Exam problems for the course
Statistical physics and thermodynamics
spring semester 2005.

These are hand in assignments for the course in "Statistical physics and thermodynamics" given at Masaryk University at the spring semester 2005. They make up the first part of the course requirements, the second part being an oral exam. The solutions to the problems should be handed in minimum one week before the oral exam. The answers to the problems can be written in English or Czech, they can be written by hand or on the computer but they should be legible. **Do not leave out any part of the calculation! Motivate your assumptions and approximations carefully**. As a minimum requirement to pass the course I have 24 points but more points of course gives higher grades. Please observe that if you hand in just enough problems to get 24 points, chances are that you will have some mistake somewhere and then you will not pass the exam!

1. Assume that a container contains an ideal gas with constant heat capacity $C_V = \frac{3}{2}kN$. Assume further that there is a small hole of area A in the container through which the gas leaks out to a surrounding vacuum. The leaking is sufficiently slow that the system inside the box can be considered to be in equilibrium all the time. Compute how

   (a) the number of particles inside the box changes with time.

   (b) the energy in the box changes with time.

   (c) the temperature of the system changes with time.

   (6 p)

2. Compute the free energy as a function of T and V of a van der Waals gas with heat capacity C_V independent of *both* temperature and volume. Assume that the equation of state is of the form

$$P = \frac{NkT}{V}\left(1 + \frac{N}{V}B_2(T)\right)$$

   (4 p)

3. Assume that the equation of state of a gas can be written as

$$PV = \alpha E(T, V)$$

   where α is a constant and $E(T, V)$ is the energy of the system.

   (a) Show that the entropy can be written as

$$S = g(TV^\alpha)$$

   and that the energy can be written as

$$E = \frac{1}{V^\alpha} f(TV^\alpha)$$

   for some arbitrary functions f and g fulfilling the relation $f'(x) = xg'(x)$.

(b) Assuming that the ratio $\frac{E}{V}$ depends only on T, show that it can be written

$$\frac{E}{V} = \sigma T^{\frac{\alpha+1}{\alpha}}$$

for some constant σ.

(c) Assume that in the neighborhood of absolute zero, $f(TV^{\alpha})$ can be represented by the power law

$$f(TV^{\alpha}) = cT^{m}V^{\alpha m} \; (m > 1)$$

Find the condition between α and m required by stability $\left(\frac{\partial P}{\partial V}\right)_T < 0$.

(d) Find V as a function of P and T in the limit when $T \to 0$. If the volume goes to zero in this limit it would be inconsistent with the quantum mechanical uncertainty relation. At zero temperature the particles do not move and if the volume goes to zero then we have them also localized which means that we know exactly both position and momentum. Use this as a requirement to find a second relation between m and α. What is the only consistent value of m?

(6 p)

4. Find the difference between C_p and C_V for a van der Waals gas with the equation of state written in the form

$$\left(P + a\left(\frac{N}{V}\right)^2\right)\left(1 - b\frac{N}{V}\right) = kT\frac{N}{V}$$

where a and b are constants.
(4 p)

5. What is the most probable kinetic energy of atoms having a Maxwellian velocity distribution? What is the average kinetic energy?
(4 p)

6. A gas of atoms, each of mass m, is maintained at the absolute temperature T inside an enclosure. The atoms emit light which passes (in the x direction) through a window of the enclosure and can then be observed as a spectral line in a spectroscope. A stationary atom would emit light at the sharply defined frequency ν_0. But, because of the Doppler effect, the frequency of the light observed from an atom having an x component of velocity v_x is not simply equal to the frequency ν_0, but is approximately given by

$$\nu = \nu_0\left(1 + \frac{v_x}{c}\right)$$

where c is the velocity of light. As a result, not all of the light arriving at the spectroscope is at the frequency ν_0; instead it is characterized by some intensity distribution $I(\nu)d\nu$ specifying the fraction of light intensity lying in the frequency range between ν and $\nu + d\nu$. Calculate

(a) The mean frequency $< \nu >$ of the light observed in the spectroscope.

(b) The root-mean square frequency shift $(\Delta\nu)_{\text{rms}} = \sqrt{\left\langle (\nu - < \nu >)^2 \right\rangle}$

(c) The relative intensity distribution $I(\nu)d\nu$ of the light observed in the spectroscope.

(4 p)

7. Consider the earth's atmosphere as an ideal gas of molecules of weight m in a uniform gravitational field. Let g denote the acceleration due to gravity.

(a) If z denotes the height above sea level, show that the change of atmospheric pressure p with height is given by

$$\frac{dp}{p} = -\frac{mg}{kT}dz$$

where T is the absolute temperature at the height z.

(b) If the decrease of pressure in (a) is due to an adiabatic expansion, show that

$$\frac{dp}{p} = \frac{\gamma}{\gamma - 1}\frac{dT}{T}$$

where γ is the ration of specific heats.

(c) From (a) and (b) calculate $\frac{dT}{dz}$ in degrees per kilometer. Assume the atmosphere to consist mostly of nitrogen (N_2) gas for which $\gamma = 1.4$. With how many degrees per kilometer does the atmosphere cool off?

(d) In an isothermal atmosphere at temperature T, express the pressure p at height z in terms of the pressure p_0 at sea level.

(e) If the sea-level pressure and temperature are p_0 and T_0 respectively, and the atmosphere is regarded as adiabatic as in part (b) find again the pressure p at height z.

(6 p)

8. When a sound wave passes through a fluid (liquid or gas), the period of vibration is short compared to the relaxation time necessary for a macroscopically small element of volume of the fluid to exchange energy with the rest of the fluid through heat flow. Hence compressions of such an element of volume can be considered adiabatic.

By analyzing one-dimensional compressions and rarefactions of the system of fluid contained in a slab of thickness dx, show that the pressure $p(x,t)$ in the fluid depends on the position x and the time t so as to satisfy the wave equation

$$\frac{\partial^2 p}{\partial t^2} = u^2 \frac{\partial^2 p}{\partial x^2}$$

where the velocity of sound propagation u is a constant given by $u = \frac{1}{\sqrt{\rho K_S}}$. Here ρ is the equilibrium density of the fluid and K_S is the adiabatic compressibility.

Calculate a numeric value for the value of the velocity of sound in nitrogen (N_2) at room temperature and pressure. Take $\gamma = 1.4$.
(4 p)

9. In some homogeneous substance at absolute temperature T_0 (e.g. a liquid or gas) focus attention on some small portion of mass M. This small portion is in equilibrium with the rest of the substance; it is large enough to be macroscopic and can be characterized by a volume V and a temperature T. Calculate the probability $\mathcal{P}(V,T)dV dT$ that the volume of this portion lies between V and $V + dV$ and that its temperature lies between T and $T + dT$. Express your answer in terms of the isothermal compressibility K_T of the substance, its density ρ_0, and its specific heat per gram c_V at constant volume.
(6 p)

10. The molar latent heat of transformation in going from phase 1 to phase 2 at the temperature T and pressure p is l. What is the latent heat of the phase transformation at a slightly different temperature (and corresponding pressure). Express your answer in terms of l and the molar specific heat c_p, coefficient of expansion $\alpha = \frac{1}{V}\frac{\partial V}{\partial T}$, and molar volume v of each phase at the original temperature T and pressure p.
(4 p)