General tools for the analysis of geodesic lenses

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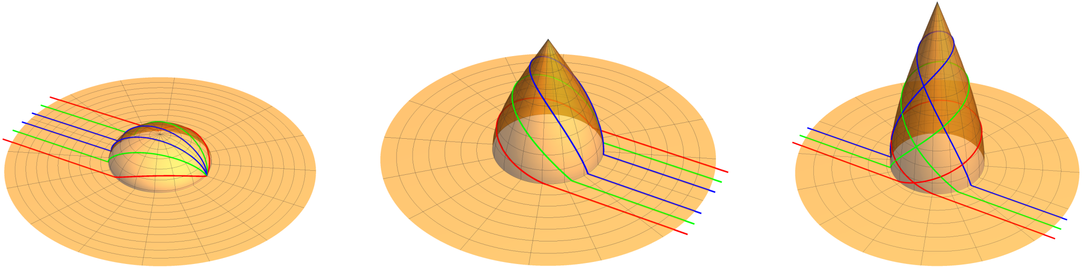
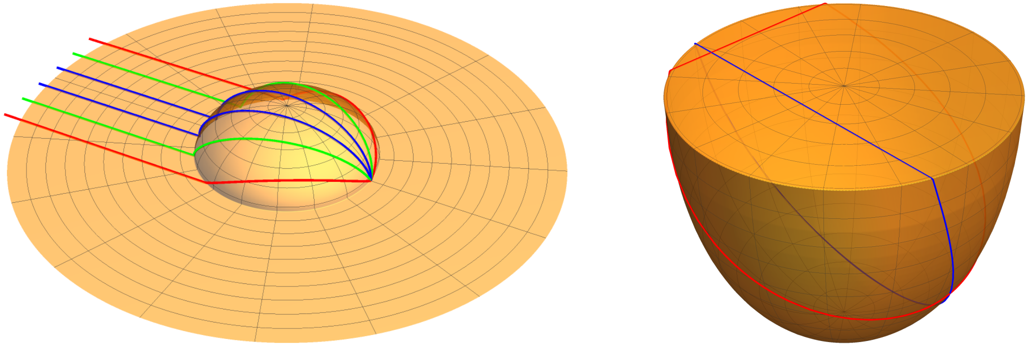


Outline

- What is a geodesic lens (GL)?
- Types and basic properties of GL
- Ray description, role of angular momentum
- Solving inverse problem to design GLs
- Relation to absolute optical instruments
- Raytracing on GL, equation for geodesics
- Conclusion

What is geodesic lens?

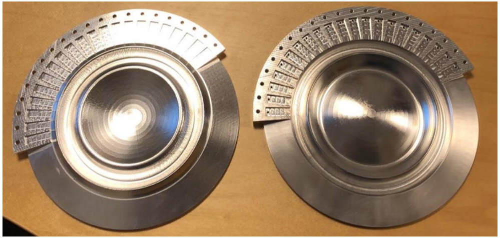
- A curved surface that is able to shape light rays
- Light rays follow geodesics – the most straight lines on the surface
- Geodesic turns neither left nor right

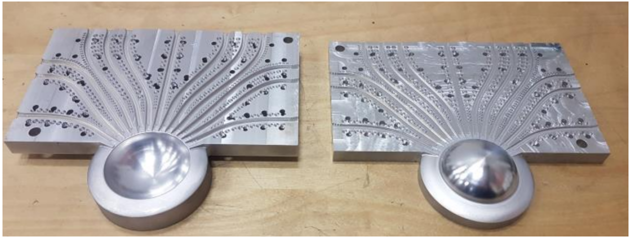


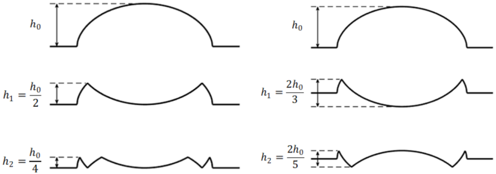
- Usually a 2D surface embedded in 3D space, but more general concepts are possible

Practical realization

Usually 2D waveguides

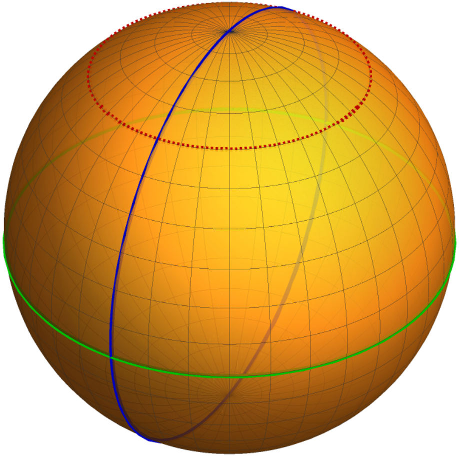




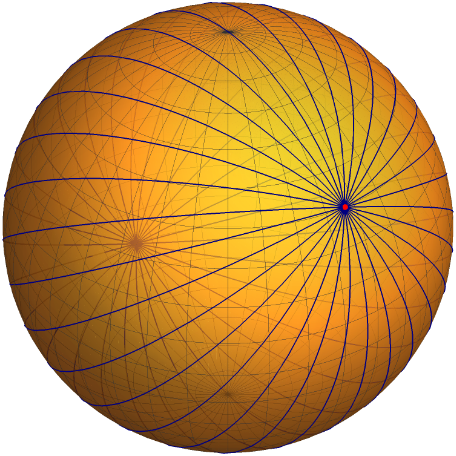


Example: sphere

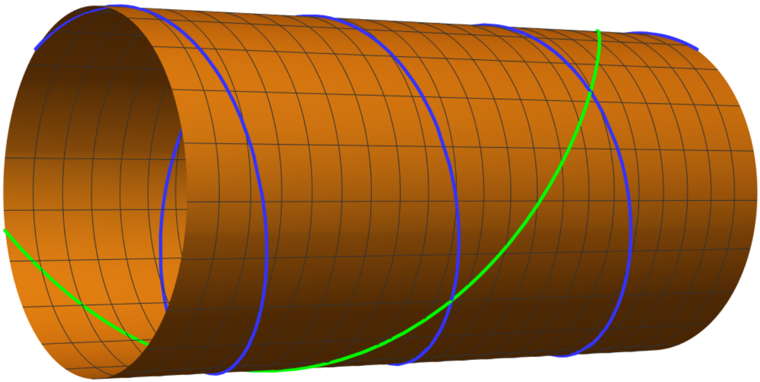
Geodesics are great circles
(the red circle is not a geodesic)



All rays emerging from one point
meet at the opposite point – imaging

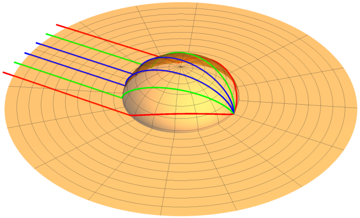


Example: cylinder

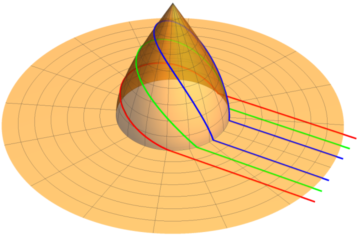


- Cylinder can be unfolded into a plane
- We get geodesics from straight lines by folding the plane into a cylinder – helix lines
- A similar procedure can be done for a cone (experiment)

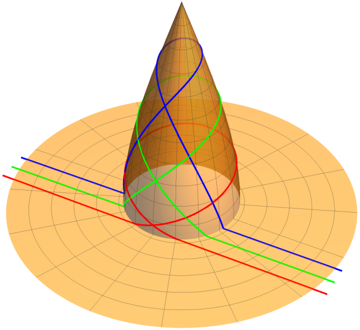
Open geodesic lenses



Luneburg GL



Eaton GL

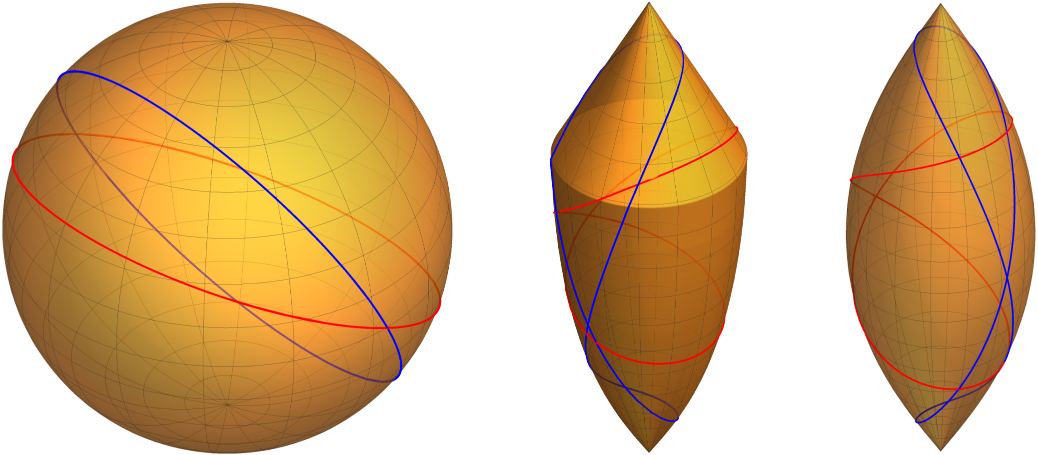


Invisible GL

- Open GLs emerge above the “equator” just on one side, then continue into a plane
- Ususally they image points at infinity
- They are important for antenna design – the plane corresponds to the environment

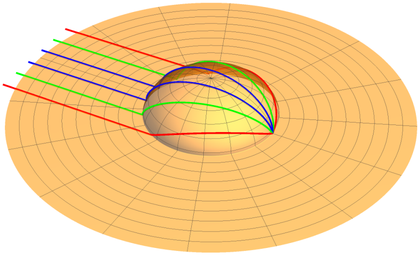
Closed geodesic lenses

Rays form closed loops, every point has its image



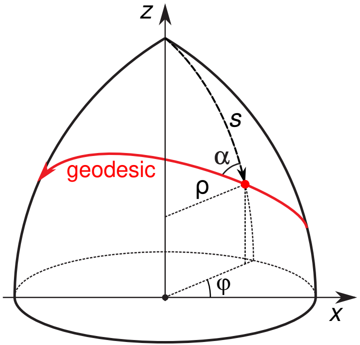
Designing rotationally-symmetric geodesic lenses

[M. Šarbort and T. Tyc, J. Opt. 14, 075705 (2012)]



- Close relation to absolute optical instruments and transformation optics
- Rotationally-symmetric GLs can be described and designed analytically
- Crucial role: conservation of angular momentum
- Inverse Abel transformation can be used for their design
- GL without rotational symmetry – a much harder problem

GL parametrisation



We use the following coordinates:

- cylindrical coordinates ρ, φ
- length measured along the meridian $s(\rho)$
- we could use the cylindrical coordinate z, but s is more useful

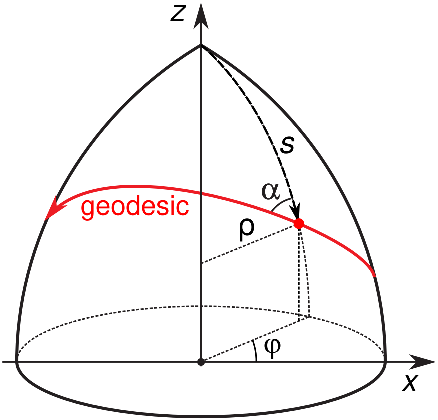
Angular momentum

- z-component of the angular momentum is conserved due to rotational symmetry

$$L_z = (\vec{r} \times \vec{v})_z = \text{const.}$$

- We could express $\vec{r}$ and $\vec{v}$ explicitly, it is a bit complicated
- A simpler argument: for $\alpha = 0$ we have $L_z = 0$, for $\alpha = \pi/2$ we have $L_z = \rho$
- For a general α we decompose velocity into the two directions and have

$$L_z = \rho \sin\alpha = \text{const.} \equiv L$$



Angle swept by the ray on GL

- From the small triangle we see

$$\rho \frac{d\varphi}{dl} = \sin\alpha = \frac{L}{\rho}$$

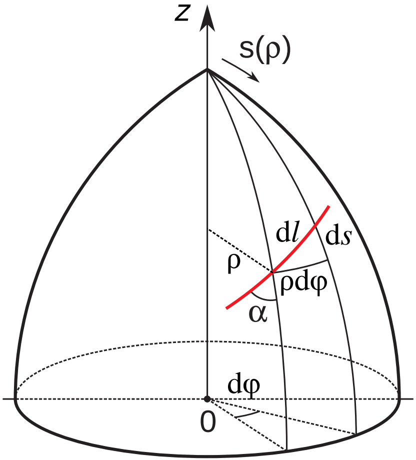
$$\frac{ds}{dl} = -\cos\alpha = \mp\sqrt{1 - \frac{L^2}{\rho^2}}$$

- From this it follows

$$\frac{d\varphi}{d\rho} = \frac{\frac{d\varphi}{dl}}{\frac{ds}{dl}} \frac{ds}{d\rho} = \pm \frac{L\, s'(\rho)}{\rho\sqrt{\rho^2 - L^2}},$$

- The angle swept by the ray

$$\Delta\varphi_{\mathrm{GL}} = 2\int_L^1 \frac{L\, s'(\rho)\, d\rho}{\rho\sqrt{\rho^2 - L^2}}$$



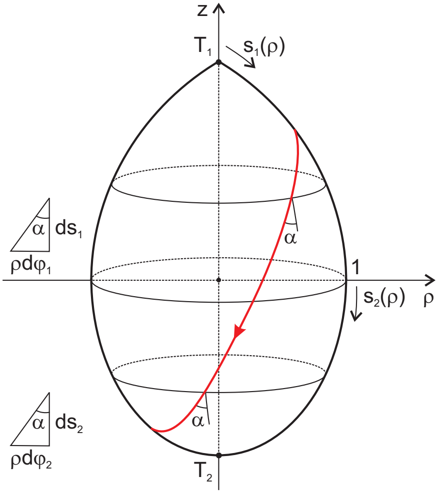
Closed (double-sided) GLs

- Closed GLs have two parts, above and below the equator
- The total swept angle for a single cycle in s is

$$\Delta\varphi_{\mathrm{GL}}(L) = 2\int_L^1 \frac{L\,[s_1'(\rho) - s_2'(\rho)]\,\mathrm{d}\rho}{\rho\sqrt{\rho^2 - L^2}}$$

(minus sign because $s_2'(\rho) \equiv \mathrm{d}s_2/\,\mathrm{d}\rho < 0$)

- Example: for a sphere, $\Delta\varphi_{\mathrm{GL}}(L) = 2\pi$
- If for some GL $\Delta\varphi_{\mathrm{GL}}(L)$ is a rational multiple of 2π for all L, rays form closed loops and we get focusing



Transforming a closed GL

- Would it be possible to deform the GL without changing $\Delta\varphi_{\mathrm{GL}}(L)$?

$$\Delta\varphi_{\mathrm{GL}}(L) = \int_L^1 \frac{L\,[s_1'(\rho) - s_2'(\rho)]\,\mathrm{d}\rho}{\rho\sqrt{\rho^2 - L^2}}$$

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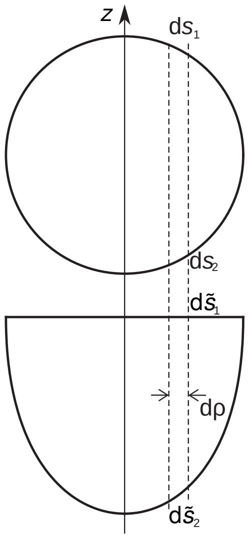
- If we change $s_1'(\rho)$ and $s_2'(\rho)$, keeping the difference $s_1'(\rho) - s_2'(\rho)$ fixed, then $\Delta\varphi_{\mathrm{GL}}(L)$ is preserved

$$\tilde{s}_2'(\rho) - \tilde{s}_1'(\rho) = s_2'(\rho) - s_1'(\rho) \quad \Rightarrow$$
$$\tilde{s}_2(\rho) - \tilde{s}_1(\rho) = s_2(\rho) - s_1(\rho)$$

- This way, we can transform any double-sided GL into infinitely many GLs with a similar functionality



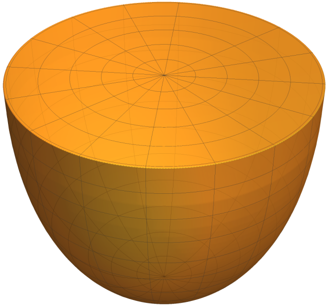
Example: transforming the sphere

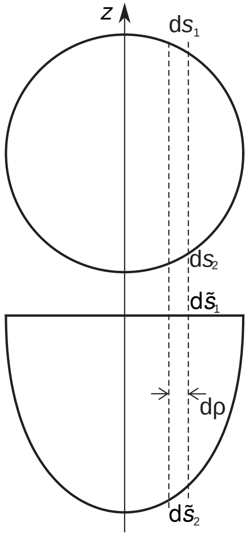
- For a sphere, $\rho(s) = \sin s$, therefore

$$s_1(\rho) = \arcsin\rho, \quad s_2(\rho) = \pi - \arcsin\rho$$

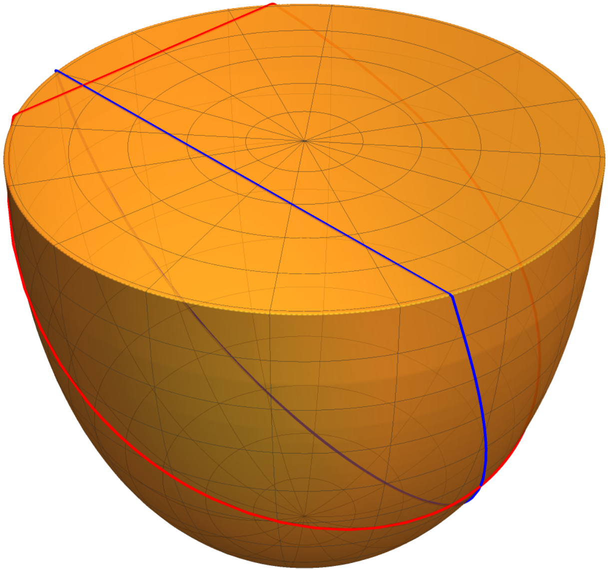
- Could we make the upper part flat? This requires

$$s_1(\rho) = \rho\,, \qquad s_2(\rho) = \pi + \rho - 2\arcsin\rho$$



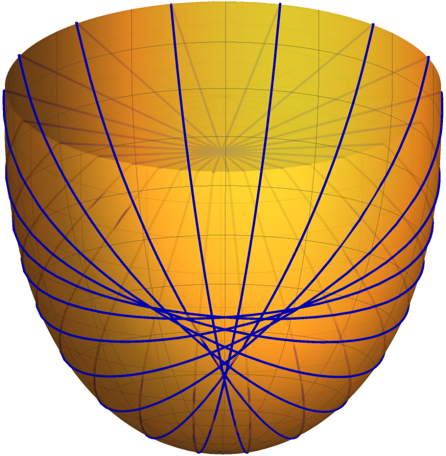


Rays on the transformed sphere

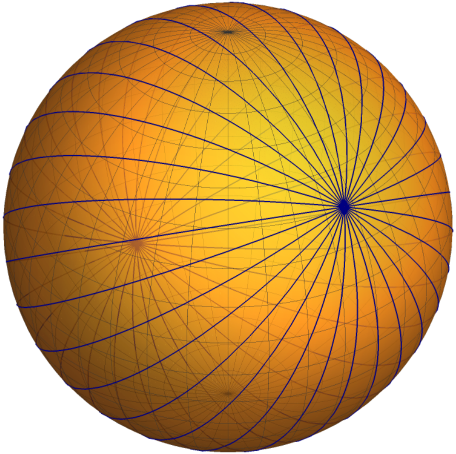


Rays on the transformed sphere

Each point has just one image – itself



Each point has just two images –
the opposite point and itself



Transmutation

- We can also transform a GL differently
- We cut the surface along the meridian and wind it, multiplying the angle φ by some $N \in \mathbb{N}$, which gives
$$\tilde{\rho}(s) = \frac{\rho(s)}{N}$$
- To keep the equator radius 1, we enlarge the whole surface by the factor N, which gives
$$\tilde{\rho}(s) = \rho(s/N)$$

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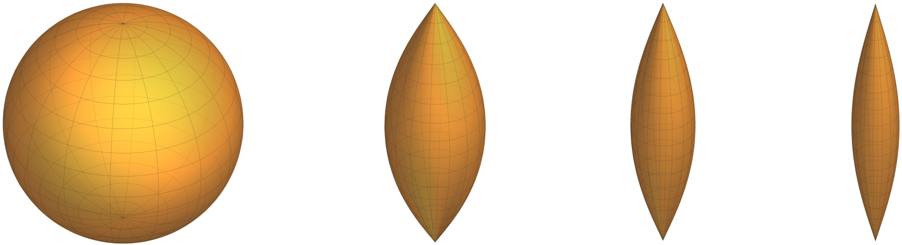
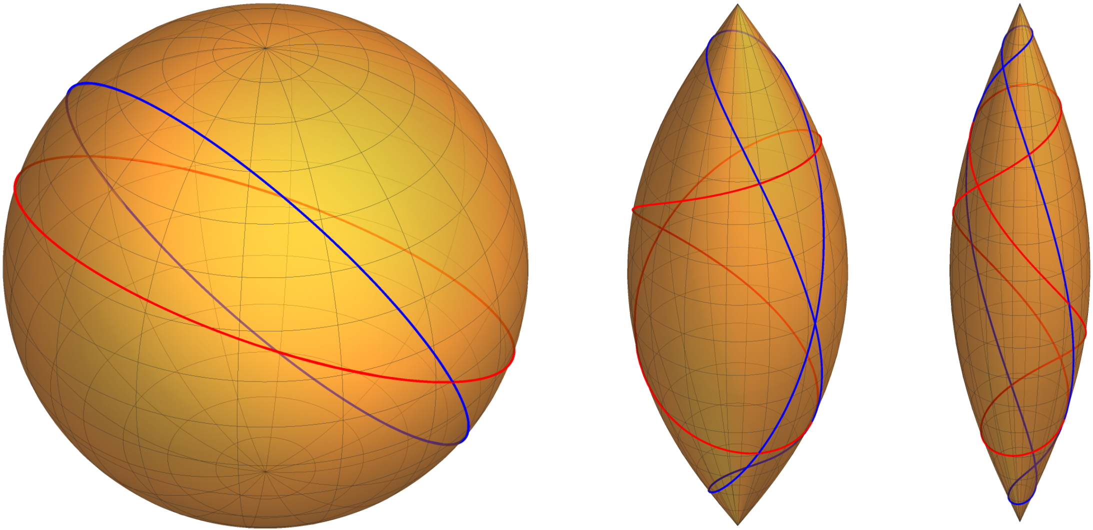


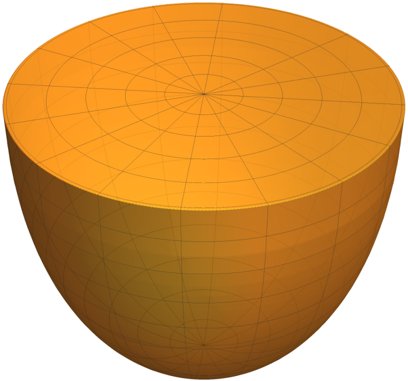
$N = 1$ $N = 2$ $N = 3$ $N = 4$

Rays on a transmuted sphere

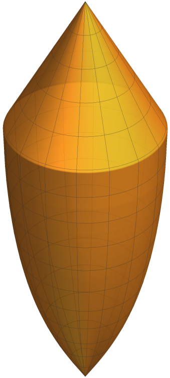


Transmuting the transformed sphere

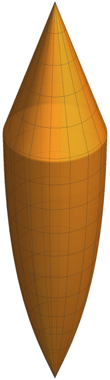
- We can transmute an arbitrary GL, e.g. the transformed sphere



$N = 1$

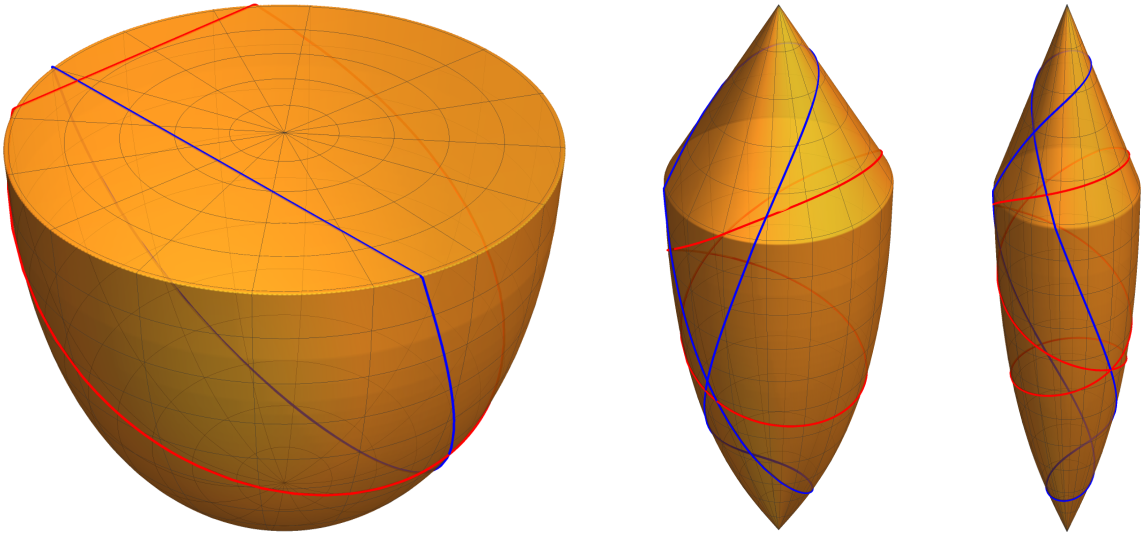


$N = 2$

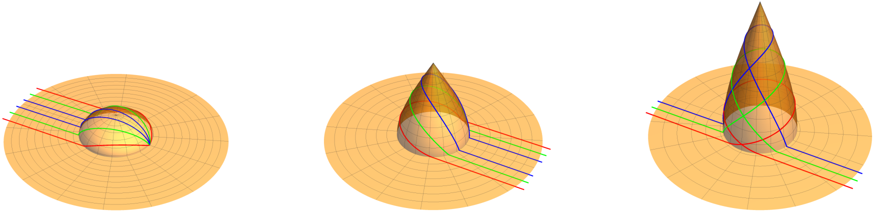


$N = 3$

Rays on the transmuted transformed sphere



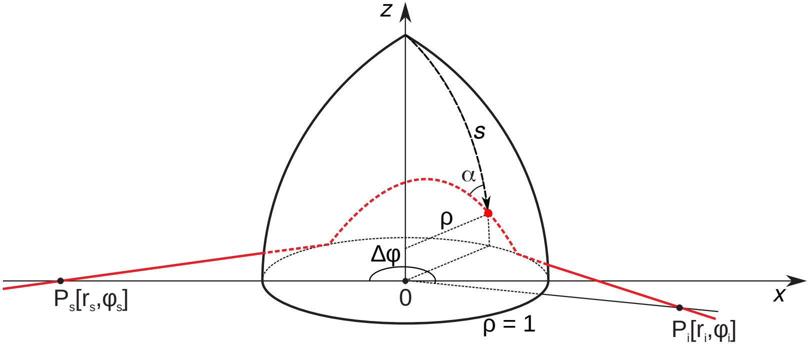
Designing open geodesic lenses



- Single-sided GLs emerge above equator just on one side, then continue into a plane
- We return to the formula

$$\Delta\varphi_{\mathrm{GL}}(L) = 2\int_L^1 \frac{L\, s'(\rho)\, \mathrm{d}\rho}{\rho\sqrt{\rho^2 - L^2}} \equiv 2g(L)$$

- We can invert this equation to find $s(\rho)$ from the known function $g(L)$ by the inverse Abel transform
- We can then design GL with some required properties, e.g. focusing



- We require that the GL focuses light rays emerging from the point P_s to the point P_i
- We denote the total azimuthal angle swept by the ray by $\Delta\varphi = M\pi$
- Angles swept from point P_s to GL and from GL to P_i are

$$\Delta\varphi_{s,i} = \int_1^{r_{s,i}} \frac{L\,d\rho}{\rho\sqrt{\rho^2 - L^2}} = -\arcsin\frac{L}{\rho}\bigg|_1^{r_{s,i}} = \arcsin L - \arcsin\frac{L}{r_{s,i}}$$

- Then it holds

$$\Delta\varphi = M\pi = \Delta\varphi_s + \Delta\varphi_i + \Delta\varphi_{GL} = \Delta\varphi_s + \Delta\varphi_i + 2g(L)$$

and therefore

$$\int_L^1 \frac{L\, s'(\rho)\,\mathrm{d}\rho}{\rho\sqrt{\rho^2 - L^2}} = g(L) = \frac{1}{2}\left(M\pi - 2\arcsin L + \arcsin\frac{L}{r_s} + \arcsin\frac{L}{r_i}\right)$$

- If we can find $s(\rho)$ from this, we can find the shape of GL that performs imaging between points P_s and P_i

Inverse Abel transform

- In the formula
$$\int_L^1 \frac{L\, s'(\rho)\, \mathrm{d}\rho}{\rho\sqrt{\rho^2 - L^2}} = g(L)$$
we relabel ρ to η, divide by $\sqrt{L^2 - \rho^2}$ and integrate with respect to L from ρ to 1:
$$\int_\rho^1 \left(\int_L^1 \frac{L\, s_1'(\eta)\, \mathrm{d}\eta}{\eta\sqrt{\eta^2 - L^2}} \right) \frac{\mathrm{d}L}{\sqrt{L^2 - \rho^2}} = \int_\rho^1 \frac{g(L)\, \mathrm{d}L}{\sqrt{L^2 - \rho^2}}.$$
- We change the integration order

Inverse Abel transform

- In the formula

$$\int_L^1 \frac{L\, s'(\rho)\,\mathrm{d}\rho}{\rho\sqrt{\rho^2 - L^2}} = g(L)$$

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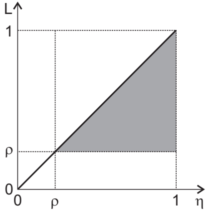


$$\int_\rho^1 \frac{s'(\eta)}{\eta} \left(\int_\rho^\eta \frac{L\,\mathrm{d}L}{\sqrt{(\eta^2 - L^2)(L^2 - \rho^2)}} \right) \mathrm{d}\eta = \int_\rho^1 \frac{g(L)\,\mathrm{d}L}{\sqrt{L^2 - \rho^2}}$$

- The integral with respect to L on the left-hand side is $\frac{\pi}{2}$, so

$$\frac{\pi}{2}\int_\rho^1 \frac{s'(\eta)\,\mathrm{d}\eta}{\eta} = \int_\rho^1 \frac{g(L)\,\mathrm{d}L}{\sqrt{L^2-\rho^2}}$$

- Differentiating with respect to ρ we find the function $s'(\rho)$:

$$s'(\rho) = -\frac{2\rho}{\pi}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\int_\rho^1 \frac{g(L)\,\mathrm{d}L}{\sqrt{L^2-\rho^2}}\right)$$

- By integrating this equation we find $s(\rho)$, which specifies the shape of GL
- We substitute the function $g(L)$:

$$s'(\rho) = -\frac{\rho}{\pi}\frac{\mathrm{d}}{\mathrm{d}\rho}\left[\int_\rho^1 \left(M\pi + \arcsin\frac{L}{r_\mathrm{s}} + \arcsin\frac{L}{r_\mathrm{i}} - 2\arcsin L\right)\frac{\mathrm{d}L}{\sqrt{L^2-\rho^2}}\right].$$

Analytic solution

- After a long calculation we find a simple result

$$s'(\rho) = A(\rho) + \frac{B}{\sqrt{1-\rho^2}}$$

where

$$\begin{aligned} A(\rho) &= 1 - \frac{1}{\pi}\left(\arcsin\sqrt{\frac{1-\rho^2}{r_{\mathrm{s}}^2-\rho^2}} + \arcsin\sqrt{\frac{1-\rho^2}{r_{\mathrm{i}}^2-\rho^2}}\right), \\ B &= (M-1) + \frac{1}{\pi}\left(\arcsin\frac{1}{r_{\mathrm{s}}} + \arcsin\frac{1}{r_{\mathrm{i}}}\right) \end{aligned}$$

- This is the solution of the Luneburg inverse problem, which can be put into the form

$$\begin{aligned} s(\rho) &= A\rho + B\arcsin\rho - \frac{1}{\pi}\left[r_{\mathrm{s}}\arcsin\left(\rho\sqrt{\frac{r_{\mathrm{s}}^2-1}{r_{\mathrm{s}}^2-\rho^2}}\right) + r_{\mathrm{i}}\arcsin\left(\rho\sqrt{\frac{r_{\mathrm{i}}^2-1}{r_{\mathrm{i}}^2-\rho^2}}\right)\right. \\ &\quad \left. - \left(\sqrt{r_{\mathrm{s}}^2-1} + \sqrt{r_{\mathrm{i}}^2-1}\right)\arcsin\rho\right] \end{aligned}$$

The case of $r_s, r_i \in \{1, \infty\}$

- If both r_s and r_i are restricted to be either 1 or ∞, the formulas get greatly simplified:

$$s(\rho) = A\rho + B \arcsin\rho\,, \qquad A + B = M$$

  with both A and B constant
- One can also express the total change of polar angle $\Delta\varphi_{GL}(L)$ swept on the geodesic lens:

$$\Delta\varphi_{GL}(L) = (A + B)\pi - 2A \arcsin L$$

- For $L = 1$ we have

$$\Delta\varphi_{GL}(1) = \pi B,$$

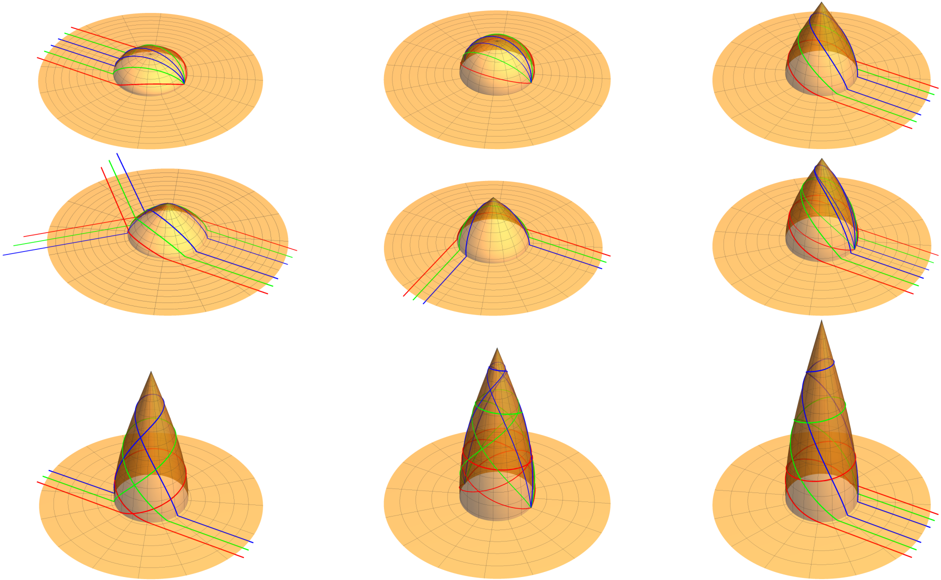
  so the ray that hits the circle $\rho = 1$ tangentially sweeps on the GL the polar angle $B\pi$
- Consequently, the polar angle swept by this ray in the plane around the GL is $A\pi$
- This geometric interpretation of A and B enables to deduce the shape of the geodesic lens quickly from the behavior of light rays

The outermost ray that reaches the curved part sweeps:

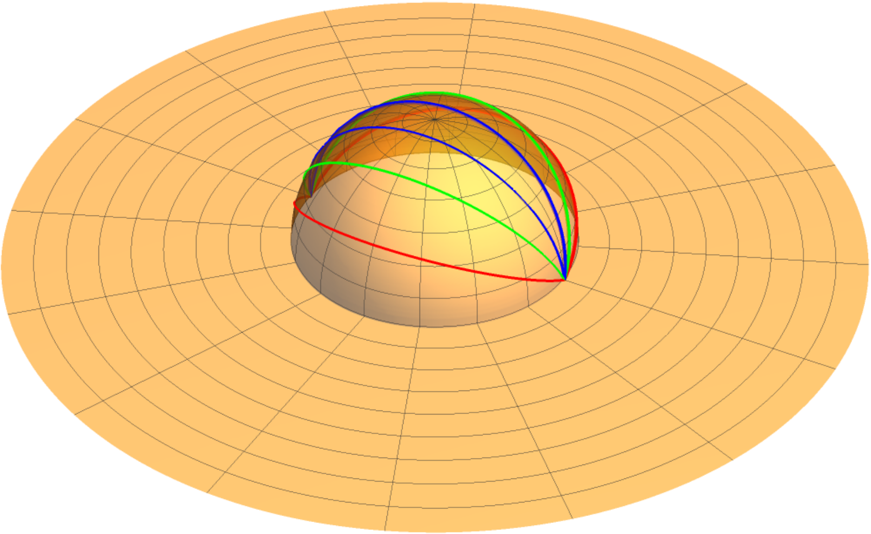
- polar angle $A\pi$ in the planar part
- polar angle $B\pi$ on the curved part

Lens	r_s	r_i	M	A	B	Ref. index of corr. lens
Maxwell's fish-eye lens	1	1	1	0	1	$n(r) = \frac{2}{1+r^2}$
Generalized Maxwell's fish-eye lens	1	1	M	0	M	$n(r) = \frac{2r^{1/M-1}}{1+r^{2/M}}$
Luneburg lens	1	∞	1	$\frac{1}{2}$	$\frac{1}{2}$	$n(r) = \sqrt{2-r^2}$
Beam divider (point source)	1	∞	M	$\frac{1}{2}$	$M-\frac{1}{2}$	
Homogeneous medium	∞	∞	1	1	0	$n(r) = 1$
90° rotating lens	∞	∞	$\frac{3}{2}$	1	$\frac{1}{2}$	$rn^4 - 2n + r = 0$
Eaton lens	∞	∞	2	1	1	$n(r) = \sqrt{\frac{2}{r}-1}$
Invisible sphere (lens)	∞	∞	3	1	2	$rn^{3/2} + rn^{1/2} - 2 = 0$
Beam divider (parallel ray source)	∞	∞	M	1	$M-1$	

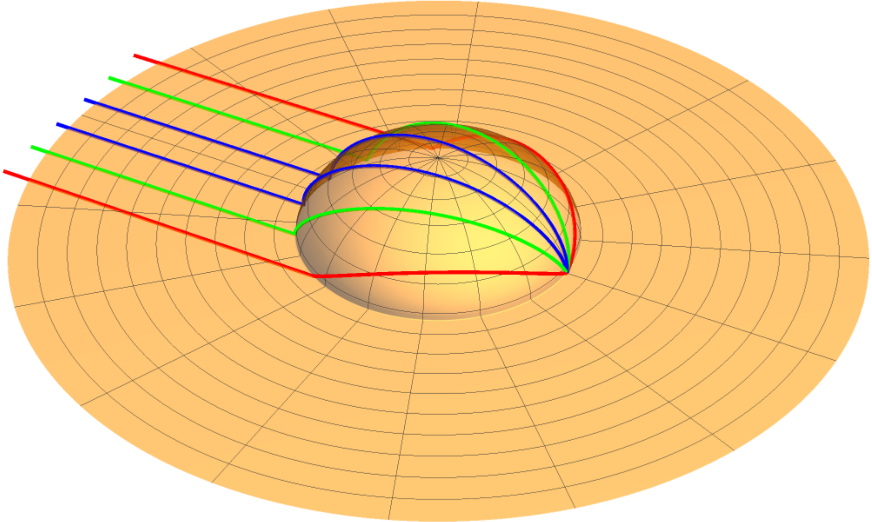
Particular examples of GLs



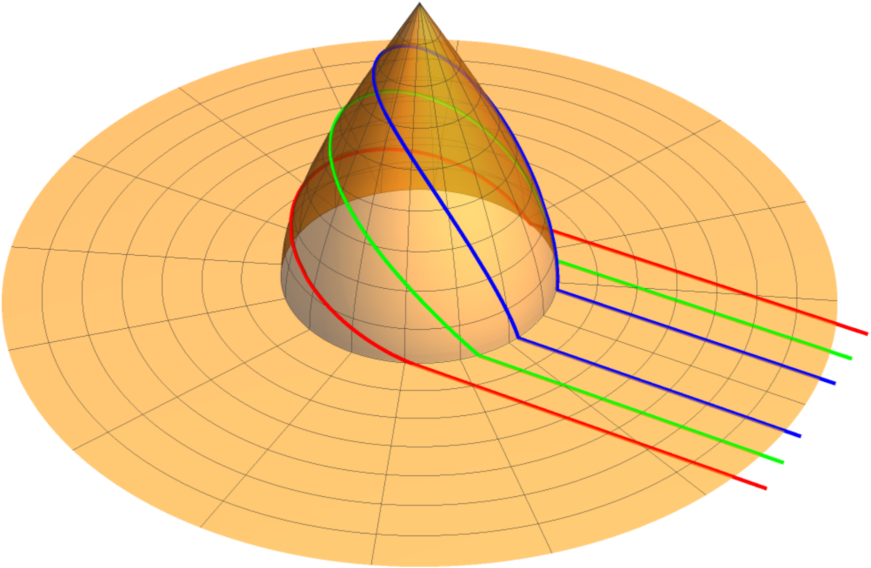
Maxwell's fish-eye lens ($r_s = 1$, $r_i = 1$, $M = 1$, $A = 0$, $B = 1$)



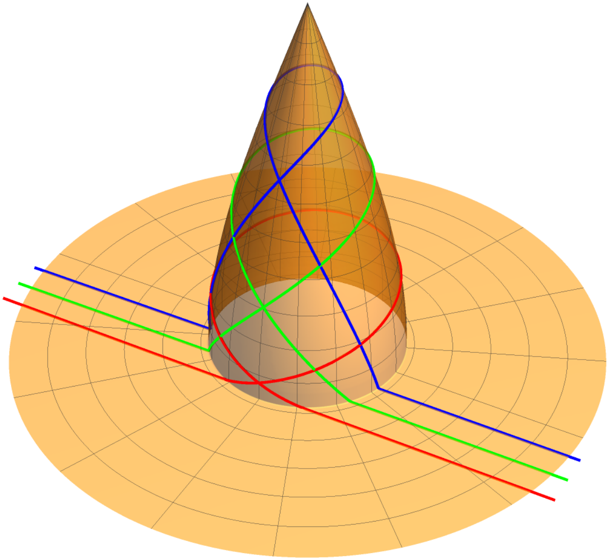
Luneburg lens ($r_s = \infty$, $r_i = 1$, $M = 1$, $A = 1/2$, $B = 1/2$)



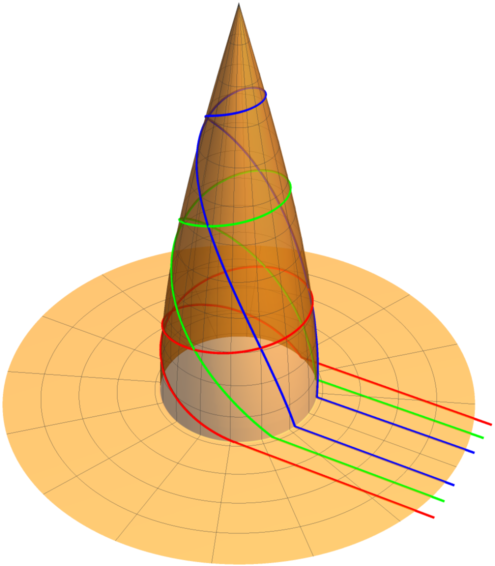
Eaton lens ($r_s = \infty$, $r_i = \infty$, $M = 2$, $A = 1$, $B = 1$)



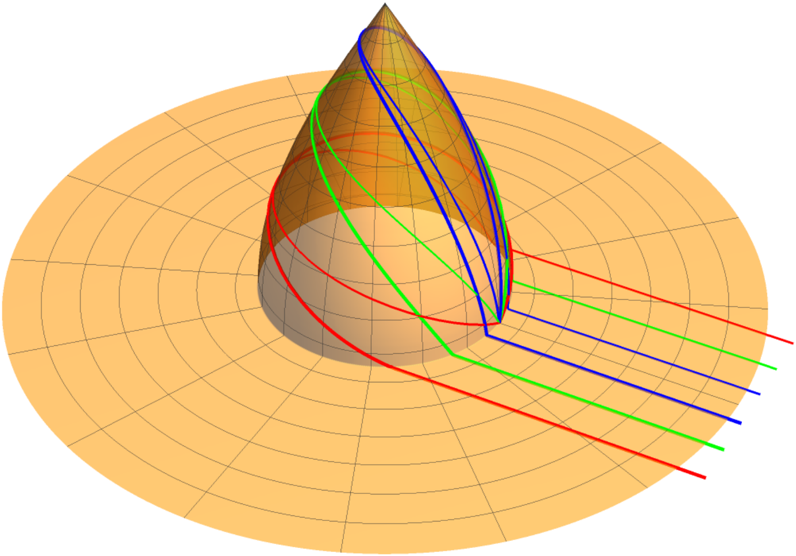
Invisible lens ($r_s = \infty$, $r_i = \infty$, $M = 3$, $A = 1$, $B = 2$)



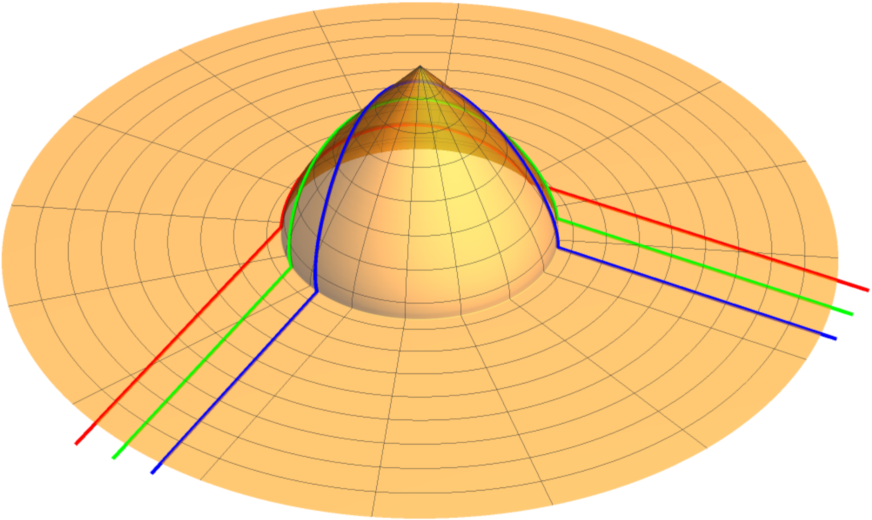
"Double Eaton lens" ($r_s = \infty$, $r_i = \infty$, $M = 4$, $A = 1$, $B = 3$)



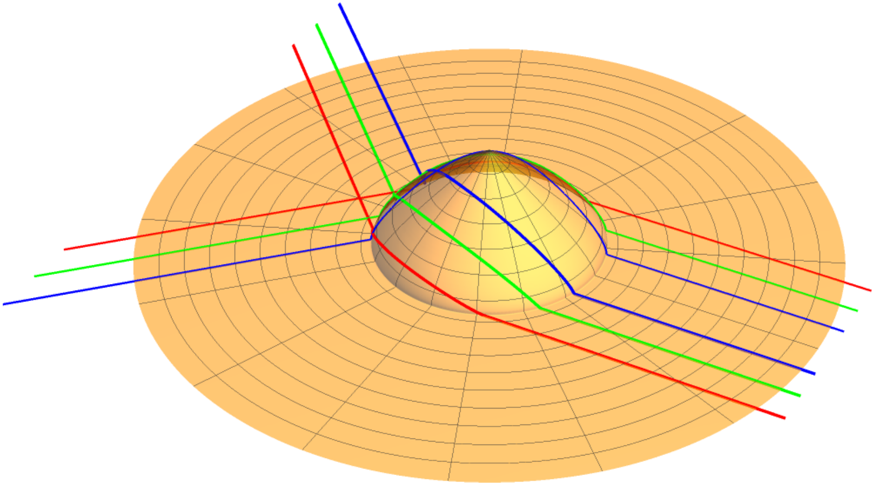
"Inverse Luneburg lens" ($r_s = \infty$, $r_i = 1$, $M = 2$, $A = 1/2$, $B = 3/2$)



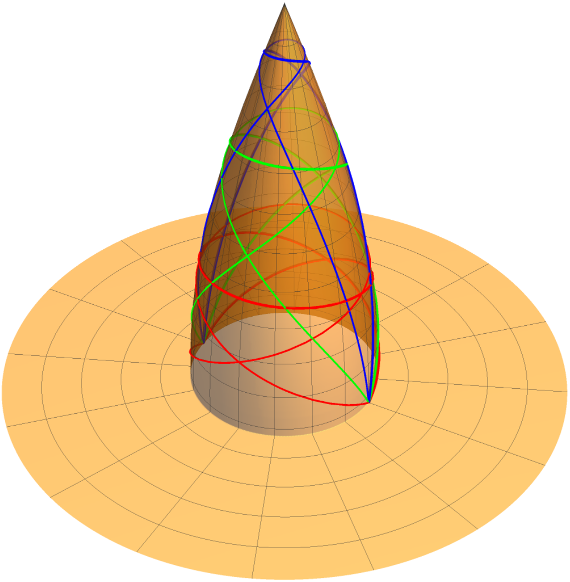
90 degree rotating lens ($r_s = \infty$, $r_i = \infty$, $M = 3/2$, $A = 1$, $B = 1/2$)



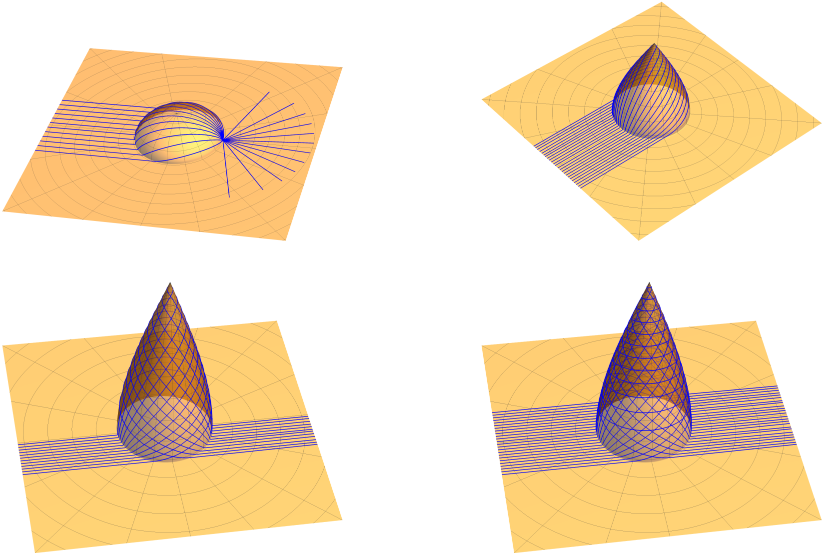
Beam divider ($r_s = \infty$, $r_i = \infty$, $M = 5/4$, $A = 1$, $B = 1/4$)



Transmuted Maxwell's fish-eye lens ($r_s = 1$, $r_i = 1$, $M = 3$, $A = 0$, $B = 3$)

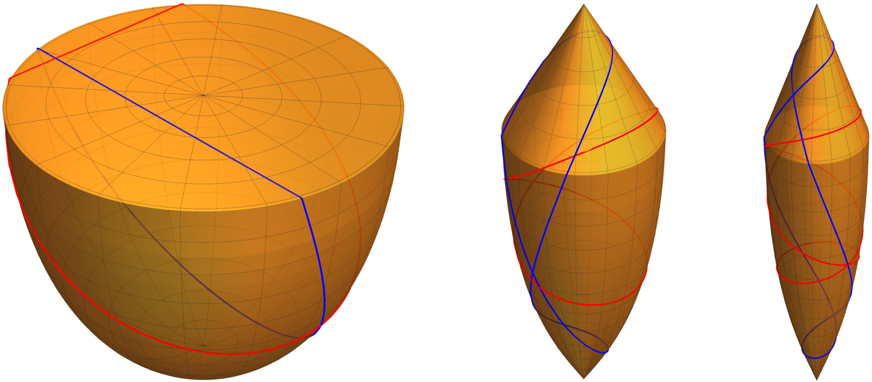


Ray videos

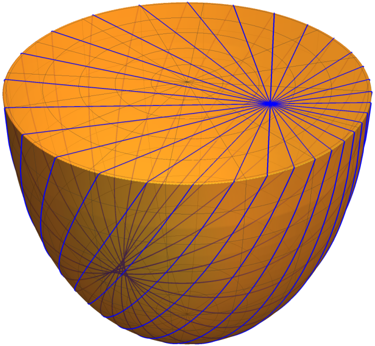


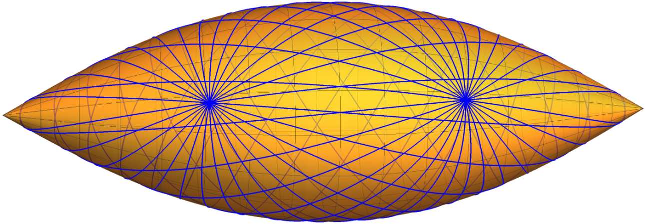
Closed GL

- A similar formalism can be applied to double-sided GLs
- There is an freedom that we discussed, therefore there are many GLs that give closed ray paths



Ray videos for closed GL





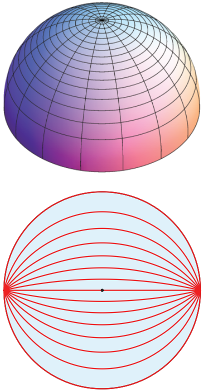
Relation to absolute optical instruments

- Each GL is equivalent to some medium (lens) with radially-symmetric refractive index $n(r)$, we take the names from these lenses
- AI image a 3D region stigmatically – [T. Tyc *et al*, New J. Phys. 13, 115004 (2011)]
- What a GL achieves via the surface curvature, AI achieves via it refractive index profile
- Optical path elements on GL and in the refractive medium must be the same:

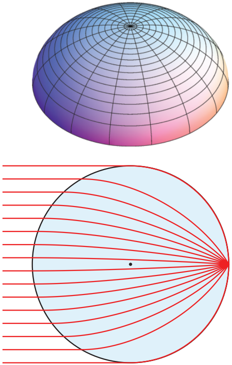
$$n^2(r)(\mathrm{d}r^2 + r^2\,\mathrm{d}\varphi^2) = \mathrm{d}s^2 + \rho^2\,\mathrm{d}\varphi^2$$

from which it follows the general relation

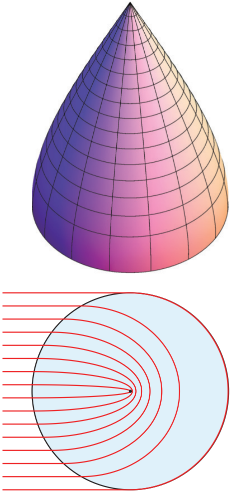
$$rn(r) = \rho\,, \qquad n\,\mathrm{d}r = \mathrm{d}s$$



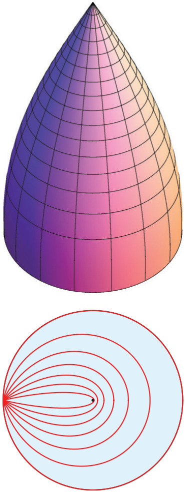
Maxwell's fisheye mirror



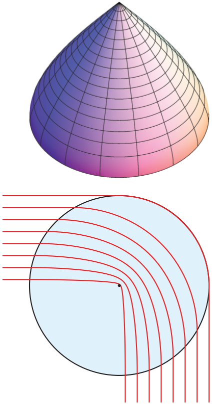
Luneburg



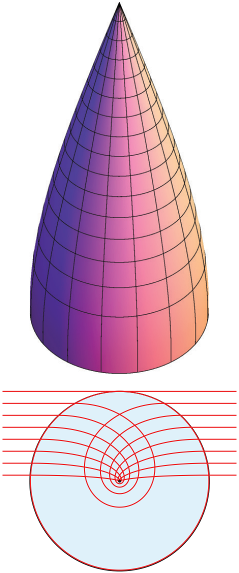
Eaton



Transmuted sphere

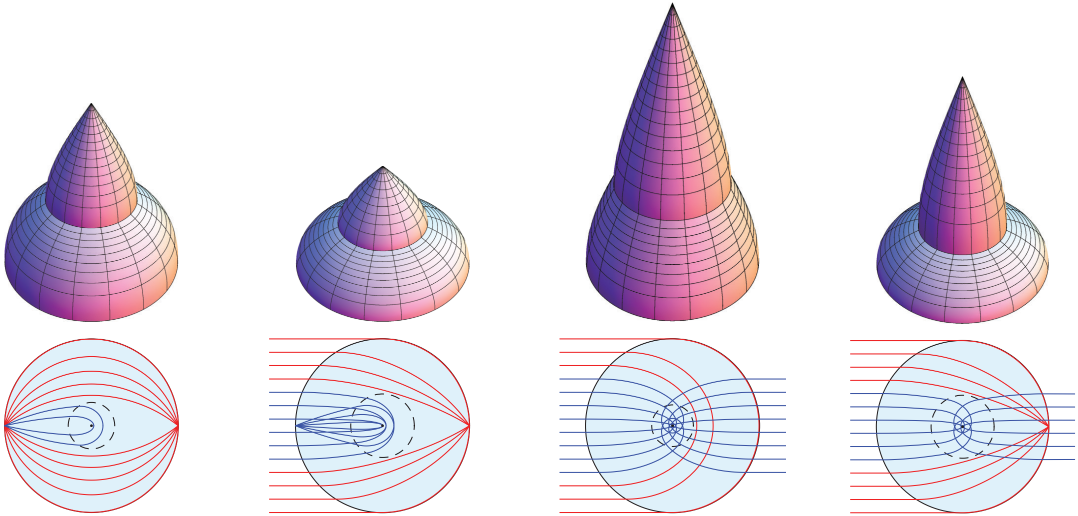


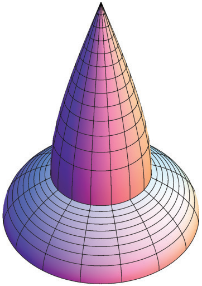
90-degree rotating

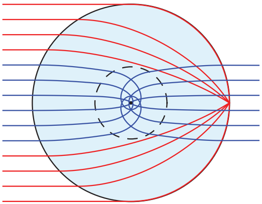


invisible

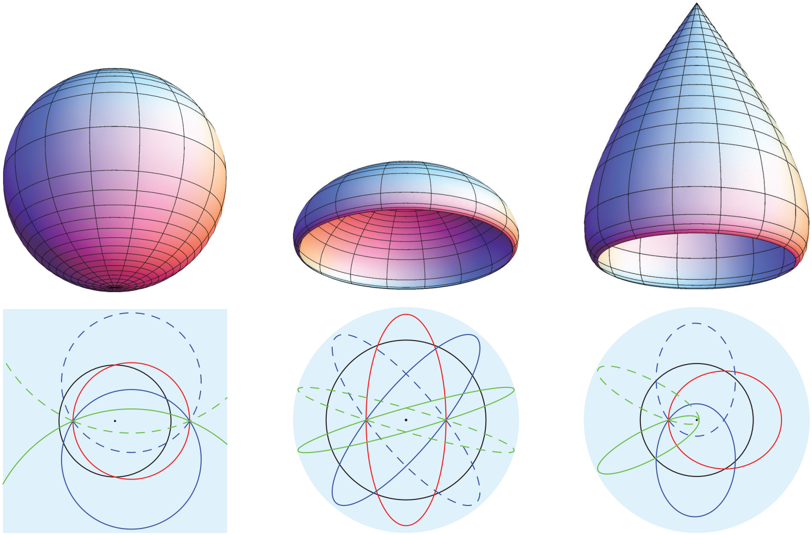
Multi-functional (or multifocal) geodesic lenses



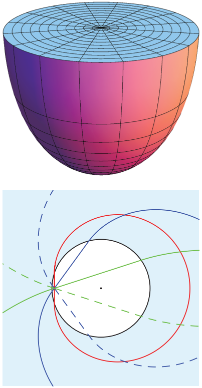




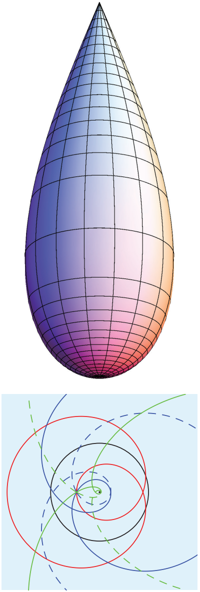
Closed GL



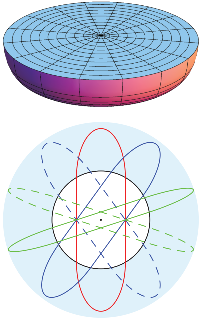
Maxwell's fisheye   Luneburg   Eaton



transformed sphere



Tannery's pear



imaging cavity

Raytracing on GL

- Once we know the shape of GL (the function $\rho(s)$), we can find ray trajectories
- They are the same as free particle trajectories on GL
- Lagrangian for such a particle is simply its kinetic energy:

$$L = \frac{1}{2}[\dot{s} + \rho^2(s)\dot{\varphi}^2]$$

- Lagrange equations for s and φ then yield

$$\ddot{s}(t) = \rho[s(t)]\,\rho'[s(t)]\dot{\varphi}^2(t)$$
$$\rho[s(t)]\ddot{\varphi}(t) + 2\rho'[s(t)]\dot{s}(t)\dot{\varphi}(t) = 0$$

  – very useful for raytracing

Geodesic equation approach

- General equation for a geodesic parametrized by curve length ξ

$$\frac{d^2x^\lambda}{d\xi^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{d\xi}\frac{dx^\nu}{d\xi} = 0$$

  $\Gamma^\lambda_{\mu\nu}$ are Christoffel symbols of coordinate system $\{x^\lambda; \lambda = 1, 2\}$
- For coordinates $(x^1, x^2) = (s, \varphi)$:

$$\Gamma^\lambda_{\mu\nu} = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & -\rho(s)\rho'(s) \end{pmatrix}_{\mu\nu} \\ \begin{pmatrix} 0 & \frac{\rho'(s)}{\rho(s)} \\ \frac{\rho'(s)}{\rho(s)} & 0 \end{pmatrix}_{\mu\nu} \end{pmatrix}_\lambda$$

$$s''(\xi) - \rho[s(\xi)]\,\rho'[s(\xi)]\varphi'^2(\xi) = 0$$

$$\varphi''(\xi) + 2\frac{\rho'[s(\xi)]}{\rho[s(\xi)]}\,s'(\xi)\varphi'(\xi) = 0$$

- The same equations as for the free particle!

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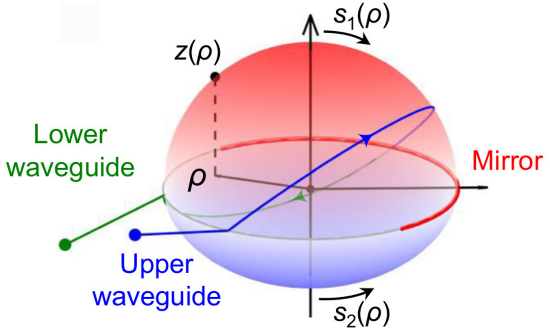
https://doi.org/10.1038/s41467-022-29587-9 OPEN

Double-layer geodesic and gradient-index lenses

Qiao Chen[1], Simon A. R. Horsley[2], Nelson J. G. Fonseca[3], Tomáš Tyc[4] & Oscar Quevedo-Teruel[1✉]



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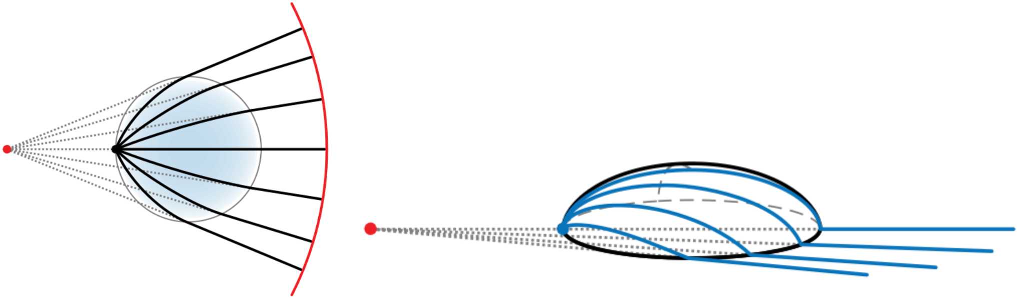
https://doi.org/10.1038/s42005-021-00774-2 OPEN

A solution to the complement of the generalized Luneburg lens problem

Nelson J. G. Fonseca [1✉], Tomáš Tyc [2] & Oscar Quevedo-Teruel [3]



GLs without rotational symmetry

- It is much more difficult to describe and design them
- Good starting point – conformal mapping between GL and a plane
- Design refractive index profile in the plane with desired functionality
- Then construct the GL surface from that

GLs without rotational symmetry

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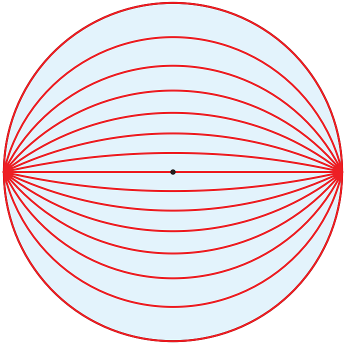

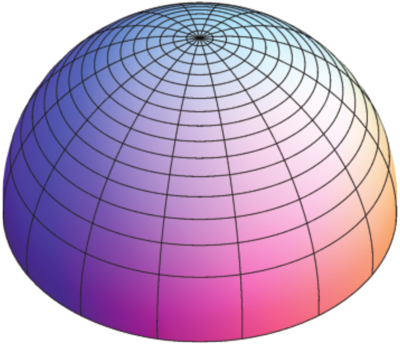

- Problem – too much freedom

- The main problem: how to embed a 2D surface whose intrinsic geometry is known into the 3D space
- Need for additional constraints that have to be finely tuned
- Still an open problem

Some useful references

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- M. Šarbort and T. Tyc, J. Opt. 14, 075705 (2012)
- M. Šarbort and T. Tyc, J. Opt. 15, 125716 (2013)
- R. C. Mitchell-Thomas, O. Quevedo-Teruel, T. M. McManus, S. A. R. Horsley and Y. Hao, Lenses on curved surfaces, Opt. Lett. 39, 3551–3554 (2014)

Conclusion

- GLs are fascinating
- They have many potential applicatons
- Can be described mathematically
- Can be also understood intuitively
- I hope you will like them
- Thank you for your attention!