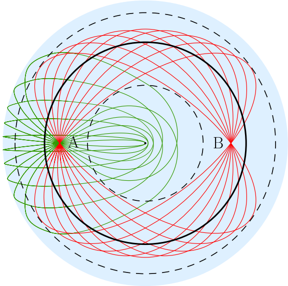


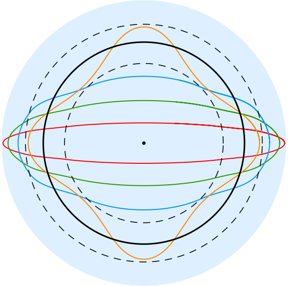


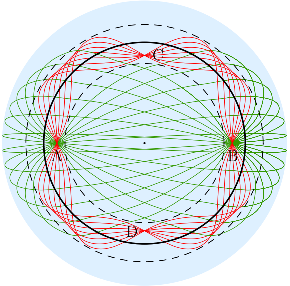


Bifocal AIs









Geodesic lenses

Just as for Maxwell's fish eye, also for other AIs we can define equivalent geodesic surfaces

Optical path in the AI corresponds to geometrical path on the surface

Rays in AI correspond to geodesics on the surface

Working directly with the geodesic lenses provides a powerful method for designing new AIs

[M. Šarbort, T. Tyc, Journal of Optics 14, 075705 (2012)]

[M. Šarbort, T. Tyc, J. Opt. 15, 125716 (2013)]

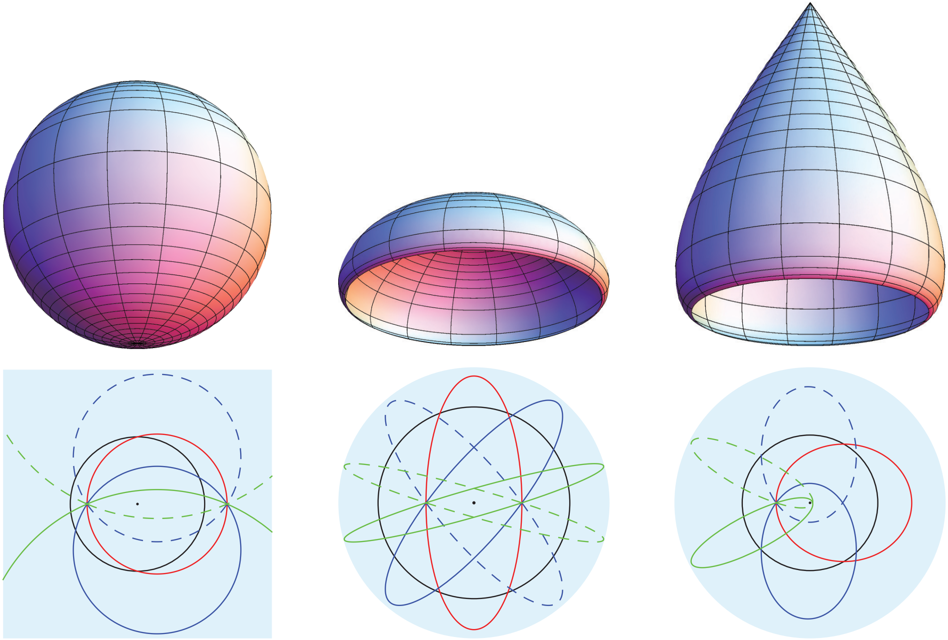
[M. Šarbort, Dissertation, Masaryk University, Brno 2013]

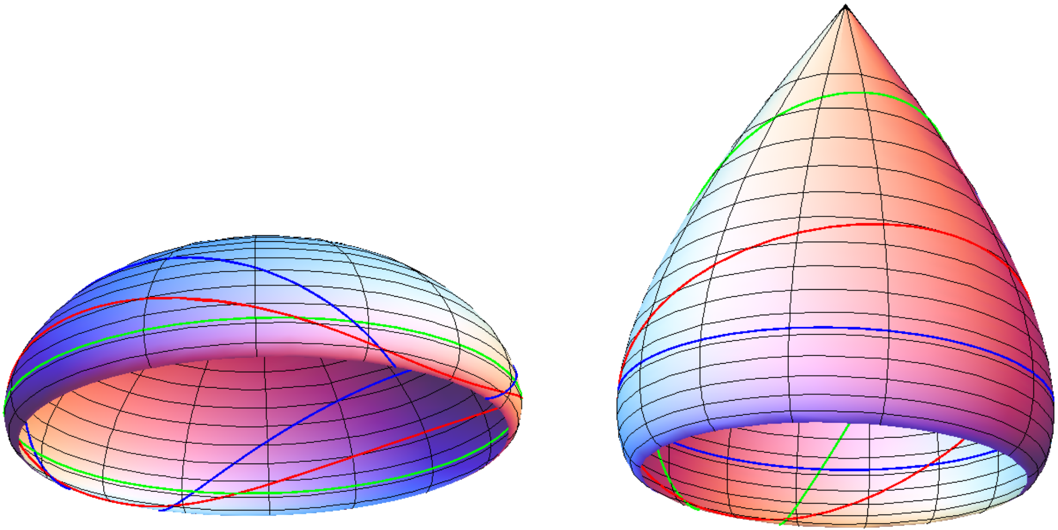
The line element $\mathrm{d}L$ on the virtual surface is

$$\mathrm{d}L^2 = \mathrm{d}R^2 + R^2\,\mathrm{d}\phi^2 + \mathrm{d}Z^2 = \left[1 + \left(\frac{\mathrm{d}Z}{\mathrm{d}R}\right)^2\right] \mathrm{d}R^2 + R^2\,\mathrm{d}\phi^2. \tag{3}$$

On the other hand, the optical path element on the physical plane is

$$\mathrm{d}s^2 = n^2\,\mathrm{d}l^2 = n^2(\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2). \tag{4}$$



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