Exam problems for the course
Introduction to General Relativity
fall semester 2020.

These are hand in assignments for the course in "Introduction to General Relativity" given at Masaryk University at the fall semester 2020. They consist the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**. As a minimum requirement to pass the course I have 40 points but more points of course gives higher grades. Please observe that if you hand in just enough problems to get 40 points, chances are that you will have some mistake somewhere and then you will not pass the exam!

1. What is the difference in proper time measured by an astronaut spending a year at the ISS and you staying at the surface of the Earth? Who measures the longest time? Why is this result strange? Can you explain how this can be? (**10p**)

2. Derive the geodesic equation for motion around the center in the Schwarzschild space time in the plane $\theta = \frac{\pi}{2}$. Show that they can be integrated to

$$\dot{t} = \frac{E}{1 - \frac{2M}{r}} \tag{1}$$

$$\dot{\phi} = \frac{L}{r^2} \tag{2}$$

$$\dot{r}^2 = E^2 - V(r) \tag{3}$$

   where E and L are integration constants. Find an explicit expression for $V(r)$ both in the massive and the massless case. The radial equation looks like the energy conservation equation for a particle in a one dimensional effective potential. If we want the orbit to be circular we need $\dot{r} = 0$ so r should be in a minimum of the potential. Analyze the different possibilities as we change the constant L (the angular momentum) both in the massive and massless case. (**10p**)

3. Assuming a static metric with spherical symmetry

$$ds^2 = -f(r)dt^2 + h(r)dr^2 + r^2\left(d\theta^2 + \sin^2(\theta)d\phi^2\right), \tag{4}$$

   solve Einstein's equations with a cosmological constant

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \Lambda g_{\mu\nu} \tag{5}$$

   to show that

$$f(r) = h^{-1}(r) = 1 - \frac{2M}{r} + \frac{\Lambda}{3}r^2 \tag{6}$$

   (**10p**)

4. Consider the space-time geometry

$$ds^2 = -dt^2 + dr^2 + (r^2 + a^2)d\Omega^2 \tag{7}$$

where $d\Omega^2$ is the round metric on S^2. Similar geometries have been proposed to describe "wormholes", tunnels between different asymptotically flat reagions of space-time. The radial coordinate r can take negative as well as positive values. What is the geometry for $r \to \infty$ and $r \to -\infty$? Investigate whether or not there are time-like geodesics traversing the wormhole (from large positive r to large negative r) in finite time. Using the equation for geodesic deviation, show that there is a maximum value for the acceleration gradient in the θ direction experienced by the observer travelling through the wormhole. (**10p**)

5. In the movie "2001: A space oddyssey" they create an artificial gravitational field by having the spaceship rotate. This can be described by the (three dimensional) metric

$$ds^2 = -(1 - \omega^2\rho^2)dt^2 + d\rho^2 + \rho^2 d\theta^2 + 2\omega\rho^2 dt d\theta, \tag{8}$$

where ω is a constant. An observer in the spaceship will think of the direction in increasing ρ as "down" and θ as "left" and "right". (The directions "backwards" and "forwards" have been left out, there is no room for them on the spaceship.) If an observer drops something from the position (ρ_0, θ_0), find the trajectory of the object. Comment on the result. How does this gravitational field differ from the one we are used to on earth? (**10p**)

6. Find for what nonzero value of the exponent a the metric

$$ds^2 = -dt^2 + t^a dx^2 + dy^2 + dz^2, \tag{9}$$

is a solution to the vacuum Einstein equations. For this value of a, assume that two observers located at the space points $(0, 0, 0)$ and $(L, 0, 0)$ are sending light signals to each other. If the sender is emitting pulses with a time separation of Δt what is the separation with which the other observer are receiveing the signals? If the emitted signals have frequency ν, what is the frequency of the detected signal? (**10p**)

7. In the theory of elasticity one uses the so called *strain tensor*

$$u_{ik} = \frac{1}{2}\left(\partial_i u_k + \partial_k u_i\right) \tag{10}$$

where u_i is the *displacement vector*. Both of these objects are tensors and the relation above holds in cartesian coordinates. However, we may choose coordinates such the the three dimensional metric takes the form

$$ds^2 = h_1^2 dx_1^2 + h_2^2 dx_2^2 + h_3^2 dx_3^2, \tag{11}$$

where h_i are (in general) functions of all the x coordinates. In this case find explicit expressions for all the components of the strain tensor in terms of the displacement vector. Compare your result to the known formulas for spherical and cylindrical coordinates. Remember that the "usual" formulas are expressed in terms of an orthonormal basis and not a coordinate basis! (**10p**)

8. A Killing vector field is a vector field satisfying the equation

$$\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0 \tag{12}$$

Show that

$$\nabla_\mu \nabla_\nu \xi_\sigma = R^\alpha_{\mu\nu\sigma} \xi_\alpha \tag{13}$$

(remember that $R^\alpha_{\mu\nu\sigma} + R^\alpha_{\nu\sigma\mu} + R^\alpha_{\sigma\mu\nu} = 0$). Killing vectors allow us to define conserved quantities in General Relativity. Assume that a geodesic observer has 4-velocity u^μ. Show that $\xi_\mu u^\mu$ is constant along his path. Show also that $j_\mu = T_{\mu\nu}\xi^\nu$ is a conserved current ($\nabla^\mu j_\mu = 0$) for any conserved energy momentum tensor ($\nabla^\mu T_{\mu\nu} = 0$). Show that $\xi = \frac{\partial}{\partial t}$ is a Killing vector for the Schwarzschild space-time. Calculate explicitly $\xi_\mu u^\mu$ where u is the 4-velocity of a radially moving geodesic observer. What is the conserved quantity? (**10p**)