Zajmava teoreticka fyzika
Tomáš Tyc, Brno 2011

Angular momentum, transmutation using logarithm, inversion problem

1 Angular momentum

In the central potential the angular momentum with respect to the centre plays a primary role. It is conserved because the Lagrangian does not change when rotating the sysem about the centre. Angular momentum is

$$|\vec{L}| = |\vec{r} \times \vec{p}| = rp\sin\alpha = rn(r)\sin\alpha \tag{1}$$

where p is the momentum of the particle and α is the angle between the particle trajectory and the radius vector. We have used tha fact that $p = \sqrt{2[E - V(r)]} = n(r)$; the momentum is equal to the refractive index.

We can also determine the maximum angular momentum that the particle or the ray can have at some radius r. Clearly, it is $L_{\max} = rn(r) \equiv \rho(r)$. We will call this value *turning parameter*.

2 Transmutation of central potentials via the logarithm

Consider physical plane with central potential $V(r)$ and the equivalent refractive index $n(r)$. We identify this plane with the complex z-plane and map it into a complex w-plane (which will be called virtual plane) by

$$w = \ln z. \tag{2}$$

This relation implies in particular that $r = |z| = |\mathrm{e}^w| = \mathrm{e}^{\mathrm{Re}\,w}$. The refractive index in w-plane is then

$$N(w) = n(|z|)\left|\frac{\mathrm{d}z}{\mathrm{d}w}\right| = n(\mathrm{e}^{\mathrm{Re}\,w})\,\mathrm{e}^{\mathrm{Re}\,w} = rn(r) \tag{3}$$

We see that the transmuted refractive index depends only on the real part of w. Putting $w = x + \mathrm{i}y$, the original problem of motion in the central potential has been transformed into the problem of motion of a particle in the virtual plane (x, y) where the potential depends only on x.

Examples:

- *Transmutation of motion in potential of fatal attraction, $V(r) = -\alpha/r^2$ with zero energy by logarithm.*
  The refractive index is $n(r) = \sqrt{2\alpha}/r$ and the transmuted refractive index will be

  $$N(w) = rn(r) = \sqrt{2\alpha} = \text{const.} \tag{4}$$

  Therefore rays in the transmuted plane will form straight lines. At the same time, we can from this deduce the form of trajectories in the original plane. Any straight line in the transmuted plane can be written as $x = ay + b$ with some real constants a, b, or as $y = c$. This, translated into the original plane where $r = \mathrm{e}^x$ and $\varphi = y$, gives $r = \mathrm{e}^{b+ay}$, which is an equation of a logarithmic spiral, or into $\varphi = c$, which is a radial line. So these are the trajectories in potential of fatal attraction with zero energy, see Fig 1. Note that for $a = 0$ and different b we get circles

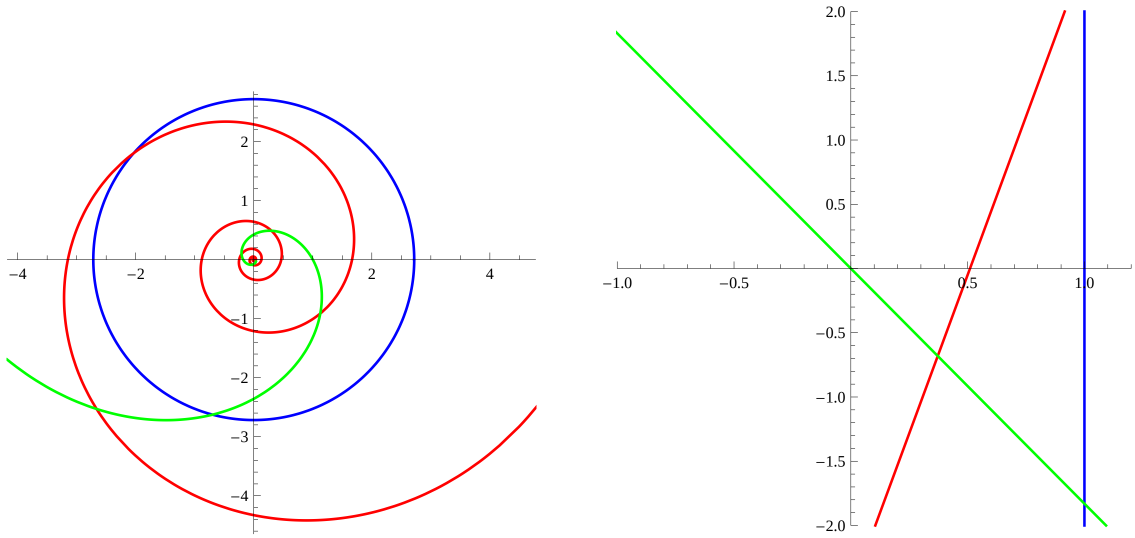

Figure 1: Transmutation of potential of fatal attraction with $E = 0$ (trajectories are logarithmic spirals) to constant potential (trajectories are straight lines). Vertical lines (such as the blue one) in the second space are mapped on circles, which shows that there are infinitely many circular orbits in the potential of fatal attraction with $E = 0$.

with an arbitrary diameter. This is because for one particular value of angular momentum, namely $L = \sqrt{2\alpha}$, the effective potential

$$V_{\mathrm{ef}} = -\frac{\alpha}{r^2} + \frac{L^2}{2r^2} \tag{5}$$

turns identically to zero, and therefore a particle with zero energy in this potential cannot move, i.e., must have $r = \text{const.}$, which corresponds to a circular trajectory, and r can be arbitrary (but fixed).

Circular orbits in physical plane

Consider a circular orbit in physical plane. This corresponds to a particle moving uniformly along the line $x = x_0$ in virtual plane. Clearly, such a motion is possible iff $\mathrm{d}N/\mathrm{d}x|_{x=x_0} = 0$ because then the particle feels no force in the x direction. Substituting $N = rn$ into this condition yields

$$n(r) + rn'(r) = 0\,, \tag{6}$$

where prime means derivative with respect to r.

On the other hand, for motion on a circular trajectory the acceleration is v^2/r, which must be equal to the force in radial direction, $F = -\mathrm{d}V/\mathrm{d}r$, that is,

$$\frac{n^2}{r} = \frac{v^2}{r} = -\frac{\mathrm{d}V}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}\frac{n^2}{2} = -nn'\,, \tag{7}$$

which gives exactly Eq. (6) and shows that the theory of transmutation via logarithm really works.

Angular momentum

How is the angular momentum reflected in the (x, y) plane? Angular momentum in We use the fact that the map (2) is conformal. Then α from Eq. (1) is at the same time the angle between the

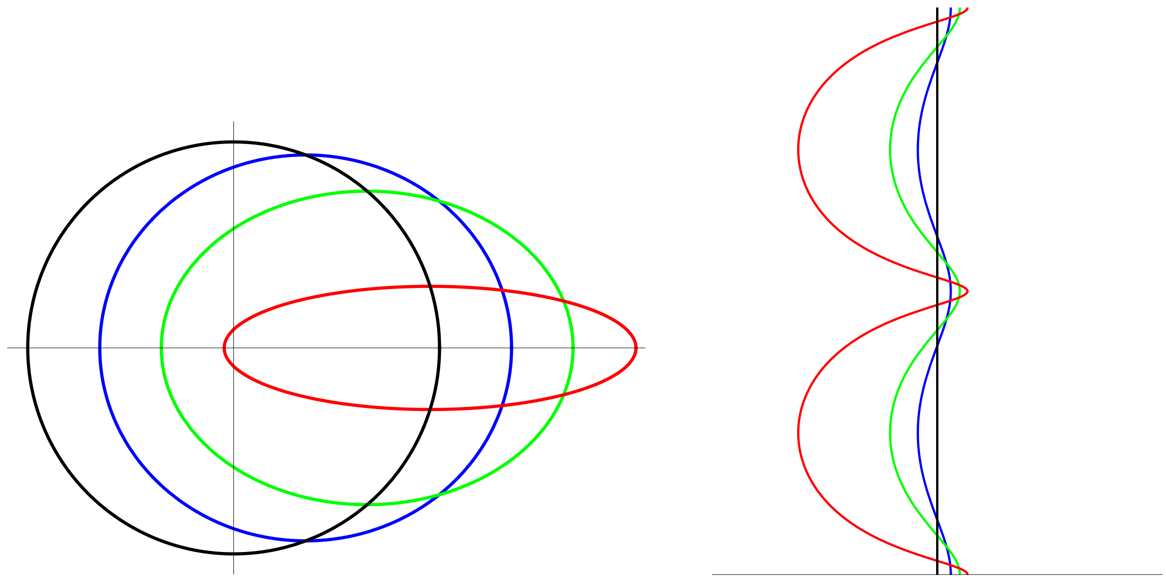

Figure 2: Transmutation of Newton potential via logarithm. The resulting potential depends only on the x-coordinate, hence solution of equations of motion in the transmuted plane is extraordinarily simple. The trajectories corresponding to transmuted Hooke potential would be almost the same, just scaled by the factor $1/2$ in both x and y directions as the Hooke potential is just a transmutation of the Newton potential.

trajectory and the x-direction in virtual plane. Now, since L is conserved in physical plane, then the quantity $N(x)\sin\alpha$ should be conserved in virtual plane. Is this reasonable? Indeed, and very much! The condition $N(x)\sin\alpha = L = \text{const.}$ is precisely the Snell's law in virtual plane. The value of L is equal to the refractive index at the point where the trajectory of the particle turns vertical, which corresponds to a turning point in the original plane. In other words, L is equal to the turning parameter ρ at that point.

Trajectories in virtual plane

To find the trajectories in virtual plane, we use the standard methods of classical mechanics. The potential $U(x)$ is related to $N(x)$ as $N = \sqrt{2(\varepsilon - U)}$. First of all, the motion in y-direction is free, so $\dot{y} = \beta = \text{const.}$ From conservation of energy we get

$$\frac{\dot{x}^2 + \beta^2}{2} + U(x) = E, \tag{8}$$

which enables to calculate $\dot{x}$ as

$$\dot{x} = \sqrt{2[E - U(x)] - \beta^2} = \sqrt{N^2(x) - \beta^2}. \tag{9}$$

Then

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\dot{x}}{\dot{y}} = \sqrt{\frac{N^2(x)}{\beta^2} - 1}. \tag{10}$$

This equation allows separation of variables. We also see another interesting thing: if the trajectory turns vertical, LHS of Eq. (10) turns to zero and hence $N = \beta$. With respect to what has been said above, β is then equal to the angular momentum L in physical plane and we can replace β by L in the above equations. Then Eq. (10) can be rewritten as

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \frac{\sqrt{N^2(x) - L^2}}{L}. \tag{11}$$

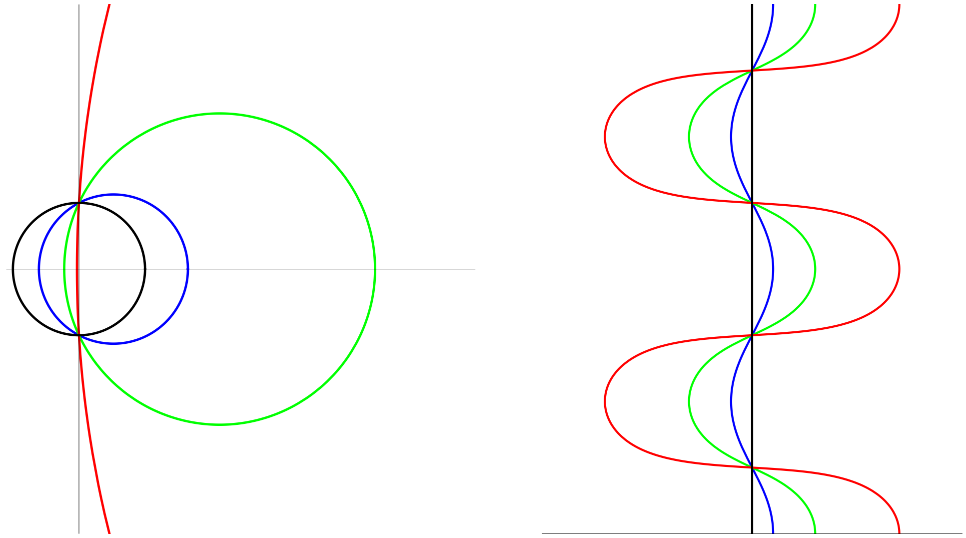

Figure 3: Transmutation of Maxwell fish eye via logarithm. The trajectories in the transmuted plane correspond to the border between day and night on the Earth at some moment of time drawn in Mercator projection.

Solution of the inverse problem

The transmutation via logarithm provides an elegant solution of the inverse problem discussed by Ostrovsky [4]. From Eq. (11) we can, for finite motion in x-direction, calculate the increment h of y corresponding to x changing from its minimum value $x_{\min}$ to its maximum value $x_{\max}$ as a function of L:

$$h(L) = L \int_{x_{\min}}^{x_{\max}} \frac{\mathrm{d}x}{\sqrt{N^2(x) - L^2}}\,. \tag{12}$$

Here $x_{\min}, x_{\max}$ denote the minimum and maximum value, respectively, of x for a given angular momentum L in physical plane; they are solutions of the equation $N(x) = L$.

The inverse problem can be solved in a very similar way to finding the 1D potential from the known period of oscillations as a function of energy [8]. We first change the integration variable in Eq. (12) from x to N:

$$\begin{aligned} h(L) =& L \int_{x_{\min}}^{x_0} \frac{\mathrm{d}x}{\sqrt{N^2(x) - L^2}} + L \int_{x_0}^{x_{\max}} \frac{\mathrm{d}x}{\sqrt{N^2(x) - L^2}} \\ =& L \int_{L}^{L_{\mathrm{m}}} \frac{\mathrm{d}x_{\min}}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}} + L \int_{L_{\mathrm{m}}}^{L} \frac{\mathrm{d}x_{\max}}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}} \\ =& - L \int_{L}^{L_{\mathrm{m}}} \frac{\mathrm{d}(x_{\max} - x_{\min})}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}}. \end{aligned} \tag{13}$$

Here we denoted x_0 the value of x between $x_{\min}$ and $x_{\max}$ for which $\mathrm{d}N/\mathrm{d}x = 0$, which corresponds to motion with constant x (and hence to circular orbit in physical plane), and L_{m} denotes the corresponding angular momentum in physical plane.

Next we divide $h(L)$ by $\sqrt{L^2 - \mathcal{L}^2}$, where $\mathcal{L}$ is a parameter, and integrate from $\mathcal{L}$ to L_{m}:

$$\xi(\mathcal{L}, L_{\mathrm{m}}) \equiv \int_{\mathcal{L}}^{L_{\mathrm{m}}} \frac{h(L)\,\mathrm{d}L}{\sqrt{L^2 - \mathcal{L}^2}} = - \int_{\mathcal{L}}^{L_{\mathrm{m}}} \frac{L\,\mathrm{d}L}{\sqrt{L^2 - \mathcal{L}^2}} \int_{L}^{L_{\mathrm{m}}} \frac{\mathrm{d}(x_{\max} - x_{\min})}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}}. \tag{14}$$

Inverting the order of integration and changing the limits appropriately, we get

$$\xi(\mathcal{L}, L_\mathrm{m}) = -\int_{\mathcal{L}}^{L_\mathrm{m}} \frac{\mathrm{d}(x_\mathrm{max} - x_\mathrm{min})}{\mathrm{d}N}\,\mathrm{d}N \int_{\mathcal{L}}^{N} \frac{L\,\mathrm{d}L}{\sqrt{N^2 - L^2}\sqrt{L^2 - \mathcal{L}^2}}. \tag{15}$$

The integral over L is equal to $\pi/2$ (by the substitution $t = L^2$ it can be transformed into integral obtained in [8]) and hence

$$\xi(\mathcal{L}, L_\mathrm{m}) = -\frac{\pi}{2}\int_{\mathcal{L}}^{L_\mathrm{m}} \frac{\mathrm{d}(x_\mathrm{max} - x_\mathrm{min})}{\mathrm{d}N}\,\mathrm{d}N = \frac{\pi}{2}[x_\mathrm{max}(\mathcal{L}) - x_\mathrm{min}(\mathcal{L})]. \tag{16}$$

Here we have used the fact that $x_\mathrm{max}(L_\mathrm{m}) = x_\mathrm{min}(L_\mathrm{m})$. Combining Eqs. (35) and (37) and relabeling $\mathcal{L}$ to L and L to L', we finally get

$$x_\mathrm{max}(L) - x_\mathrm{min}(L) = \ln\frac{r_\mathrm{max}(L)}{r_\mathrm{min}(L)} = \frac{2}{\pi}\int_{L}^{L_\mathrm{m}} \frac{h(L')\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}}\,, \tag{17}$$

which can be re-written as

$$\frac{r_\mathrm{max}(L)}{r_\mathrm{min}(L)} = \exp\left(\frac{2}{\pi}\int_{L}^{L_\mathrm{m}} \frac{h(L')\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}}\right). \tag{18}$$

In addition, we must add the following equations that express the fact that r_min and r_max are really turning points:

$$r_\mathrm{min} n(r_\mathrm{min}) = r_\mathrm{min}\sqrt{2[E - V(r_\mathrm{min})]} = L\,, \tag{19}$$
$$r_\mathrm{max} n(r_\mathrm{max}) = r_\mathrm{max}\sqrt{2[E - V(r_\mathrm{max})]} = L\,. \tag{20}$$

This way, transmuting the potential via logarithm provided a nice solution of the inverse problem and reproduced the result of Ostrovsky [4].

Construction of focusing potentials

Consider the situation when the turning angle φ is independent of L and equal to $h = \pi/m$. If m is a rational number, then the trajectories will be closed. The integral in Eq. (18) then leads to $\operatorname{arccosh}\frac{L'}{L}$ and we get

$$\frac{r_\mathrm{max}(L)}{r_\mathrm{min}(L)} = \exp\left(\frac{2}{m}\operatorname{arccosh}\frac{L_\mathrm{m}}{L}\right) = \left(\frac{L_\mathrm{m}}{L} + \sqrt{\frac{L_\mathrm{m}^2}{L^2} - 1}\right)^{2/m}. \tag{21}$$

From this equation we can express the ratio L_m/L as follows:

$$\frac{L_\mathrm{m}}{L} = \cosh\left(\frac{m}{2}\ln\frac{r_\mathrm{max}(L)}{r_\mathrm{min}(L)}\right) = \frac{1}{2}\left[\left(\frac{r_\mathrm{max}}{r_\mathrm{min}}\right)^{m/2} + \left(\frac{r_\mathrm{min}}{r_\mathrm{max}}\right)^{m/2}\right]. \tag{22}$$

Now the equations (19), (20) and (22) provide a simple recipe for generating focusing potentials. Suppose we choose a function f that, for given r_min, gives $r_\mathrm{max} = f(r_\mathrm{min})$ and at the same time for given r_max, it gives $r_\mathrm{min} = f(r_\mathrm{max})$. This means that the graph of the function f is symmetric with respect to the axis of the first quadrant and the point where the graph and the axis intersect corresponds to $r_{\max} = r_{\min}$ and hence a circular trajectory. With the function f defined this way, we can express L from Eq. (22):

$$L = 2L_{\mathrm{m}} \left[\left(\frac{r}{f(r)} \right)^{m/2} + \left(\frac{f(r)}{r} \right)^{m/2} \right]^{-1} \tag{23}$$

and then substitute this into either of Eqs. (19), (20) to get the refractive index follows:

$$n(r) = \frac{L}{r} = \frac{2L_{\mathrm{m}}}{r} \left[\left(\frac{r}{f(r)} \right)^{m/2} + \left(\frac{f(r)}{r} \right)^{m/2} \right]^{-1} \tag{24}$$

The potential becomes

$$V(r) = E - \frac{2L_{\mathrm{m}}^2}{r^2} \left[\left(\frac{r}{f(r)} \right)^{m/2} + \left(\frac{f(r)}{r} \right)^{m/2} \right]^{-2} \tag{25}$$

We illustrate this method on a few specific examples.

- *Harmonic (or Hooke) potential*

  Take $m = 2$, $L_{\mathrm{m}} = 1$ and $f(r) = \sqrt{2 - r^2}$, which corresponds to the condition of semiaxes of the ellipses $a^2 + b^2 = 2$. Then Eq. (24) yields

$$n(r) = \sqrt{2 - r^2} \tag{26}$$

  and Eq. (25) gives

$$V(r) = E - 1 + \frac{r^2}{2} \tag{27}$$

  Setting $E = 1$, we arrive at the quadratic potential $V(r) = r^2/2$.

- *Newton potential*

  Take $m = 1$, $L_{\mathrm{m}} = 1$ and $f(r) = 2 - r$ since now both the turning points are on the main axis and $a = 1$. Then Eq. (24) yields

$$n(r) = \sqrt{\frac{2}{r} - 1} \tag{28}$$

  and Eq. (25) gives

$$V(r) = E + \frac{1}{2} - \frac{1}{r} \tag{29}$$

  Setting $E = -1/2$, we arrive at the Newton potential $V(r) = -1/r$.

- *Maxwell fish eye*

  Take $m = 1$, $L_{\mathrm{m}} = 1$ and $f(r) = 1/r$. Then Eq. (24) yields

$$n(r) = \frac{2}{1 + r^2}\,, \tag{30}$$

  which is the famous Maxwell fish eye profile. The potencial (when using $E = 0$) is

$$V(r) = -\frac{2}{(1 + r^2)^2} \tag{31}$$

  If we use the same formula $f(r) = a^2/r$ but take a general m, then the general Demkov refractive index is obtained:

$$n(r) = \frac{2}{r\left[r^m + r^{-m}\right]} \tag{32}$$

The inverse problem for infinite motion

The method described above can be adapted also to infinite motion in a potential (scattering) and one can obtain the generally known result [9], Eq. (8).

We start with the scattering angle $\chi = \pi - 2\alpha$, where α is now the angle corresponding to motion from infinity to turning point $r_{\min} = \mathrm{e}^{x_{\min}}$. Now α is given by a formula analogous to Eq. (12), just the upper limit is infinity. We also represent π by a suitable integral. This way we get

$$\chi(L) = \pi - 2L \int_{x_{\min}}^{\infty} \frac{\mathrm{d}x}{\sqrt{N^2(x) - L^2}} = 2L \left(\int_L^{\infty} \frac{\mathrm{d}N}{N\sqrt{N^2 - L^2}} - \int_{x_{\min}}^{\infty} \frac{\mathrm{d}x}{\sqrt{N^2(x) - L^2}} \right) \tag{33}$$

Introducing a new variable $z = \ln N$, using the fact that $\mathrm{d}z/\mathrm{d}N = 1/N$ and changing integration variable in the second integral from x to N, we get

$$\chi(L) = 2L \left(\int_L^{\infty} \frac{\mathrm{d}z}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}} - \int_L^{\infty} \frac{\mathrm{d}x}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}} \right) = 2L \int_L^{\infty} \frac{\mathrm{d}(z - x)}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}} \tag{34}$$

The variable z has a nice interpretation. Since $N = \mathrm{e}^z$, we see that if z were the coordinate in the transmuted plane instead of x, then the refractive index in the original plane would be unity. Indeed, the refractive index in the log-plane is $N = rn(r) = \mathrm{e}^x n(\mathrm{e}^x)$, so if $n = 1$, then $N = \mathrm{e}^x$. This way Eq. (34) compares the motion in the actual potential with the motion that would occur if the potential were constant.

In a similar way as in the case of finite motion, we now divide $\chi(L)$ by $\sqrt{L^2 - \mathcal{L}^2}$, where $\mathcal{L}$ is a parameter, and integrate from $\mathcal{L}$ to infinity:

$$\xi(\mathcal{L}) \equiv \int_{\mathcal{L}}^{\infty} \frac{\chi(L)\,\mathrm{d}L}{\sqrt{L^2 - \mathcal{L}^2}} = 2 \int_{\mathcal{L}}^{\infty} \frac{L\,\mathrm{d}L}{\sqrt{L^2 - \mathcal{L}^2}} \int_L^{\infty} \frac{\mathrm{d}(z - x)}{\mathrm{d}N} \frac{\mathrm{d}N}{\sqrt{N^2 - L^2}}. \tag{35}$$

Inverting the order of integration and changing the limits appropriately, we get

$$\xi(\mathcal{L}) = 2 \int_{\mathcal{L}}^{\infty} \frac{\mathrm{d}(z - x)}{\mathrm{d}N}\,\mathrm{d}N \int_{\mathcal{L}}^{N} \frac{L\,\mathrm{d}L}{\sqrt{N^2 - L^2}\sqrt{L^2 - \mathcal{L}^2}}. \tag{36}$$

Again, the integral over L is $\pi/2$ and hence

$$\xi(\mathcal{L}) = \pi \int_{\mathcal{L}}^{\infty} \frac{\mathrm{d}(z - x)}{\mathrm{d}N}\,\mathrm{d}N = \pi[x_{\min}(\mathcal{L}) - z(\mathcal{L})] \tag{37}$$

Here we have used the fact that in the upper limit the difference between z and x is zero, i.e., $z(\infty) = x(\infty)$. In other words, the refractive index $n(r)$ converges to a constant for $r \to \infty$. This is reasonable, otherwise the scattering problem cannot even be well defined.

So finally we get, after renaming some variables,

$$x_{\min}(L) = \ln L + \frac{1}{\pi} \int_L^{\infty} \frac{\chi(L')\,\mathrm{d}L'}{\sqrt{L'^2 - L^2}}, \tag{38}$$

which is equivalent to Eq. (8) of [9].

We see that the scattering problem and the inverse problem for finite motion can be treated almost identically. Is there more beyond this? How about a situation when $h(L)$ is given for finite motion in some interval of angular momentum and also $\chi(L)$ is given for infinite motion. Can the two cases be somehow combined?

References

[1] U. Leonhardt, *Science* **312**, 1777 (2006).

[2] Yu. N. Demkov, V. N. Ostrovsky, N. B. Berezina, Sov. Phys.-JETP **33**, 867 (1971).

[3] V. N. Ostrovsky, J. Phys. B: At. Mol. Phys. **14**, 4425 (1981).

[4] V. N. Ostrovsky, *Phys. Rev. A* **56**, 526 (1997).

[5] U. Leonhardt and T. Tyc, *Science* **323**, 110 (2009).

[6] J. Bertrand, C. R. Acad. Sci., Paris **77**, 849 (1873).

[7] T. Needham, Visual complex analysis, Oxford Univeristy Press, Oxford 2000.

[8] L. D. Landau, E. M. Lifshitz, Mechanics.

[9] U. Leonhardt, *Phys. Rev. Lett.* **97**, 073902 (2006).